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AVERAGE SHADOWING PROPERTIES ON COMPACT METRIC SPACES

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ABSTRACT. We prove that if a continuous surjective map f on a compact metric space X has the average shadowing property, then every point x is chain recurrent. We also show that if a homeomorphism f has more than two fixed points on S^1, then f does not satisfy the average shadowing property. Moreover, we construct a homeomorphism on a circle which satisfies the shadowing property but not the average shadowing property. This shows that the converse of the theorem 1.1 in [6] is not true.

1. Introduction

The shadowing property(also called the pseudo-orbit tracing property) is one of the most important notions in dynamical systems(see [1]). In [2], Blank introduced the notion of average-shadowing property (see [3]). In [5], Sakai proved that, on a closed C^∞ surface, the C^1 interior of the set of C^1 diffeomorphisms with the average-shadowing property is characterized by the set of Anosov diffeomorphisms. In [6], Zhang proved that whenever a homeomorphism f on a compact metric space X has the average-shadowing property, every point x in X is chain recurrent.

In this paper, we extend the property of homeomorphisms of the theorem 1.1 in [6] to the property of continuous surjective maps.

THEOREM. [6] *If a homeomorphism f on a compact metric space X has the average shadowing property, then every point x is chain recurrent. Moreover f has only one chain component which is the whole space.*

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And we show that if a homeomorphism f has more than two fixed points on S^1, then f does not satisfy the average shadowing property:

THEOREM. *Let S^1 be a circle and let $f : S^1 \to S^1$ be a self-homeomorphism. If $N[fix(f)] \geq 2$, then f does not satisfy the average shadowing property on S^1.*

Moreover, we show that the converse of the theorem 1.1 in [6] is not true by constructing a homeomorphism on a circle which satisfy the shadowing property, but not the average shadowing property.

2. Notions

Let (X, d) be a compact metric space and let $f : X \to X$ be a homeomorphism of X onto itself. A subset $Fix(f)$ of X is called a fixed point set of f on X. Let $N[Fix(f)]$ denote the number of the subset $Fix(f)$. A sequence $\{x_n\}_{n\in\mathbb{Z}}$ is called an *orbit* of f if $x_{n+1} = f(x_n)$ for all $n \in \mathbb{Z}$ and a *δ-pseudo-orbit* of f if

$$d(f(x_n), x_{n+1}) \leq \delta, \text{ for all } n \in \mathbb{Z}.$$

We say that the homeomorphism f has the *shadowing property* if for each $\epsilon > 0$ there exists δ such that every δ-pseudo-orbit $\{x_n\}_{n\in\mathbb{Z}}$ is ϵ-shadowed by an orbit $\{f^n(y)|\ n \in \mathbb{Z}\}$ of for some $y \in X$, *i.e.*

$$d(f^n(y),\ x_n) \leq \epsilon \text{ for all } n \in \mathbb{Z}.$$

Let $x,\ y \in X$ be given. We say $x \xrightarrow{f} y$ if and only if for each δ >0, there is a δ-pseudo-orbit $\{x_i\}_0^l$ of some length $l+1$ of f, such that $x = x_0, x_1, \ldots, x_l = y$. It is said that $x \overset{f}{\sim} y$ if and only if $x \xrightarrow{f} y$ and $y \xrightarrow{f} x$. It is denoted $R(f) = \{x \in X | x \overset{f}{\sim} x\}$. It is easy to see that $\overset{f}{\sim}$ is an equivalent relation on $R(f)$. It is called an equivalent class with respect to $\overset{f}{\sim}$ a chain component of f.

For $\delta > 0$, a sequence $\{x_i\}_{i=-\infty}^{\infty}$ of points in X is called an *δ-average-pseudo-orbit* of f if there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$, and k $\in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=1}^{n} d(f(x_{i+k}),\ x_{i+k+1}) < \delta.$$

The average-pseudo-orbits are a certain generalization of the notion of pseudo-orbits. It is said that f has the *average-shadowing property* if for all $\epsilon > 0$, there exists $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i=-\infty}^{\infty}$ is ϵ-shadowed in average by some $z \in X$, that is

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=1}^{n} d(f^i(z),\ x_i) < \epsilon.$$

Let (X, d) be a compact metric space and let f be a continuous map of X onto itself. For $\delta > 0$, a sequence $\{x_i\}_{i=0}^{\infty}$ of points in X is called a δ-average pseudo-orbit of f if there is a natural number $N = N(\delta) > 0$ such that for all $n \geq N$ and $k \geq 0$,

$$\frac{1}{n} \sum_{i=0}^{n-1} d(f(x_{i+k}),\ x_{i+k+1}) < \delta.$$

We say that f has the average shadowing property if there is a metric d for X with the following property: for every $\epsilon > 0$, there is $\delta > 0$ such that every δ-average-pseudo-orbit $\{x_i\}_{i=0}^{\infty}$ is ϵ-shadowed in average by some point $y \in X$; that is

$$\lim_{n\to\infty} \sup \frac{1}{n} \sum_{i=0}^{n-1} d(f^i(y),\ x_i) < \epsilon.$$

We use by $B(x,\ \epsilon)$ the open ball with the center x and the radius ϵ.

3. Average shadowing property

Consider a circle S^1 with coordinate $x \in [0,\ 1)$, and we denote by d the metric on S^1 induced by the usual distance on the real line. When we study the theory of shadowing and average shadowing, usually we only consider the homeomorphisms on S^1 which preserve orientation.

Let $\Pi(x) : \mathbb{R} \to S^1$ be the covering projection defined by the relations

$$\Pi(x) \in [0,\ 1) \text{ and } \Pi(x) \equiv x (x \bmod 1)$$

with respect to the considered coordinates on S^1.

Let $f : S^1 \to S^1$ be a homeomorphism and let a homeomorphism $F : \mathbb{R} \to \mathbb{R}$ be a lifting of f.

THEOREM 3.1. *Let S^1 be a circle and let $f : S^1 \to S^1$ be a self-homeomorphism. If $N[fix(f)] \geq 2$, then f does not satisfy the average shadowing property on S^1.*

PROOF. Let S^1 be a circle and let $f : S^1 \to S^1$ be a homeomorphism. Take two fixed points $\{a,\ b\} \in Fix(f)$ and $\epsilon > 0$ such that $min\{d(a,\ b),\ d(b,\ a)\} > 3\epsilon$. We denote D by the diameter of S^1, that is, $D = max_{(x,y)\in S^1\times S^1} d(x,\ y)$. Consider $\delta > 0$. Take a natural number N such that $\frac{3D}{N} < \delta$. Define a sequence $\{x_i\}_{i=-\infty}^{\infty}$ by

$$x_i = \begin{cases} a, & \text{if } 0 \le i \le N \\ a, & \text{if } 3\cdot 2^j \cdot N + 1 \le i \le 3\cdot 2^{j+1}\cdot N,\ j = 0, 2, 4, \ldots \\ b, & \text{if } N+1 \le i \le 3N \\ b, & \text{if } 3\cdot 2^j \cdot N + 1 \le i \le 3\cdot 2^{j+1}\cdot N,\ j = 1, 3, 5, \ldots . \end{cases}$$

and

$$x_i = \begin{cases} a, & \text{if } -3N \le i \le -N-1 \\ a, & \text{if } -3N\cdot 2^{j+1} \le i \le -3N\cdot 2^{|j|} - 1,\ j = 2, 4, 6, \ldots \\ b, & \text{if} -N \le i \le -1 \\ b, & \text{if } -3N\cdot 2^{j+1} \le i \le -3N\cdot 2^j - 1,\ j = 0, 1, 3, 5, \ldots . \end{cases}$$

Then it is easy to see that for $n > N$ and $k \in \mathbb{Z}$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}),\ x_{i+k+1}) < \frac{1}{n}\cdot\frac{n}{N}\cdot 3D < \delta.$$

Thus $\{x_i\}_{i=0}^{\infty}$ is a δ-average-pseudo orbit of f. We assume that there is a point z in S^1 such that $\{x_i\}_{i=-\infty}^{\infty}$ is ϵ-shadowed in average by z. Then there is a natural number t and a fixed point c of f such that for $n > t$, $f^n(z) \in B(c,\ \epsilon)$ and since $d(a,\ b) > 3\epsilon$,

$$d(f^n(z),\ a) > \epsilon \text{ or } d(f^n(z),\ b) > \epsilon.$$

Hence

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} d(f^i(z),\ x_i) \ge \epsilon.$$

It is a contradiction and so we complete the proof of Theorem 3.1. □

The following Corollary 3.2 shows that the converse of the theorem 1.1 in [6] is not true.

COROLLARY 3.2. *There is a homeomorphism f on S^1 satisfying following:*

1. *any point x in S^1 is chain recurrent;*
2. *f does not satisfy the average shadowing property.*

PROOF. Let $F : [0,\ 1] \to [0,\ 1]$ be a homeomorphism defined by

$$F(t) = \begin{cases} t + (\frac{1}{2} - t)t & \text{if } 0 \le t \le \frac{1}{2} \\ t + (1-t)(t - \frac{1}{2}) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

F induces a homeomorphism $f : S^1 \to S^1$. Obviously $a = \Pi(0)$ and $b = \Pi(\frac{1}{2})$ are fixed points of f. Then any point x in S^1 is chain recurrent of f and by Theorem 3.1, f does not satisfy the average shadowing property. □

The following Remark shows that there is a homeomorphism on S^1 which has the shadowing property, but not the average shadowing property.

REMARK 3.3. Let $F : [0,\ 1] \to [0,\ 1]$ be a homeomorphism defined by

$$F(t) = \begin{cases} t + (\frac{1}{2} - t)t & \text{if } 0 \leq \frac{1}{2} \\ t - (t - \frac{1}{2})(1-t) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

F induces a homeomorphism $f : S^1 \to S^1$. Then $a = \Pi(0)$ and $b = \Pi(\frac{1}{2})$ are fixed points of f. If x is not a and b in S^1, then $\lim_{n\to\infty} f^n(x) = b$. By [4], f has the shadowing property. But the point $\Pi(\frac{1}{2})$ is not chain recurrent point, and by the theorem 1.1 in [6], f does not satisfy the average shadowing property.

We use the theorem 1.1 in [6] to drive another characterization of the average shadowing property.

THEOREM 3.4. *If a continuous surjective map f on a compact metric space X has the average shadowing property, then every point x is chain recurrent. Moreover f has only one chain component which is the whole space.*

PROOF. Let $(X,\ d)$ be a compact metric space and let $f : X \to X$ be a continuous surjective map with the average shadowing property. It is sufficient to prove that for any two different points $x,\ y \in X$, $x \xrightarrow{f} y$.

Let $x,\ y$ be any two different points of X. We denote D by the diameter of X, that is, $D = max_{(x,y)\in X\times X} d(x,\ y)$. If y is in the positive orbit of X, then $x \xrightarrow{f} y$. So we assume that y is not in the positive orbit of x. For any $\epsilon > 0$, take $0 < \epsilon \leq \frac{\epsilon_0}{2}$ such that if $d(x,\ y) < 2\epsilon$, then $d(f(x),\ f(y)) < \epsilon_0$. Let $\delta = \delta(\epsilon) > 0$ be a number as in the definition of the average shadowing property f, that is, every δ-average-pseudo orbit $\{x_i\}_{i=0}^{\infty}$ is ϵ-shadowed in average by some z in X. Fix a sufficient large integer $N_0 > 0$ which $\frac{3D}{N_0} < \delta$.

Consider a subset S_1 of X which satisfy $f(S_1) = \{y\}$. Take a point $y_1 \in S_1$. Then $f(y_1) = y$, for $y_1 \in S$. Again we consider a subset S_i of

X and take a point y_i in S_i satisfying

$$f(S_i) = \{y_{i-1}\},\ 1 < i \leq N_0 - 2.$$

Define a cyclic sequence $\{x_i\}_{i=0}^{\infty}$ by

$$\begin{cases} x_i = f^{[i \ mod \ 2N_0]} & \text{if } [i \ mod \ 2N_0] \in [0,\ N_0] \\ x_i = y_{2N_0-([i \ mod \ 2N_0]+1)} & \text{if } [i \ mod \ 2N_0] \in [N_0+1,\ 2N_0-2] \\ x_i = y & \text{if } [i \ mod \ 2N_0] = 2N_0 - 1. \end{cases}$$

Then for $n \geq N_0$ and $k > 0$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(f(x_{i+k}),\ x_{i+k+1}) < \frac{1}{n}\cdot\frac{n}{N_0}\cdot 3D \leq \frac{3D}{N_0} < \delta.$$

Thus $\{x_i\}_{i=0}^{\infty}$ is a cyclic δ-average-pseudo-orbit of f. Hence it is ϵ-shadowed in average by some $z \in X$, that is,

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z),\ x_i) < \epsilon.$$

Put $P_1 = \{x, f(x), \ldots, f^{N_0}(x)\}$. Then we have a following result.

Claim: There exists an infinite sequence $\{i_1, i_2, \ldots\}$, $i_s < i_k$ for $s < k$ such that

$$B(f^{i_j}(z),\ 2\epsilon) \cap P_1 \neq \phi \text{ and } d(f^{i_j}(z),\ x_{i_j}) < 2\epsilon \text{ for all } i_j \in \{i_1, i_2, \ldots\}.$$

Otherwise, there exists a natural number N such that for all $i > N$, $d(f^i(z),\ x_i) \geq 2\epsilon$. Then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(f^i(z),\ x_i) \geq 2\epsilon,$$

which is a contradiction. Put $P_2 = \{y_{N_0-2}, \ldots, y_1, y\}$. Then similar result hold for P_2. There is an infinite sequence $\{l_1, l_2, \ldots\}$, $l_s < l_k$ if $s < k$ such that

$$B(f^{l_j}(z),\ 2\epsilon) \cap P_2 \neq \phi \text{ and } d(f^{l_j}(z),\ x_{l_j}) < 2\epsilon.$$

Now choose

$$i_0 \in \{i_1, i_2, \ldots\} \text{and } l_0 \in \{l_1, l_2, \ldots\} \text{ with } i_0 < l_0$$

such that $x_{i_0} \in P_1$ and $x_{l_0} \in P_2$. Then

$$d(f^{i_0}(z),\ x_{i_0}) < 2\epsilon \text{ and } d(f^{l_0}(z),\ x_{l_0}) < 2\epsilon.$$

By assuming

$$\begin{cases} x_{i_0} = f^{j_1}(x) & \text{for some } j_1 > 0 \\ x_{l_0} = y_{j_2} & \text{for some } j_2 > 0, \end{cases}$$

we have the following an ϵ_0-pseudo-orbit from x to y

$$x,\ f(x),\ \ldots,\ f^{j_1}(x) = x_{i_0}$$
$$f^{i_0+1}(z),\ f^{i_0+2}(z),\ \ldots,\ f^{l_0-1}(z)$$
$$x_{l_0} = y_{j_2},\ y_{j_2-1},\ \ldots,\ y.$$

This proves $x \xrightarrow{f} y$ and we complete the proof of Theorem 3.4. □

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