Commun. Korean Math. Soc. 21 (2006), No. 2, pp. 397–407

POLYNOMIAL-FITTING INTERPOLATION RULES GENERATED BY A LINEAR FUNCTIONAL

Kyung Joong Kim

Abstract. We construct polynomial-fitting interpolation rules to agree with a function f and its first derivative f' at equally spaced nodes on the interval of interest by introducing a linear functional with which we produce systems of linear equations. We also introduce a matrix whose determinant is not zero. Such a property makes it possible to solve the linear systems and then leads to a conclusion that the rules are uniquely determined for the nodes. An example is investigated to compare the rules with Hermite interpolating polynomials.

1. Introduction

In many scientific fields, a simple and convenient formula to approximately represent a function f or to reproduce a given table of numerical values of the function may be needed. Once such a simple formula has been obtained, it can be used in place of f or the table. In particular, polynomials are often used for approximating continuous functions. One reason is that there exist some polynomials to uniformly converge to the continuous functions. This fact is guaranteed by the Stone-Weierstrass approximation theorem [11]. The theorem says that given any function, defined and continuous on a closed and bounded interval, there exists a polynomial that is as close to the given function as desired. Another important reason is that the derivative and indefinite integral of a polynomial are easy to determine and are also polynomials. Therefore, polynomial interpolation is usually used to provide the value of f at a certain point in the interval of interest when the values of f at the mesh points on the interval are assumed to be known. A good-interpolating

Received November 11, 2005.
2000 Mathematics Subject Classification: 65D05.
Key words and phrases: Interpolation rule, Hermite polynomial.
This work was supported by 2004 HanKuk Aviation University Faculty Research Grant.

polynomial needs to provide an accurate approximation over the entire interval, not near a specific point on the interval. Lagrange or Hermite interpolating polynomials are well known as such interpolating polynomials [2].

On occasion, it may happen that the user knows not only the function value f at the mesh points but also its derivative f' at the same mesh points. For example, physical problems that are position-dependent rather than time-dependent are often described in terms of differential equations with conditions imposed at more than one point. In particular, the two-point boundary-value problems involving a second-order differential equation may be solved by a numerical technique which is called a "shooting" method, by analogue to the procedure of firing objects at a stationary target. Then the user disposes not only of the pointwise solution but also of its first derivative. So it makes sense to search for an evaluation of f on the basis of the whole available information because this way the quality of the interpolating polynomials will be normally better than before. This is why Hermite interpolating polynomials are superior in accuracy to Lagrange interpolating polynomials.

The Hermite interpolating polynomials are usually constructed by extending Lagrange interpolating polynomials or by using the divided difference for more computable form [1, 2]. The existence and uniqueness of the Hermite interpolating polynomials are easily obtained by simplifying Hermite interpolation theory with multiple nodes (Chap. 3 in [10]). In this paper, the Hermite interpolating polynomials set up at equally spaced nodes are generated by new approach using a linear functional with which we produce interpolation rules linearly transformed to the Hermite interpolating polynomials. That is, we construct polynomial-fitting interpolating rules by the linear functional which leads to the Hermite interpolating polynomials. With this background, the paper is organized as follows.

In Section 2, we present the form of the polynomial-fitting interpolation rules involving first derivatives and make systems of linear equations from it. In Section 3, a matrix whose determinant is not zero, is introduced to solve the linear systems. In Section 4, we compare the obtained interpolation rules with the Hermite interpolating polynomials and discuss some results.

2. Polynomial-fitting interpolation rules involving first derivatives

Consider a function f and its interpolation rule, denoted by I, which involves not only pointwise values of the function but also of its derivative at equidistant nodes, viz.:

$$
\begin{aligned}
& f(x_0+Nht) \approx I(t) \\
= \; & \alpha_{-N} f(x_0-Nh) + \cdots + \alpha_{-1} f(x_0-h) + \alpha_0 f(x_0) \\
& + \alpha_1 f(x_0+h) + \cdots + \alpha_N f(x_0+Nh) \\
& + h[\beta_{-N} f'(x_0-Nh) + \cdots + \beta_{-1} f'(x_0-h) \\
& + \beta_0 f'(x_0) + \beta_1 f'(x_0+h) + \cdots + \beta_N f'(x_0+Nh)]
\end{aligned} \tag{1}
$$

where N is a positive integer, x_0 is the middle node on the interval of interest, the other nodes on the interval are equally spaced by h and $-1 \le t \le 1$. Note that, for each t in $[-1,1]$, x_0+Nht corresponds to a certain value in $[x_0-Nh, x_0+Nh]$. Therefore, $I(t)$, defined on $[-1,1]$, approximates the function f on the whole range of the closed interval $[x_0-Nh, x_0+Nh]$ by using the function value and its first derivative at nodes $x_0-Nh, \ldots, x_0-h, x_0, x_0+h, \ldots, x_0+Nh$. For convenience, keep taking the notations α_k and β_k instead of $\alpha_k(t)$ and $\beta_k(t)$ indicating that α_k and β_k depend on t.

Based on the ideas which were introduced in [3] and then more investigated in [4, 5, 6, 7, 8], we take a linear functional $L(f(x), h, C)$,

$$
\begin{aligned}
& L(f(x), h, C) \\
= \; & f(x+Nht) \\
& -[\alpha_{-N} f(x-Nh) + \cdots + \alpha_{-1} f(x-h) + \alpha_0 f(x) \\
& \quad + \alpha_1 f(x+h) + \cdots + \alpha_N f(x+Nh)] \\
& -h[\beta_{-N} f'(x-Nh) + \cdots + \beta_{-1} f'(x-h) + \beta_0 f'(x) \\
& \quad + \beta_1 f'(x+h) + \cdots + \beta_N f'(x+Nh)]
\end{aligned} \tag{2}
$$

where C is the vector of coefficients α_k and β_k which have to be determined, $C = (\alpha_{-N}, \alpha_{-N+1}, \ldots, \alpha_N, \beta_{-N}, \beta_{-N+1}, \ldots, \beta_N)$. When the values of the function f and its first derivative f' at the $2N+1$ nodes are assumed to be known, our problem is to determine the values of coefficients α_k and β_k from the conditions

$$
L(x^{n-1}, h, C) = 0 \quad (n = 1, 2, \ldots). \tag{3}
$$

By inserting each monomial $f(x) = 1, x, x^2, \ldots$ into (2), we get

(4)

$$\begin{aligned} L(1,h,\mathcal{C}) &= 1-(\alpha_{-N}+\cdots+\alpha_{-1}+\alpha_0+\alpha_1+\cdots+\alpha_N), \\ L(x,h,\mathcal{C}) &= x[1-(\alpha_{-N}+\cdots+\alpha_{-1}+\alpha_0+\alpha_1+\cdots+\alpha_N)] + \\ &\quad h[Nt+(\alpha_{-N}N+\cdots+\alpha_{-1}-\alpha_1-\cdots-\alpha_N N) \\ &\quad -(\beta_{-N}+\cdots+\beta_{-1}+\beta_0+\beta_1+\cdots+\beta_N)], \\ L(x^2,h,\mathcal{C}) &= x^2[1-(\alpha_{-N}+\cdots+\alpha_{-1}+\alpha_0+\alpha_1+\cdots+\alpha_N)] + \\ &\quad 2hx[Nt+(\alpha_{-N}N+\cdots+\alpha_{-1}-\alpha_1-\cdots-\alpha_N N) \\ &\quad -(\beta_{-N}+\cdots+\beta_{-1}+\beta_0+\beta_1+\cdots+\beta_N)] + \\ &\quad h^2[(Nt)^2-(\alpha_{-N}N^2+\cdots+\alpha_{-1}+\alpha_1+\cdots+\alpha_N N^2) \\ &\quad +2(\beta_{-N}N+\cdots+\beta_{-1}-\beta_1-\cdots-\beta_N N)], \\ &\vdots \end{aligned}$$

The values of $L(x^m, h, \mathcal{C})\,(m = 0, 1, 2, \ldots)$ at $x = 0$, will be denoted by $L_m(h, \mathcal{C})$ and called moments. Then we have

$$\begin{aligned} L_0(h,\mathcal{C}) &= 1-\textstyle\sum_{k=1}^N \alpha_k^+ - \alpha_0, \\ L_1(h,\mathcal{C}) &= h\left(Nt+\textstyle\sum_{k=1}^N \alpha_k^-(N+1-k)-\sum_{k=1}^N \beta_k^+ - \beta_0\right), \\ L_2(h,\mathcal{C}) &= h^2\Big((Nt)^2-\textstyle\sum_{k=1}^N \alpha_k^+(N+1-k)^2 \\ &\quad +2\textstyle\sum_{k=1}^N \beta_k^-(N+1-k)\Big), \\ &\vdots \end{aligned} \tag{5}$$

or, in general, for even $m \geq 2$

$$\begin{aligned} L_m(h,\mathcal{C}) &= h^m\Big((Nt)^m-\textstyle\sum_{k=1}^N \alpha_k^+(N+1-k)^m \\ &\quad +m\textstyle\sum_{k=1}^N \beta_k^-(N+1-k)^{m-1}\Big), \end{aligned} \tag{6}$$

and for odd $m \geq 3$

$$\begin{aligned} L_m(h,\mathcal{C}) &= h^m\Big((Nt)^m+\textstyle\sum_{k=1}^N \alpha_k^-(N+1-k)^m \\ &\quad -m\textstyle\sum_{k=1}^N \beta_k^+(N+1-k)^{m-1}\Big), \end{aligned} \tag{7}$$

where

$$\begin{aligned} \alpha_k^+ &= \alpha_{-N-1+k}+\alpha_{N+1-k}, & \alpha_k^- &= \alpha_{-N-1+k}-\alpha_{N+1-k}, \\ \beta_k^+ &= \beta_{-N-1+k}+\beta_{N+1-k}, & \beta_k^- &= \beta_{-N-1+k}-\beta_{N+1-k}. \end{aligned} \tag{8}$$

With the moments, (4) can be rewritten as follows:

$$(9)\qquad \begin{aligned} L(1,h,\mathcal{C}) &= L_0(h,\mathcal{C}),\\ L(x,h,\mathcal{C}) &= xL_0(h,\mathcal{C})+L_1(h,\mathcal{C}),\\ L(x^2,h,\mathcal{C}) &= x^2L_0(h,\mathcal{C})+2xL_1(h,\mathcal{C})+L_2(h,\mathcal{C}),\\ &\ \vdots \end{aligned}$$

Since L in (2) is a linear functional, it follows that, upon taking $f(x)$ as an expansion of power functions, $f(x)=a_0+a_1x+a_2x^2+\cdots$, we have

$$(10)\qquad \begin{aligned} L(f(x),h,\mathcal{C}) &= \textstyle\sum_{m=0}^{\infty} a_m L(x^m,h,\mathcal{C})\\ &= L_0(h,\mathcal{C})(a_0+a_1x+a_2x^2+\cdots)+\\ &\quad L_1(h,\mathcal{C})(a_1+2a_2x+3a_3x^2+\cdots)+\\ &\quad L_2(h,\mathcal{C})(a_2+3a_3x+6a_4x^2+\cdots)+\cdots\\ &= L_0(h,\mathcal{C})f(x)+\tfrac{1}{1!}L_1(h,\mathcal{C})f^{(1)}(x)\\ &\quad +\tfrac{1}{2!}L_2(h,\mathcal{C})f^{(2)}(x)+\cdots\\ &= \textstyle\sum_{m=0}^{\infty}\frac{1}{m!}L_m(h,\mathcal{C})f^{(m)}(x). \end{aligned}$$

We now address the problem of determining the values of the coefficients α_k and β_k such that the functional L is identically vanishing at any x and $h \neq 0$ for as many successive terms as the number of parameters. For such purpose it is natural to impose that

$$(11)\qquad L_m(h,\mathcal{C})=0,\quad m=0,1,\ldots,4N+1,$$

since the rule I in (1) has $4N+2$ parameters which consist of the $2N+1$ coefficients α_k and the other $2N+1$ coefficients β_k. Thus, the number of parameters equals the number of conditions to be imposed. We now obtain a system of $4N+2$ linear equations in α_k and β_k (or $\alpha_k^{\pm}$ and $\beta_k^{\pm}$). But, instead of handling the system directly to find its solution α_k and β_k, we break the linear system into two "smaller" linear systems,

$$(12)\qquad AX=P \text{ and } BY=Q,$$

which are easier to handle individually. The former governs coefficients α_0, α_k^+ and β_k^- while the latter does α_k^-, β_0 and β_k^+. In detail, we have

(13)

$$A=\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & 0 & 0 & \cdots & 0 & 0\\ N^2 & (N-1)^2 & \cdots & 2^2 & 1 & 0 & H_0^1 & H_1^1 & \cdots & H_{N-2}^1 & H_{N-1}^1\\ N^4 & (N-1)^4 & \cdots & 2^4 & 1 & 0 & H_0^2 & H_1^2 & \cdots & H_{N-2}^2 & H_{N-1}^2\\ & & & & & & & & & & \\ N^{4N} & (N-1)^{4N} & \ldots & 2^{4N} & 1 & 0 & H_0^{2N} & H_1^{2N} & \cdots & H_{N-2}^{2N} & H_{N-1}^{2N} \end{pmatrix},$$

$$(14)\qquad X=\ (\alpha_1^+\ \ \alpha_2^+\ \ \cdots\ \ \alpha_N^+\ \ \alpha_0\ \ \beta_1^-\ \ \beta_2^-\ \ \cdots\ \ \beta_N^-)^T$$

and

$$P = \begin{pmatrix} 1 & (Nt)^2 & (Nt)^4 & \cdots & (Nt)^{4N} \end{pmatrix}^T, \tag{15}$$

where

$$H_k^\eta = -2\eta(N-k)^{2\eta-1} \ (k = 0, 1, \ldots, N-1 \text{ and } \eta = 1, 2, \ldots, 2N).$$

Likewise,

(16)

$$B = \begin{pmatrix} F_N^0 & F_{N-1}^0 & \cdots & F_2^0 & F_1^0 & 1 & 1 & \cdots & 1 & 1 & 1 \\ F_N^1 & F_{N-1}^1 & \cdots & F_2^1 & F_1^1 & T_0^1 & T_1^1 & \cdots & T_{N-2}^1 & T_{N-1}^1 & 0 \\ F_N^2 & F_{N-1}^2 & \cdots & F_2^2 & F_1^2 & T_0^2 & T_1^2 & \cdots & T_{N-2}^2 & T_{N-1}^2 & 0 \\ & & & & & & & & & & \\ F_N^{2N} & F_{N-1}^{2N} & \cdots & F_2^{2N} & F_1^{2N} & T_0^{2N} & T_1^{2N} & \cdots & T_{N-2}^{2N} & T_{N-1}^{2N} & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} \alpha_1^- & \alpha_2^- & \cdots & \alpha_N^- & \beta_1^+ & \beta_2^+ & \cdots & \beta_N^+ & \beta_0 \end{pmatrix}^T \tag{17}$$

and

$$Q = \begin{pmatrix} Nt & (Nt)^3 & (Nt)^5 & \cdots & (Nt)^{4N+1} \end{pmatrix}^T, \tag{18}$$

where

$$F_m^\mu = -m^{2\mu+1} \ (1 \le m \le N \text{ and } 0 \le \mu \le 2N)$$

and

$$T_k^\eta = (2\eta+1)(N-k)^{2\eta} \ (0 \le k \le N-1 \text{ and } 1 \le \eta \le 2N).$$

The existence of the unique solution of each linear system, $AX = P$ or $BY = Q$, is investigated next section. This will be done by first newly constructing a matrix whose special cases involve the above matrices A and B and then by showing that its determinant is never zero.

3. Determinant

We now improve the techniques that are considered in [9] to construct classical integration formulas. Assume that a and b are positive integers. For distinct real number w_j, let W_j denote the column vector $(w_j^a, w_j^{a+2}, \ldots, w_j^{a+2k}, \ldots, w_j^{a+2(2b-1)})^T$. Define a $2b \times 2b$ matrix W as

$$W = (W_1, W_2, \ldots, W_b, W_1', W_2', \ldots, W_b'), \tag{19}$$

where the superscript on W_j means the first derivative of W_j with respect to w_j $(j = 1, 2, \ldots, b)$, that is $W_j' = dW_j/dw_j$. Then, the matrix is written as

(20)
$$W = \begin{pmatrix} w_1^a & \cdots & w_b^a & aw_1^{a-1} & \cdots & aw_b^{a-1} \\ w_1^{a+2} & \cdots & w_b^{a+2} & (a+2)w_1^{a+1} & \cdots & (a+2)w_b^{a+1} \\ \\ w_1^{a+2k} & \cdots & w_b^{a+2k} & (a+2k)w_1^{a+2k-1} & \cdots & (a+2k)w_b^{a+2k-1} \\ \\ w_1^{a+2(2b-1)} & \cdots & w_b^{a+2(2b-1)} & \widetilde{W}_1 & \cdots & \widetilde{W}_b \end{pmatrix},$$

where
$$\widetilde{W}_k = (a+2(2b-1))w_k^{a+2(2b-1)-1} (k = 1, 2, \ldots, b).$$

For the matrix W, we have

THEOREM 1. *The determinant of the matrix W is of the form,*

$$\det(W) = K \prod_{j=1}^{b} w_j^{2a+1} \prod_{i>j} (w_j^2 - w_i^2)^4, \tag{21}$$

where K is a constant.

PROOF. Consider $\det(W)$ as a polynomial $P(w_1)$ in w_1 and expand $\det(W)$ using both the first column and the $(b+1)$th column of W. Then, the lowest degree term in $P(w_1)$ has degree $2a+1$. That is, we have $P(w_1) = w_1^{2a+1}\tilde{P}(w_1)$, where $\tilde{P}(w_1)$ is a polynomial in w_1 with coefficients consisting of polynomials in $w_2, \ldots, w_b$. Moreover, w_j and $-w_j$ $(j = 2, 3, \ldots, b)$ are zeros of $P(w_1)$ with multiplicity four, respectively. Such results come from the fact that the determinant of a matrix with two equal columns is zero. Therefore

$$w_1^{2a+1} \prod_{i=2}^{b} (w_1^2 - w_i^2)^4 \tag{22}$$

is a factor of $\det(W)$. Repeat the above procedure to $\det(W)$ for each w_j and then get other factors of it,

$$w_j^{2a+1} \prod_{i=j+1}^{b} (w_j^2 - w_i^2)^4 \quad \text{for } j = 2, 3, \ldots, b. \tag{23}$$

As a result, the determinant, $\det(W)$, has a factor

$$\prod_{j=1}^{b} w_j^{2a+1} \prod_{i=j+1}^{b} (w_j^2 - w_i^2)^4 \tag{24}$$

so that its degree in all w_j is at least $2ab + 4b^2 - 3b$ because
(25)

$$\sum_{j=1}^{b} (2a + 1 + 8(b - j)) = (2a + 1)b + 8b^2 - \frac{8b(1 + b)}{2} = 2ab + 4b^2 - 3b.$$

On the other hand, a direct calculation of the determinant of W easily shows that the degree of $\det(W)$ in all w_j is exactly the same as (25). For example, by taking w_j out of each column W_j between the first column and the bth column of the $\det(W)$, the degree of $\det(W)$ in all w_j becomes

$$b + \sum_{k=0}^{2b-1} (a + 2k - 1) = 2ab + 4b^2 - 3b. \tag{26}$$

Therefore, we finally have

$$\det(W) = K \prod_{j=1}^{b} w_j^{2a+1} \prod_{i=j+1}^{b} (w_j^2 - w_i^2)^4, \tag{27}$$

where K is a constant which is independent of $w_1, w_2, \ldots, w_b$.

The fact that the determinant of A in (13) is not zero, can be easily obtained by Theorem 1 if we expand $\det(A)$ according to the $(N + 1)$th column of the matrix A. From the expansion of it, only one minor whose size is $2N \times 2N$ survives while other minors all vanish. This nonvanishing minor is exactly the same as the determinant of the matrix W when $a = 2$, $b = N$ and $w_j = N + 1 - j$ in the matrix W. Likewise, expand $\det(B)$ according to the $(2N + 1)$th column of the matrix B to get the same conclusion that $\det(B)$ is not zero. In this case, substitute $a = 3$, $b = N$ and $w_j = N + 1 - j$ into the matrix W. Hence each linear system has the unique solution, respectively. It implies that all the coefficients α_k and β_k of the rule (1) can be determined by algebraically manipulating the relations given in (8).

4. Discussion

In this section, we will discuss the relation between the Hermite interpolating polynomials and the polynomial-fitting interpolation rule $I(t)$ given in (1). By using change of variables, the Hermite interpolating polynomials which are constructed using both function value f and its first derivative f' at equally spaced nodes, can be linearly transformed into the polynomial-fitting interpolation rules. For easy understanding, consider a Hermite interpolating polynomial, denoted by $H_5(x)$, of degree at most five agreeing with f and f' at three nodes $x_0 - h, x_0$ and $x_0 + h$. Then $H_5(x)$ is written as

(28)
$$\begin{aligned} & H_5(x) \\ = & f(x_0-h)H_{2,0}(x) + f(x_0)H_{2,1}(x) + f(x_0+h)H_{2,2}(x) \\ & + f'(x_0-h)\hat{H}_{2,0}(x) + f'(x_0)\hat{H}_{2,1}(x) + f'(x_0+h)\hat{H}_{2,2}(x), \end{aligned}$$

where

(29)
$$\begin{aligned} H_{2,0}(x) &= \left(1-2(x-(x_0-h))L'_{2,0}(x_0-h)\right)L^2_{2,0}(x), \\ H_{2,1}(x) &= \left(1-2(x-x_0)L'_{2,1}(x_0)\right)L^2_{2,1}(x), \\ H_{2,2}(x) &= \left(1-2(x-(x_0+h))L'_{2,2}(x_0+h)\right)L^2_{2,2}(x) \\ \hat{H}_{2,0}(x) &= (x-(x_0-h))\,L^2_{2,0}(x), \\ \hat{H}_{2,1}(x) &= (x-x_0)L^2_{2,1}(x), \\ \hat{H}_{2,2}(x) &= (x-(x_0+h))\,L^2_{2,2}(x), \\ L_{2,0}(x) &= \tfrac{1}{2h^2}(x-x_0)(x-(x_0+h)), \\ L_{2,1}(x) &= -\tfrac{1}{h^2}(x-(x_0-h))(x-(x_0+h)), \\ L_{2,2}(x) &= \tfrac{1}{2h^2}(x-(x_0-h))(x-x_0). \end{aligned}$$

As might be expected, it is easily checked that

(30)
$$\begin{aligned} & H_5(x_0-h) = f(x_0-h), \quad H_5(x_0) = f(x_0), \quad H_5(x_0+h) = f(x_0+h), \\ & H'_5(x_0-h) = f'(x_0-h), \quad H'_5(x_0) = f'(x_0), \quad H'_5(x_0+h) = f'(x_0+h). \end{aligned}$$

By using the change of variables,

$$x = x_0 + ht, \tag{31}$$

the Hermite interpolating polynomial $H_5(x)$, defined on $[x_0 - h, x_0 + h]$, is transformed into a t-dependent function as follows:

(32)
$$\begin{aligned} & H_5(x) \\ = & H_5(x_0+ht) \\ = & \tfrac{1}{4}t^2(4+3t)(t-1)^2 f(x_0-h) + (t+1)^2(t-1)^2 f(x_0) \\ & + \tfrac{1}{4}t^2(4-3t)(t+1)^2 f(x_0+h) + \tfrac{1}{4}ht^2(t+1)(t-1)^2 f'(x_0-h) \\ & + ht(t+1)^2(t-1)^2 f'(x_0) + \tfrac{1}{4}ht^2(t-1)(t+1)^2 f'(x_0+h), \end{aligned}$$

where t is in $[-1, 1]$. This function is exactly the same as the polynomial-fitting interpolation rule I which is obtained by (1) after determining the coefficients corresponding to f and f' at three nodes $x_0 - h$, x_0 and $x_0 + h$. Moreover, the same results as (30) can be obtained from (32). Therefore the t-dependent function is expected to become the Hermite interpolating polynomial (28) without performing the change of variables because of the existence and uniqueness of the Hermite interpolating polynomial. In detail, note that, in this case, we have

$$(33)\qquad A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 1 & 0 & -4 \end{pmatrix}, \qquad X = \begin{pmatrix} \alpha_1^+ \\ \alpha_0 \\ \beta_1^- \end{pmatrix}, \qquad P = \begin{pmatrix} 1 \\ t^2 \\ t^4 \end{pmatrix}$$

and

$$(34)\qquad B = \begin{pmatrix} -1 & 1 & 1 \\ -1 & 3 & 0 \\ -1 & 5 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} \alpha_1^- \\ \beta_1^+ \\ \beta_0 \end{pmatrix}, \qquad Q = \begin{pmatrix} t \\ t^3 \\ t^5 \end{pmatrix}.$$

After solving the associated linear systems,

$$AX = P \text{ and } BY = Q,$$

all α_k and β_k are computed from Eqs. (8). Thus, for $N = 1$ we obtain the form of the rule I which is exactly the same as (32). This way the Hermite interpolating polynomial is linearly transformed to the interpolation rule I. Likewise, the rule I can be also linearly transformed to the Hermite interpolating polynomial.

In order to get the t-dependent interpolation rule I, we use matrix computations and simple algebraic calculations with Eqs. (8) through which the rule I produces the Hermite interpolating polynomial.

References

[1] K. E. Atkinson, *An Introduction to Numerical Analysis*, John Wiley and Sons, 1989.

[2] R. L. Burden and J. D. Faires, *Numerical Analysis*, Brooks/Cole, 2001.

[3] L. Gr. Ixaru, *Operations on Oscillatory Functions*, Comput. Phys. Comm. **105** (1997), 1–19.

[4] L. Gr. Ixaru and B. Paternoster, *A Gauss Quadrature Rule for Oscillatory Integrands*, Comput. Phys. Comm. **133** (2001), 177–188.

[5] L. Gr. Ixaru, M. Rizea, H. De Meyer, G. Vanden Berghe, *Weights of the exponential fitting multistep algorithms for first order ODEs*, J. Comput. Appl. Math. **132** (2001), 83–93.

[6] L. Gr. Ixaru, G. Vanden Berghe and M. De Meyer, *Exponentially fitted variable two-step BDF algorithm for first order ODEs*, Comput. Phys. Comm. **150** (2003), 116–128.

[7] ———, *Frequency evaluation in exponential fitting multistep algorithms for ODEs*, J. Comput. Appl. Math. **140** (2002), 423–434.

[8] K. J. Kim, *Two-frequency-dependent Gauss quadrature rules*, J. Comput. Appl. Math. **174** (2005), 43–55.

[9] K. J. Kim, R. Cools and L. Gr. Ixaru, *Quadrature rules using first derivatives for oscillatory integrands*, J. Comput. Appl. Math. **140** (2002), 479–497.

[10] V. I. Krylov, *Approximate Calculation of Integrals*, Macmillan, New York, 1962.

[11] W. Rudin, *Principles of Mathematical Analysis*, McGRAW-Hill, Singapore, 1976.

School of General Studies
Hankuk Aviation University
Goyang 412-791, Korea
E-mail: kj_kim@hau.ac.kr