

AN EXTENSION OF GENERALIZED VECTOR QUASI-VARIATIONAL INEQUALITY

SANGHO KUM* AND WON KYU KIM

ABSTRACT. In this paper, we shall give an affirmative answer to the question raised by Kim and Tan [1] dealing with generalized vector quasi-variational inequalities which generalize many existence results on (VVI) and (GVQVI) in the literature. Using the maximal element theorem, we derive two theorems on the existence of weak solutions of (GVQVI), one theorem on the existence of strong solution of (GVQVI), and one theorem on strong solution in the 1-dimensional case.

1. Introduction and preliminaries

Let X be a nonempty set. We shall denote by 2^X the family of all subsets of X . A multifunction of X into another nonempty set Y is a function from X into 2^Y . Let E and F be real Hausdorff topological vector spaces, C a multifunction of a set $X \subset E$ into F such that for any $x \in X$, $C(x)$ is a (not necessarily closed) convex cone in F , $\text{int } C(x) \neq \emptyset$ and $C(x) \neq F$, where $\text{int } C(x)$ denotes the interior of $C(x)$. Let $T : X \rightarrow 2^{L(E,F)}$ be a multifunction on $X \subset E$ into $L(E, F)$, the space of all linear continuous operators from E to F , $g : X \rightarrow X$ be a single-valued function, and $A : X \rightarrow 2^X$ be a multifunction.

Consider a generalized vector quasi-variational inequality (in short, GVQVI) with multifunctions :

$$(1.1) \quad \begin{aligned} & \text{(GVQVI) Find } \hat{x} \in X \text{ such that } \hat{x} \in clA(\hat{x}) \text{ and} \\ & \langle T(\hat{x}), x - g(\hat{x}) \rangle \not\subseteq -\text{int } C(\hat{x}) \text{ for every } x \in A(\hat{x}), \end{aligned}$$

Received December 6, 2005.

2000 Mathematics Subject Classification: 47J20, 49J40.

Key words and phrases: generalized vector quasi-variational inequality, equilibrium.

*This work was supported by Korea Research Foundation Grant(KRF-2002-041-C00037).

where $clA(\hat{x})$ denotes the closure of $A(\hat{x})$ in X and $\langle \cdot, \cdot \rangle$ is the evaluation between $L(E, F)$ and E . The motivation for considering this kind of (GVQVI) comes from a recent work of Kim and Tan [6]. They obtained existence results on (GVQVI) with a single valued function T . Then they raised a question :

“ Can we extend the results to the multi-valued case T ? ”

Let us temporarily explain properties of solutions of (GVQVI) : A solution \hat{x} of (GVQVI) is usually called a *weak solution* while a solution \hat{x} is said to be a *strong solution* of (GVQVI) if there exists $\hat{y} \in T(\hat{x})$ (not depending on $x \in A(\hat{x})$) such that

$$(1.2) \quad \langle \hat{y}, x - g(\hat{x}) \rangle \notin -\text{int } C(\hat{x}) \text{ for all } x \in A(\hat{x}).$$

In the case of T being a single valued function, it is meaningless to distinguish a strong solution from a weak one. As remarked in [5], the strong solutions of (GVQVI) have not been systematically investigated until now, and so existence theorems for strong solutions are of real interests since strong solutions can be interpreted as the optimal solutions in numerous practical models (e.g., in mathematical economics, control theory, network problems, et al.). Also we would like to point out that in a recent paper, Chiang et al. [2] discussed strong solutions for a generalized vector quasi-equilibrium problem (in short, GVQEP) under suitable conditions.

The aim of this paper is to give an affirmative answer to the above question raised by Kim and Tan [6] so that we can generalize many existence results on (VVI) and (GVQVI) in the literature (e.g., see [5, 6] and references therein). To be more specific, we derive two theorems on the existence of weak solutions of (GVQVI), one theorem on the existence of strong solution of (GVQVI), and one theorem on strong solution in the special case $F = \mathbb{R}$, hence $L(E, F) = E^*$ the dual space of E .

In relation to this work, the following comment should be made: Actually, in their recent work, Khanh and Luu [5] extended Theorem 1 in [6], one of the main results in [6], to the multi-valued case. However, the contents of their paper and our results are different in several respects. One of them is the proving method. That is, we insist on the original method in [6], say, maximal element theorem whereas they used Fan's KKM Lemma. Also topological assumptions and examples are different. In particular, we need not the closedness of the convex cone $C(x)$

while they assumed it. But, motivated by [5], we could add Theorem 3 regarding the existence of strong solution of (GVQVI). To do so, we shall need the monotone type condition on T as follows:

For any $t_1, t_2 \in T(x)$ and $y_1, y_2 \in X$ with

$$\langle t_1, y_1 - g(x) \rangle \in -\text{int } C(x) \quad \text{and} \quad \langle t_2, y_2 - g(x) \rangle \in -\text{int } C(x);$$

then T satisfies the condition

$$(1.3) \quad \langle t_1, y_2 - g(x) \rangle + \langle t_2, y_1 - g(x) \rangle \in -\text{int } C(x).$$

It should be noted that if T is single-valued, then the condition (1.3) is automatically satisfied since $-\text{int } C(x)$ is a convex cone.

Let us recall some basic terminologies. A multifunction $T : X \rightarrow 2^Y$ is said to be *upper semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \subset V$, then there exists an open neighborhood U of x in X such that $T(y) \subset V$ for each $y \in U$; and a multifunction $T : X \rightarrow 2^Y$ is said to be *lower semicontinuous* if for each $x \in X$ and each open set V in Y with $T(x) \cap V \neq \emptyset$, then there exists an open neighborhood U of x in X such that $T(y) \cap V \neq \emptyset$ for each $y \in U$. A multifunction T is said to be *continuous* if T is both upper semicontinuous and lower semicontinuous.

We shall need the following lemma, which is a special case of Theorem 2 of Ding-Kim-Tan [4]:

LEMMA 1. Let $\Gamma = (X, A, P)$ be an 1-person game such that

- (1) X is a nonempty compact convex subset of a Hausdorff topological vector space,
- (2) $A : X \rightarrow 2^X$ is a multifunction such that for each $x \in X$, $A(x)$ is nonempty convex and for each $y \in X$, $A^{-1}(y)$ is open in X ,
- (3) the multifunction $cl A : X \rightarrow 2^X$ is upper semicontinuous,
- (4) $P : X \rightarrow 2^X$ is a multifunction such that $P^{-1}(y)$ is open in X for each $y \in X$,
- (5) for each $x \in X$, $x \notin co P(x)$, where $co P(x)$ denotes the convex hull of $P(x)$.

Then Γ has an equilibrium choice $\hat{x} \in X$; i.e.,

$$\hat{x} \in cl A(\hat{x}) \quad \text{and} \quad A(\hat{x}) \cap P(\hat{x}) = \emptyset.$$

For non-compact settings, we shall need the following lemma, which is a particular form of Theorem 2 of Ding-Kim-Tan [3]:

LEMMA 2. Let $\Gamma = (X, A, P)$ be an 1-person game such that

- (1) X is a nonempty convex subset of a locally convex Hausdorff topological vector space and D is a nonempty compact subset of X ,
- (2) $A : X \rightarrow 2^D$ is a multifunction such that for each $x \in X$, $A(x)$ is nonempty convex and for each $y \in D$, $A^{-1}(y)$ is open in X ,
- (3) the multifunction $cl A : X \rightarrow 2^X$ is upper semicontinuous,
- (4) $P : X \rightarrow 2^D$ is a multifunction such that $P^{-1}(y)$ is open in X for each $y \in D$,
- (5) for each $x \in X$, $x \notin co P(x)$.

Then Γ has an equilibrium choice $\hat{x} \in D$; i.e.,

$$\hat{x} \in cl A(\hat{x}) \quad \text{and} \quad A(\hat{x}) \cap P(\hat{x}) = \emptyset.$$

2. Main results

We begin with the following result on (GVQVI) :

THEOREM 1. Let X be a nonempty compact convex subset of a Hausdorff topological vector space E and let F be a Hausdorff topological vector space. Let $T : X \rightarrow 2^{L(E,F)}$ be a nonempty compact-valued and upper semicontinuous multifunction, where $L(E, F)$ is equipped with the topology of compact convergence. Let $C : X \rightarrow 2^F$ be a multifunction such that for each $x \in X$, $C(x)$ is a convex cone in F with $-\text{int } C(x) \neq \emptyset$ and $C(x) \neq F$. Suppose that $g : X \rightarrow X$ is continuous and $A : X \rightarrow 2^X$ is a multifunction such that each $A(x)$ is nonempty convex, each $A^{-1}(y)$ is open in X and $cl A : X \rightarrow 2^X$ is upper semicontinuous. Furthermore, assume that the multifunction $W : X \rightarrow 2^F$, $Wx = F \setminus (-\text{int } C(x))$ is closed, that is, the graph $Gr(W)$ of W is closed in $X \times F$ and

$$(2.1) \quad \langle T(x), x - g(x) \rangle \not\subseteq -\text{int } C(x) \text{ for each } x \in X,$$

where $\langle T(x), x - g(x) \rangle$ denotes the set $\{\langle s, x - g(x) \rangle : s \in T(x)\}$.

Then there exists a point $\hat{x} \in X$ such that

$$\hat{x} \in cl A(\hat{x}) \quad \text{and} \quad \langle T(\hat{x}), x - g(\hat{x}) \rangle \not\subseteq -\text{int } C(\hat{x}) \quad \text{for every } x \in A(\hat{x}).$$

PROOF. Define $P : X \rightarrow 2^X$ by

$$P(x) := \{y \in X : \langle T(x), y - g(x) \rangle \subset -\text{int } C(x)\} \quad \text{for each } x \in X.$$

Then $P(x)$ is convex, hence $P(x) = \text{co}P(x)$ for each $x \in X$. Indeed, let $y_1, y_2 \in P(x)$ and $t \in [0, 1]$. Since $\langle T(x), y_i - g(x) \rangle \subset -\text{int } C(x)$ for $i = 1, 2$, we have

$$\begin{aligned} & \langle T(x), ty_1 + (1-t)y_2 - g(x) \rangle \\ & \subset t\langle T(x), y_1 - g(x) \rangle + (1-t)\langle T(x), y_2 - g(x) \rangle \\ & \subset -\text{int } C(x) \end{aligned}$$

because $-\text{int } C(x)$ is a convex cone. By (2.1), $x \notin P(x) = \text{co}P(x)$ for all $x \in X$. This shows that the condition (5) of Lemma 1 is satisfied. Next we shall show that $P^{-1}(y)$ is open in X for each $y \in X$. Note that

$$P^{-1}(y) = \{x \in X : \langle T(x), y - g(x) \rangle \subset -\text{int } C(x)\}.$$

Let $(x_\lambda)_{\lambda \in \Gamma}$ be a net in $(P^{-1}(y))^c = X \setminus P^{-1}(y)$ converging to $\bar{x} \in X$. Since $x_\lambda \in (P^{-1}(y))^c$, we have

$$\langle T(x_\lambda), y - g(x_\lambda) \rangle \cap W(x_\lambda) \neq \emptyset.$$

Thus there exists $s_\lambda \in T(x_\lambda)$ such that $\langle s_\lambda, y - g(x_\lambda) \rangle \in W(x_\lambda)$. Since $T(X)$ is a compact subset of $L(E, F)$, we may assume without loss of generality that there exists $\bar{s} \in T(X)$ such that $s_\lambda \rightarrow \bar{s}$. As T is upper semicontinuous, we get $\bar{s} \in T(\bar{x})$. Hence we have

$$\langle s_\lambda, y - g(x_\lambda) \rangle \rightarrow \langle \bar{s}, y - g(\bar{x}) \rangle \in \langle T(\bar{x}), y - g(\bar{x}) \rangle$$

because $L(E, F)$ is endowed with the topology of compact convergence. Also $\langle \bar{s}, y - g(\bar{x}) \rangle \in W(\bar{x})$ by the closedness of the graph of W . Thus

$$\langle T(\bar{x}), y - g(\bar{x}) \rangle \cap W(\bar{x}) \neq \emptyset.$$

Therefore $\bar{x} \in (P^{-1}(y))^c$, so that $P^{-1}(y)$ is open in X . This shows that the condition (4) of Lemma 1 is also satisfied. By assumptions, the rest of the hypotheses of Lemma 1 are also satisfied so that by Lemma 1, there exists $\hat{x} \in X$ such that $\hat{x} \in \text{cl}A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. Therefore we have $\hat{x} \in \text{cl}A(\hat{x})$ and

$$\langle T(\hat{x}), x - g(\hat{x}) \rangle \not\subset -\text{int } C(\hat{x}) \quad \text{for every } x \in A(\hat{x}).$$

This completes the proof. \square

REMARK 1. Let's look at the minimal topological assumption on T and $L(E, F)$ in [6, Theorem 1]:

“Let the single valued operator $T : X \rightarrow L(E, F)$ be such that
(2.2)

$$\langle T(x_\alpha), y_\alpha \rangle \rightarrow \langle T(x), y \rangle \text{ in } F \text{ whenever } x_\alpha \rightarrow x \text{ and } y_\alpha \rightarrow y \text{ in } X.”$$

We have $\langle T(x_\alpha), y_\alpha \rangle - \langle T(x), y \rangle = \langle T(x_\alpha) - T(x), y_\alpha \rangle + \langle T(x), y_\alpha - y \rangle$, hence (2.2) implies that $T(x_\alpha) \rightarrow T(x)$ uniformly on the net y_α in the compact set X because obviously $\langle T(x), y_\alpha - y \rangle \rightarrow 0$ the origin of F . From this observation it is very natural to assume that $L(E, F)$ is equipped with the topology of compact convergence and $T : X \rightarrow L(E, F)$ is continuous. Actually, among well-known topologies on $L(E, F)$ (for the various topologies on the space $L(E, F)$, refer to the book of Bourbaki [1, III, p. 13-14]) the topology of compact convergence is the most suitable one to ensure the convergence (2.2). Also it can be easily checked that the convergence (2.2) is stronger than the topology of pointwise convergence. Based on these facts, for the development to a multifunction case in Theorem 1, we assumed that $T : X \rightarrow 2^{L(E, F)}$ is compact-valued and upper semicontinuous, where $L(E, F)$ is equipped with the topology of compact convergence. In this respect, we may regard Theorem 1 as a real generalization of [6, Theorem 1] to the multifunction case T . On the other hand, it is mentioned in [5] that the topological assumption corresponding to (2.2) in multi-valued case is fulfilled if E and F are Banach spaces and $T : X \rightarrow 2^{L(E, F)}$ is compact-valued and upper semicontinuous, where $L(E, F)$ is equipped with the usual norm topology.

Using Lemma 2, we can obtain the following existence theorem for (GVQVI) in non-compact sets in a locally convex Hausdorff topological vector space :

THEOREM 2. *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and let D be a nonempty compact subset of X , and let F be a Hausdorff topological vector space. Let $T : X \rightarrow 2^{L(E, F)}$ be a nonempty compact-valued and upper semicontinuous multifunction such that the image $T(X)$ is contained in a compact subset of $L(E, F)$, where $L(E, F)$ is equipped with the topology of compact convergence. Let $C : X \rightarrow 2^F$ be a multifunction such that for each $x \in X$, $C(x)$ is a convex cone in F with $-\text{int}C(x) \neq \emptyset$ and $C(x) \neq F$. Suppose that $g : X \rightarrow D$ is continuous and $A : X \rightarrow 2^D$ is a*

multifunction such that each $A(x)$ is nonempty convex, each $A^{-1}(y)$ is open in X and $cl A : X \rightarrow 2^D$ is upper semicontinuous. Furthermore, assume that the multifunction $W : X \rightarrow 2^F$, $Wx = F \setminus (-int C(x))$ is closed, that is, the graph $Gr(W)$ of W is closed in $X \times F$ and

$$(2.3) \quad \langle T(x), x - g(x) \rangle \not\subseteq -int C(x) \text{ for each } x \in X.$$

Here $\langle T(x), x - g(x) \rangle$ denotes the set $\{ \langle s, x - g(x) \rangle : s \in T(x) \}$.

Then there exists a point $\hat{x} \in D$ such that

$$\hat{x} \in clA(\hat{x}) \quad \text{and} \quad \langle T(\hat{x}), x - g(\hat{x}) \rangle \not\subseteq -int C(\hat{x}) \quad \text{for every } x \in A(\hat{x}).$$

PROOF. Define $P : X \rightarrow 2^D$ by

$$P(x) := \{ y \in D : \langle T(x), y - g(x) \rangle \subset -int C(x) \} \quad \text{for each } x \in X.$$

Then the same argument in proving Theorem 1 shows that $x \notin P(x) = coP(x)$ for all $x \in X$ and $P^{-1}(y)$ is open for each $y \in D$ because the image $T(X)$ is contained in a compact subset of $L(E, F)$. Thus all the hypotheses of Lemma 2 are satisfied so that by Lemma 2, there exists $\hat{x} \in D$ such that $\hat{x} \in clA(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. Hence we have $\hat{x} \in clA(\hat{x})$ and

$$\langle T(\hat{x}), x - g(\hat{x}) \rangle \not\subseteq -int C(\hat{x}) \quad \text{for every } x \in A(\hat{x}).$$

□

REMARK 2. In Theorem 2, we still use the topology of compact convergence on $L(E, F)$ and the upper semicontinuity of T in comparison to the topological assumption (2.2) for a single-valued case in [6, Theorem 2]. Also we supposed that the image $T(X)$ is contained in a compact subset of $L(E, F)$. The reason why we are concerned with those circumstances is that the condition (2.2) mixing the topology of $L(E, F)$ and the continuity of T is not very convenient in dealing with the multi-valued case. It is rather useful to impose those conditions separately.

As an application, we can obtain the single-valued version of Theorem 2 as follows.

COROLLARY 1. (cf. [6, Theorem 2]) *Let X be a nonempty convex subset of a locally convex Hausdorff topological vector space E and*

let D be a nonempty compact subset of X , and let F be a Hausdorff topological vector space. Let $T : X \rightarrow L(E, F)$ be a continuous function such that the image $T(X)$ is contained in a compact subset of $L(E, F)$, where $L(E, F)$ is equipped with the topology of compact convergence. Let $C : X \rightarrow 2^F$ be a multifunction such that for each $x \in X$, $C(x)$ is a convex cone in F with $-\text{int} C(x) \neq \emptyset$ and $C(x) \neq F$. Suppose that $g : X \rightarrow D$ is continuous and $A : X \rightarrow 2^D$ is a multifunction such that each $A(x)$ is nonempty convex, each $A^{-1}(y)$ is open in X and $\text{cl} A : X \rightarrow 2^D$ is upper semicontinuous. Furthermore, assume that the multifunction $W : X \rightarrow 2^F$, $Wx = F \setminus (-\text{int} C(x))$ is closed, that is, the graph $\text{Gr}(W)$ of W is closed in $X \times F$ and

$$\langle T(x), x - g(x) \rangle \notin -\text{int} C(x) \text{ for each } x \in X.$$

Then there exists a point $\hat{x} \in D$ such that

$$\hat{x} \in \text{cl} A(\hat{x}) \quad \text{and} \quad \langle T(\hat{x}), x - g(\hat{x}) \rangle \notin -\text{int} C(\hat{x}) \quad \text{for every } x \in A(\hat{x}).$$

Now we give an example of non-compact setting in which Theorem 2 can be applied, but the corresponding results due to Kim and Tan [6], Chiang et al. [2] and related existence theorems on compact settings are not applicable.

EXAMPLE 1. Let $E = F = \mathbb{R}$, $X = [0, \infty)$, $D = [0, 1]$, and $L(E, F) = E^*$, $C(x) = [0, \infty)$, $g(x) = \{0\}$ for each $x \in X$. Now we consider the following (GVQVI) problem for multifunctions

$$A(x) := \begin{cases} (\sqrt{x}, 1], & \text{if } 0 \leq x < 1, \\ \{1\}, & \text{if } x = 1, \\ (\frac{1}{x}, 1], & \text{if } x > 1, \end{cases}$$

$$T(x) := \begin{cases} [0, x], & \text{if } 0 \leq x \leq 1, \\ [0, \frac{1}{x}], & \text{if } x > 1. \end{cases}$$

Then it is easy to see that the whole assumptions of Theorem 2 are satisfied, so we can obtain the solution $\bar{x} = 1 \in X$ such that $\bar{x} \in \text{cl} A(\bar{x}) = \{1\}$ and $\langle T(\bar{x}), x - g(\bar{x}) \rangle \geq 0$ for every $x \in A(\bar{x})$.

In a recent paper, as an application of the KKM-theorem, Khanh and Luu [5] obtain an existence theorem on a strong solution of (GVQVI) using the pseudomonotone and generalized lower semicontinuous assumptions on T , and they give some applications on traffic equilibrium problems. Motivated by their work we consider the following. If a multifunction T satisfy the previous monotone type condition (1.3), then a strong solution for the (GVQVI) can be obtained :

THEOREM 3. *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E and let F be a Hausdorff topological vector space. Let $T : X \rightarrow 2^{L(E,F)}$ be a nonempty convex-valued and lower semicontinuous multifunction, where $L(E, F)$ is equipped with the topology of compact convergence. Let $C : X \rightarrow 2^F$ be a multifunction such that for each $x \in X$, $C(x)$ is a convex cone in F with $-\text{int } C(x) \neq \emptyset$ and $C(x) \neq F$. Suppose that $g : X \rightarrow X$ is continuous and $A : X \rightarrow 2^X$ is a multifunction such that each $A(x)$ is nonempty convex, each $A^{-1}(y)$ is open in X and $\text{cl } A : X \rightarrow 2^X$ is upper semicontinuous. Furthermore, assume that the multifunction $W : X \rightarrow 2^F$, $Wx = F \setminus (-\text{int } C(x))$ is closed, that is, the graph $\text{Gr}(W)$ of W is closed in $X \times F$ and*

$$(2.4) \quad \langle T(x), x - g(x) \rangle \subseteq F \setminus -\text{int } C(x) \text{ for each } x \in X,$$

where $\langle T(x), x - g(x) \rangle$ denotes the set $\{ \langle s, x - g(x) \rangle : s \in T(x) \}$.

If T satisfy the condition (1.3), then there exists a point $\hat{x} \in X$ such that

$$\hat{x} \in \text{cl } A(\hat{x}) \text{ and } \langle T(\hat{x}), x - g(\hat{x}) \rangle \subseteq F \setminus -\text{int } C(\hat{x}) \text{ for every } x \in A(\hat{x}).$$

PROOF. We first define $P : X \rightarrow 2^X$ by

$$P(x) := \{ y \in X : \exists t \in T(x), \langle t, y - g(x) \rangle \in -\text{int } C(x) \} \text{ for each } x \in X.$$

Then $P(x)$ is convex, hence $P(x) = \text{co}P(x)$ for each $x \in X$. Indeed, let $y_1, y_2 \in P(x)$ and $\lambda \in [0, 1]$. Then there exist $t_1, t_2 \in T(x)$ such that $\langle t_i, y_i - g(x) \rangle \in -\text{int } C(x)$ for $i = 1, 2$. Since $T(x)$ is convex, $\bar{t} := \lambda t_1 + (1 - \lambda)t_2 \in T(x)$ and let $\bar{y} := \lambda y_1 + (1 - \lambda)y_2 \in X$. Then we have

$$\begin{aligned} \langle \bar{t}, \bar{y} - g(x) \rangle &= \langle \lambda t_1 + (1 - \lambda)t_2, \lambda(y_1 - g(x)) + (1 - \lambda)(y_2 - g(x)) \rangle \\ &= \lambda^2 \langle t_1, y_1 - g(x) \rangle + (1 - \lambda)^2 \langle t_2, y_2 - g(x) \rangle \\ &\quad + \lambda(1 - \lambda) (\langle t_1, y_2 - g(x) \rangle + \langle t_2, y_1 - g(x) \rangle). \end{aligned}$$

Since $-\text{int}C(x)$ is a convex cone, by the condition (1.3), we have $\langle \bar{t}, \bar{y} - g(x) \rangle \in -\text{int}C(x)$. Hence $\bar{y} \in P(x)$ so that $P(x)$ is convex. Also, for each $x \in X$, we have $x \notin P(x)$ because of (2.4).

Next we shall show that $P^{-1}(y)$ is open in X for each $y \in X$. Note that

$$P^{-1}(y) = \{x \in X : \exists t \in T(x), \langle t, y - g(x) \rangle \in -\text{int}C(x)\}.$$

We shall show that $(P^{-1}(y))^c$ is closed. Let $(x_\lambda)_{\lambda \in \Gamma}$ be a net in $(P^{-1}(y))^c = X \setminus P^{-1}(y)$ converging to $\bar{x} \in X$. Since $x_\lambda \in (P^{-1}(y))^c$, we have

$$\langle T(x_\lambda), y - g(x_\lambda) \rangle \subseteq F \setminus -\text{int}C(x_\lambda) = W(x_\lambda).$$

Therefore $\langle t_\lambda, y - g(x_\lambda) \rangle \in W(x_\lambda)$ for all $t_\lambda \in T(x_\lambda)$. Let $\bar{s} \in T(\bar{x})$ be arbitrarily fixed. Since T is lower semicontinuous, we get a net $(s_\lambda)_{\lambda \in \Gamma}$ such that $s_\lambda \in T(x_\lambda)$ converging to \bar{s} . Hence we have

$$\langle s_\lambda, y - g(x_\lambda) \rangle \rightarrow \langle \bar{s}, y - g(\bar{x}) \rangle \in \langle T(\bar{x}), y - g(\bar{x}) \rangle$$

because $L(E, F)$ is endowed with the topology of compact convergence. Also $\langle \bar{s}, y - g(\bar{x}) \rangle \in W(\bar{x})$ by the closedness of the graph of W . Thus

$$\langle \bar{s}, y - g(\bar{x}) \rangle \in W(\bar{x}) = F \setminus -\text{int}C(\bar{x}).$$

Since $\bar{s} \in T(\bar{x})$ is arbitrary, we have

$$\langle T(\bar{x}), y - g(\bar{x}) \rangle \subseteq W(\bar{x}) = F \setminus -\text{int}C(\bar{x});$$

so that $\bar{x} \in (P^{-1}(y))^c$; and hence $P^{-1}(y)$ is open in X . This shows that the condition (4) of Lemma 1 is also satisfied. The rest of the hypotheses of Lemma 1 are also satisfied so that there exists $\hat{x} \in X$ such that $\hat{x} \in \text{cl}A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. Therefore we have $\hat{x} \in \text{cl}A(\hat{x})$ and

$$\langle T(\hat{x}), x - g(\hat{x}) \rangle \subseteq F \setminus -\text{int}C(\hat{x}) \quad \text{for every } x \in A(\hat{x}).$$

□

REMARK 3. In contrast to Theorem 1, we need the lower semicontinuity on T and the monotone type assumption (1.3) to obtain a strong solution for (GVQVI) in Theorem 3; however, we do need neither the compactness of $T(x)$ nor the upper semicontinuity on T . Observe that

every solution in Theorem 3 is much more stronger than a strong solution.

Next we give an example in which Theorem 3 can be applied, but the corresponding results in [2, 5, 6] and related existence theorems on compact settings are not applicable.

EXAMPLE 2. Let $E = F = \mathbb{R}$, $X = [0, 2]$, and $L(E, F) = E^*$, $C(x) = [0, \infty)$, $g(x) \equiv x$ for each $x \in X$. Now we consider the following (GVQVI) problem for multifunctions

$$A(x) := [0, 1 - \frac{1}{3}x), \quad \text{for each } x \in X;$$

$$T(x) := \begin{cases} [-\frac{1}{2}x, \frac{1}{2}x + \frac{1}{4}], & \text{if } 0 \leq x \leq 1, \\ (-1, 1), & \text{if } 1 < x \leq 2. \end{cases}$$

Then it is easy to see that T is lower semicontinuous on X . Also, the condition (2.4) is automatically satisfied so that the whole assumptions of Theorem 3 are satisfied. Therefore, we can obtain the strong solution $\bar{x} = 0 \in X$ such that $\bar{x} \in clA(\bar{x})$ and $\langle T(\bar{x}), x - g(\bar{x}) \rangle \geq 0$ for every $x \in A(\bar{x})$. However, T is neither pseudomonotone on X nor upper semicontinuous on X . Hence the previous results in [2, 5, 6] are not available. When T is single-valued, we can obtain the single-valued version of Theorem 3, which is comparable to Theorem 1 in [6] as follows:

COROLLARY 2. (cf. [6, Theorem 1]) *Let X be a nonempty compact convex subset of a Hausdorff topological vector space E and let F be a Hausdorff topological vector space. Let $T : X \rightarrow L(E, F)$ be a continuous function, where $L(E, F)$ is equipped with the topology of compact convergence. Let $C : X \rightarrow 2^F$ be a multifunction such that for each $x \in X$, $C(x)$ is a convex cone in F with $-int C(x) \neq \emptyset$ and $C(x) \neq F$. Suppose that $g : X \rightarrow X$ is continuous and $A : X \rightarrow 2^X$ is a multifunction such that each $A(x)$ is nonempty convex, each $A^{-1}(y)$ is open in X and $cl A : X \rightarrow 2^X$ is upper semicontinuous. Furthermore, assume that the multifunction $W : X \rightarrow 2^F$, $Wx = F \setminus (-int C(x))$ is closed, that is, the graph $Gr(W)$ of W is closed in $X \times F$ and*

$$\langle T(x), x - g(x) \rangle \notin -int C(x) \text{ for each } x \in X.$$

Then there exists a point $\hat{x} \in X$ such that

$$\hat{x} \in clA(\hat{x}) \text{ and } \langle T(\hat{x}), x - g(\hat{x}) \rangle \notin -int C(\hat{x}) \text{ for every } x \in A(\hat{x}).$$

Finally, as another consequence of Theorem 2, we obtain the following.

THEOREM 4. *Let E be a locally convex Hausdorff topological vector space, X be a nonempty convex subset of E and D be a nonempty compact subset of X . Let $T : X \rightarrow 2^{E^*}$ be upper semicontinuous from the relative topology on X to the topology of compact convergence on E^* such that each $T(x)$ is nonempty compact convex and $T(X)$ is contained in a compact subset of E^* . Suppose that $g : X \rightarrow D$ is continuous such that*

$$(2.5) \quad \inf_{w \in T(x)} \langle w, g(x) - x \rangle \leq 0 \quad \text{for all } x \in X;$$

and $A : X \rightarrow 2^D$ is a multifunction such that each $A(x)$ is nonempty convex, each $A^{-1}(y)$ is open in X , and $cl A : X \rightarrow 2^D$ is upper semicontinuous.

Then there exist $\hat{x} \in D$ and $\hat{w} \in T(\hat{x})$ such that

- (1) $\hat{x} \in cl A(\hat{x})$,
- (2) $\langle \hat{w}, g(\hat{x}) - x \rangle \leq 0$ for all $x \in A(\hat{x})$.

PROOF. Putting $F = \mathbb{R}$, $C(x) = [0, \infty)$ and $W(x) = [0, \infty)$ for each $x \in X$, we can easily check that all the assumptions of Theorem 2 are satisfied. Indeed, define $P : X \rightarrow 2^D$ by

$$P(x) := \{y \in D : \inf_{w \in T(x)} \langle w, g(x) - y \rangle > 0\} \quad \text{for all } x \in X.$$

Also observe that (2.5) does correspond to (2.3) of Theorem 2. Then by Theorem 2, there exists $\hat{x} \in D$ such that $\hat{x} \in cl A(\hat{x})$ and $A(\hat{x}) \cap P(\hat{x}) = \emptyset$. Hence $\hat{x} \in cl A(\hat{x})$, and

$$\inf_{w \in T(\hat{x})} \langle w, g(\hat{x}) - y \rangle \leq 0 \quad \text{for all } y \in A(\hat{x}).$$

We now define $f : A(\hat{x}) \times T(\hat{x}) \rightarrow \mathbb{R}$ by

$$f(y, w) := \langle w, g(\hat{x}) - y \rangle \quad \text{for each } (y, w) \in A(\hat{x}) \times T(\hat{x}).$$

Note that for each fixed $y \in A(\hat{x})$, $w \mapsto f(y, w)$ is continuous and affine, and for each $w \in T(\hat{x})$, $y \mapsto f(y, w)$ is affine. Thus, by Kneser's minimax theorem [7], we have

$$\inf_{w \in T(\hat{x})} \sup_{y \in A(\hat{x})} \langle w, g(\hat{x}) - y \rangle = \sup_{y \in A(\hat{x})} \inf_{w \in T(\hat{x})} \langle w, g(\hat{x}) - y \rangle \leq 0.$$

Since $T(\hat{x})$ is compact, there exists $\hat{w} \in T(\hat{x})$ such that

$$\sup_{x \in A(\hat{x})} \langle \hat{w}, g(\hat{x}) - y \rangle = \inf_{w \in T(\hat{x})} \sup_{y \in A(\hat{x})} \langle w, g(\hat{x}) - y \rangle \leq 0.$$

Therefore $\langle \hat{w}, g(\hat{x}) - y \rangle \leq 0$ for all $y \in A(\hat{x})$. \square

REMARK 4. Let us compare the conditions imposed on T between Theorem 4 above and [6, Theorem 3]. The continuity of multifunction T in [6, Theorem 3] is weakened to be upper semicontinuous in Theorem 4. However, the condition that $T(X)$ is contained in a compact set is assumed. Of course, if X is compact, clearly $T(X)$ is compact because T is upper semicontinuous. In this case, Theorem 4 is a strict generalization of [6, Theorem 3] without assuming the lower semicontinuity of T . In the meantime, the proof of Theorem 4 can serve as a simpler one for [6, Theorem 3] in compact settings. Indeed, they adopted an auxiliary lemma [4, Lemma 3] concerning continuity of marginal function to show that $P^{-1}(y)$ is open for each $y \in D$. However, this step is redundant in Theorem 4 because it is enough to check whether all the assumptions of Theorem 2 are satisfied.

References

- [1] N. Bourbaki, *Topological Vector Spaces (translated by H. G. Eggleston and S. Madan)*, Springer-Verlag, New York, 1987.
- [2] Y. Chiang, O. Chadli, and J. C. Yao, *Existence of solutions to implicit vector variational inequalities*, J. Optim. Theory Appl. **116** (2003), 251–264.
- [3] X. -P. Ding, W. K. Kim, and K. -K. Tan, *Equilibria of non-compact generalized games with L^* -majorized preferences*, J. Math. Anal. Appl. **164** (1992), 508–517.
- [4] ———, *Equilibria of generalized games with L -majorized multifunctions*, Int. J. Math. Math. Sci. **17** (1994), 783–790.
- [5] P. Q. Khanh and L. M. Luu, *On the existence of solutions to vector quasi-variational inequalities and quasi-complementarity problems with applications to traffic network equilibria*, J. Optim. Theory Appl. **123** (2004), 533–548.
- [6] W. K. Kim and K. -K. Tan, *On generalized vector quasi-variational inequalities*, Optimization **46** (1999), 185–198.
- [7] H. Kneser, *Sur un théorème fondamental de la théorie des jeux*, C. R. Acad. Sci. Paris **234** (1952), 2418–2420.

Department of Mathematics Education
 Chungbuk National University
 Cheongju 361-763, Korea
E-mail: shkum@chungbuk.ac.kr
 wkkim@chungbuk.ac.kr

