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ON A GENERALIZED DIFFERENCE SEQUENCE SPACES DEFINED BY A MODULUS FUNCTION AND STATISTICAL CONVERGENCE

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ABSTRACT. In this paper, we define the sequence spaces: $[V,\lambda,f,p]_0(\Delta^r,E,u)$, $[V,\lambda,f,p]_1(\Delta^r,E,u)$, $[V,\lambda,f,p]_\infty(\Delta^r,E,u)$, $S_\lambda(\Delta^r,E,u)$, and $S_{\lambda 0}(\Delta^r,E,u)$, where E is any Banach space, and $u=(u_k)$ be any sequence such that $u_k\neq 0$ for any k , examine them and give various properties and inclusion relations on these spaces. We also show that the space $S_\lambda(\Delta^r,E,u)$ may be represented as a $[V,\lambda,f,p]_1(\Delta^r,E,u)$ space. These are generalizations of those defined and studied by M. Et., Y. Altin and H. Altinok [7].

1. Introduction

Let ω be the set of all sequences, real or complex numbers and l_∞, c and c_0 be respectively the Banach spaces of bounded, convergent and null sequences $x=(x_k)$, normed by $\| x \|=\sup_k | x_k |$, where $k\in\mathbb{N}$, the set of positive integers.

Let $\lambda=(\lambda_n)$ be a non-decreasing sequence of positive numbers tending to ∞ such that $\lambda_{n+1}\leq\lambda_n+1$ and $\lambda_1=1$.

The generalized de la Vallée-Poussin means is defined by:

$$t_n(x)=\frac{1}{\lambda_n}\sum_{k\in I_n} x_k,$$

where $I_n=[n-\lambda_n+1,n]$ for $n=1,2,\ldots$.

A sequence $x=(x_k)$ is said to be (V,λ)-summable to a number L(see [12]) if $t_n(x)\to L$, as $n\to\infty$.

If $\lambda_n=n$, (V,λ)-summability and strong (V,λ)-summability are reduced to $(C,1)$-summability and $[C,1]$-summability, respectively.

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The idea of difference sequence spaces was introduced by Kizmaz [10]. In 1981, Kizmaz [10] defined the sequence spaces:

$$X(\Delta) = \{x = (x_k) : \Delta x \in X\}$$

for $X = l_\infty, c$ and c_0, where $\Delta x = (x_k - x_{k+1})$.

Et and Colak [6] generalized the above sequence space to the sequence spaces:

$$X(\Delta^r) = \{x = (x_k) : \Delta^r x \in X\}$$

for $X = l_\infty, c$ and c_0, where $r \in \mathbb{N}$, $\Delta^0 x = (x_k)$, $\Delta x = (x_k - x_{k+1})$, $\Delta^r x = (\Delta^r x_k - \Delta^r x_{k+1})$ and so $\Delta^r x_k = \sum_{v=0}^{r} (-1)^v \binom{r}{v} x_{k+v}$.

Later on, difference sequence spaces were studied by Malkowsky and Parashar [16], Et and Basarir [4], Et and Bektas [5].

We recall that a modulus function f is a function from $[0, \infty)$ to $[0, \infty)$ such that

(i) $f(x) = 0$ if and only if $x = 0$,

(ii) $f(x + y) \leq f(x) + f(y)$ for all $x, y \geq 0$,

(iii) f is increasing,

(iv) f is continuous from the right at 0.

It follows that f must be continuous everywhere on $[0, \infty)$. A modulus may be unbounded or bounded. Ruckle [18] and Maddox [15], used a modulus f to construct some sequence spaces.

Subsequently, modulus function has been discussed in [1], [17], [20] and many others.

Let $X, Y \in \omega$. Then we shall write (see [21])

$$M(X, Y) = \cap_{x \in X} x^{-1} * Y = \{a \in \omega : ax \in Y \text{ for all } x \in X\}.$$

The set $X^\alpha = M(X, l_1)$ is called Kothe-Toeplitz dual space or α-dual of X.

Let X be a sequence space. Then X is called

(i) solid (or normal), if $(\alpha_k x_k) \in X$ for all sequences (α_k) of scalars with $| \alpha_k | \leq 1$ for all $k \in \mathbb{N}$, whenever $(x_k) \in X$.

(ii) symmetric, if $(x_k) \in X$ implies $(x_{\pi(k)}) \in X$, where $\pi(k)$ is permutation of $\mathbb{N}$.

(iii) perfect if $X = X^{\alpha\alpha}$.

(iv) sequence algebra if $x \cdot y \in X$, whenever $x, y \in X$.

It is well known that if X is perfect then X is normal (see [9]).

The following inequality will be used throughout this paper (see [14]):

$$| a_k + b_k |^{p_k} \leq C\{| a_k |^{p_k} + | b_k |^{p_k}\}, \tag{1.1}$$

where $a_k, b_k \in \mathbb{C}$, $0 < p_k \leq \sup_k p = H$, $C = \max(1, 2^{H-1})$.

2. Main results

We prove some results concerning the spaces: $[V,\lambda,f,p]_0(\Delta^r,E,u)$, $[V,\lambda,f,p]_1(\Delta^r,E,u)$, $[V,\lambda,f,p]_\infty(\Delta^r,E,u)$.

DEFINITION 2.1. Let E be a Banach space, we define $\omega(E)$ to be the vector space of all E-valued sequences that is

$$\omega(E)=\{x=(x_k):x_k\in E\}.$$

Let f be a modulus function and $p=(p_k)$ be a sequence of strictly positive real numbers. Then we define the following sequence spaces:

$$\begin{aligned}&[V,\lambda,f,p]_1(\Delta^r,E,u)\\ =\ &\{x\in\omega(E):\lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^r u_kx_k-Le\|)]^{p_k}=0,\ \text{for some } L\},\end{aligned}$$

$$[V,\lambda,f,p]_0(\Delta^r,E,u)=\{x\in\omega(E):\lim_n\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^r u_kx_k\|)]^{p_k}=0\},$$

and

$$\begin{aligned}&[V,\lambda,f,p]_\infty(\Delta^r,E,u)\\ =\ &\{x\in\omega(E):\sup_n\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\|\Delta^r u_kx_k\|)]^{p_k}<\infty\},\end{aligned}$$

where $e=(1,1,1,\ldots)$.

If $u=e$, then these spaces reduce to those defined and studied by M. Et., Y. Altin and H. Altinok [7].

If $x\in[V,\lambda,f,p]_1(\Delta^r,E,u)$, then we will write $x_k\to L[V,\lambda,f,p]_1(\Delta^r,E,u)$ and L will be called the $\lambda_{E,u}$-difference limit of x with respect to the modulus f. Throughout this paper, Z will denote any one of the notations $0,1$, or ∞. In the case $f(x)=x$, $p_k=1$ for all $k\in\mathbb{N}$ we shall write $[V,\lambda]_Z(\Delta^r,E,u)$ instead of $[V,\lambda,f,p]_Z(\Delta^r,E,u)$ and if $p_k=1$ for all $k\in\mathbb{N}$, we shall write $[V,\lambda,f]_Z(\Delta^r,E,u)$ instead of $[V,\lambda,f,p]_Z(\Delta^r,E,u)$.

We prove the following theorems:

THEOREM 2.2. *Let the sequence (p_k) be bounded. Then the sequence spaces $[V,\lambda,f,p]_Z(\Delta^r,E,u)$ are linear spaces.*

PROOF. We shall prove it for $[V,\lambda,f,p]_0(\Delta^r,E,u)$. The others can be proved by the same way. Let $x,y\in[V,\lambda,f,p]_0(\Delta^r,E,u)$ and $\beta,\mu\in\mathbb{C}$.

Then there exists positive numbers M_β and N_μ such that $|\beta| \leq M_\beta$ and $|\mu| \leq N_\mu$. Since f is subadditive and Δ^r is linear, we get:

$$\begin{aligned} & \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r \beta u_k x_k + \mu u_k y_k \|)]^{p_k} \\ \leq & \frac{1}{\lambda_n}\sum_{k\in I_n}[f(|\beta| \| \Delta^r u_k x_k \| + f(|\mu| \| \Delta^r u_k y_k \|))]^{p_k} \\ \leq & C(M_\beta)^H \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k \|)]^{p_k} \\ & + C(N_\mu)^H \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k y_k \|)]^{p_k} \to 0, \text{ as } n\to\infty. \end{aligned}$$

This proves that $[V,\lambda,f,p]_0(\Delta^r,E,u)$ is a linear spaces. □

THEOREM 2.3. *Let f be a modulus function. Then*

$$[V,\lambda,f,p]_0(\Delta^r,E,u) \subset [V,\lambda,f,p]_1(\Delta^r,E,u) \subset [V,\lambda,f,p]_\infty(\Delta^r,E,u).$$

PROOF. The first inclusion is obvious. We establish the second one. Let $x \in [V,\lambda,f,p]_1(\Delta^r,E,u)$. Then by definition of f we have

$$\begin{aligned} & \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k \|)]^{p_k} = \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k - Le + Le \|)]^{p_k} \\ \leq & C\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k - Le \|)]^{p_k} + C\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| Le \|)]^{p_k}. \end{aligned}$$

Now, there exists a positive number K_L such that $\| Le \| \leq K_L$. Hence we have

$$\begin{aligned} & \frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k \|)]^{p_k} \\ \leq & \frac{C}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k - Le \|)]^{p_k} + \frac{C}{\lambda_n}[K_L f(1)]^H \lambda_n. \end{aligned}$$

Since $x \in [V,\lambda,f,p]_1(\Delta^r,E,u)$ we have $x \in [V,\lambda,f,p]_\infty(\Delta^r,E,u)$. This completes the proof. □

THEOREM 2.4. *$[V,\lambda,f,p]_0(\Delta^r,E,u)$ is a paranormed space with*

$$g_\Delta(x) = \sup_n\left(\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k \|)]^{p_k}\right)^{\frac{1}{M}},$$

where $M = \max(1, \sup p_k)$.

PROOF. From Theorem 2.3, for each $x \in [V,\lambda,f,p]_\infty(\Delta^r,E,u)$, $g_\Delta(x)$ exists. Clearly $g_\Delta(x) = g_\Delta(-x)$. It is trivial that $\Delta^r u_k x_k = 0$ for $x = 0$. Since $f(0) = 0$, we get $g_\Delta(x) = 0$ for $x = 0$. Since $^{p_k}/M \leq 1$ and $M \geq 1$, using the Minkowski's inequality and definition of f, for each n, we have

$$\begin{aligned}
&(\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k + \Delta^r u_k y_k \|)]^{p_k})^{\frac{1}{M}} \\
\leq\ & (\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k \|) + f(\| \Delta^r u_k y_k \|)]^{p_k})^{\frac{1}{M}} \\
\leq\ & (\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k x_k \|)]^{p_k})^{\frac{1}{M}} + (\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r u_k y_k \|)]^{p_k})^{\frac{1}{M}}.
\end{aligned}$$

Hence $g_\Delta(x)$ is subadditive. Finally, to check the continuity of multiplication, let us take any complex number β. By definition of f, we have

$$g_\Delta(\beta x) = \sup_n(\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \Delta^r \beta u_k x_k \|)]^{p_k})^{\frac{1}{M}} \leq K_\beta^{\frac{H}{M}} g_\Delta(x),$$

where K_β is a positive integer such that $| \beta | \leq K_\beta$. Now, let $\beta \to 0$ for any fixed x with $g_\Delta(x) \neq 0$. By definition of f for $| \beta | < 1$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \beta\Delta^r u_k x_k \|)]^{p_k} < \epsilon \text{ for } n > n_0(\epsilon). \tag{2.1}$$

Also, for $1 \leq n \leq n_0$, taking β small enough, since f is continuous, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n}[f(\| \beta\Delta^r u_k x_k \|)]^{p_k} < \epsilon\ . \tag{2.2}$$

□

Now, (2.2) and (2.3) together imply that $g_\Delta(\beta x) \to 0$ as $\beta \to 0$.

THEOREM 2.5. *If $r \geq 1$, then the inclusion*

$$[[V,\lambda,f]_Z(\Delta^{r-1},E,u) \subset [V,\lambda,f]_Z(\Delta^r,E,u)$$

is strict. In general $[V,\lambda,f]_Z(\Delta^i,E,u) \subset [V,\lambda,f]_Z(\Delta^r,E,u)$ for all $i = 1,2,\ldots,r-1$ and the inclusion is strict.

PROOF. We give the proof for $Z = \infty$ only. It can be proved in a similar way for $Z = 0$ and $Z = 1$. Let $x \in [V,\lambda,f]_\infty(\Delta^{r-1},E,u)$. Then we have

$$\sup_n \frac{1}{\lambda_n} \sum_{k\in I_n} [f(\| \Delta^{r-1} u_k x_k \|)] < \infty.$$

By definition of f, we have

$$\begin{aligned} & \frac{1}{\lambda_n} \sum_{k\in I_n} [f(\| \Delta^{r} u_k x_k \|)] \\ \leq \ & \frac{1}{\lambda_n} \sum_{k\in I_n} [f(\| \Delta^{r-1} u_k x_k \|)] + \frac{1}{\lambda_n} \sum_{k\in I_n} [f(\| \Delta^{r-1} u_{k+1} x_{k+1} \|)] < \infty. \end{aligned}$$

Thus $[V,\lambda,f]_\infty(\Delta^{r-1},E,u) \subset [V,\lambda,f]_\infty(\Delta^{r},E,u)$. Proceeding in this way, one will have $[V,\lambda,f]_\infty(\Delta^{i},E,u) \subset [V,\lambda,f]_\infty(\Delta^{r},E,u)$ for all $i = 1,2,\dots,r-1$. Let $E = \mathbb{C}$ and $\lambda_n = n$ for each $n \in \mathbb{N}$. Then the sequence $x = (x^r)$, for example, belongs to $[V,\lambda,f]_\infty(\Delta^{r},E,u)$, but does not belong to $[V,\lambda,f]_\infty(\Delta^{r-1},E,u)$ for $f(x) = x$. (If $x = (x^r)$, then $\Delta^r x_k = (-1)^r r!$ and $\Delta^{r-1} x_k = (-1)^{r+1} r!(k + \frac{r-1}{2})$ for all $k \in \mathbb{N}$).

The proof of the following result is a routine work. □

PROPOSITION 2.6. $[V,\lambda,f,p]_1(\Delta^{r-1},E,u) \subset [V,\lambda,f]_0(\Delta^{r},E,u)$.

THEOREM 2.7. *Let f, f_1, and f_2 be modulus functions. Then we have*

(i) $[V,\lambda,f_1,p]_Z(\Delta^r,E,u) \subset [V,\lambda,f\circ f_1,p]_Z(\Delta^r,E,u)$,

(ii) $[V,\lambda,f_1,p]_Z(\Delta^r,E,u) \cap [V,\lambda,f_2,p]_Z(\Delta^r,E,u)$
$\subset [V,\lambda,f+f_1,p]_Z(\Delta^r,E,u)$.

PROOF. We shall only prove (i). Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $f(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_k = f_1(\| \Delta^r u_k x_k \|)$ and consider

$$\sum_{k\in I_n} [f(y_k)]^{p_k} = \sum_1 [f(y_k)]^{p_k} + \sum_2 [f(y_k)]^{p_k},$$

where the first summation is over $y_k \leq \delta$ and the second summation is over $y_k > \delta$. Since f is continuous, we have

$$\sum_1 [f(y_k)]^{p_k} < \lambda_n \epsilon^H \tag{2.3}$$

and for $y_k > \delta$, we use the fact that

$$y_k < \frac{y_k}{\delta} \leq 1 + \frac{y_k}{\delta}.$$

By the definition of f, we have for $y_k > \delta$,

$$f(y_k) < 2f(1)\frac{y_k}{\delta}.$$

Hence

$$\frac{1}{\lambda_n}\sum_2 [f(y_k)]^{p_k} \leq \max(1,(2f(1)\delta^{-1})^H)\frac{1}{\lambda_n}\sum_{k\in I_n} y_k. \tag{2.4}$$

From (2.4) and (2.5), we obtain $[V,\lambda,f,p]_0(\Delta^r,E,u) \subset [V,\lambda,f\circ f_1,p]_0(\Delta^r,E,u)$.

The proof of (ii) follows from the following inequality:

$$[(f_1+f_2)(\| \Delta^r u_k x_k \|)]^{p_k} \leq C[f_1(\| \Delta^r u_k x_k \|)]^{p_k} + C[f_2(\| \Delta^r u_k x_k \|)]^{p_k}.$$

□

The following result is a consequence of Theorem 2.7 (i).

PROPOSITION 2.8. *Let f be a modulus function. Then $[V,\lambda,p]_Z(\Delta^r, E,u) \subset [V,\lambda.f,p]_Z(\Delta^r,E,u)$.*

3. Statistical convergence

The notion of statistical convergence was introduced by Fast [3] and studied by various authors ([2], [8],[11], [13], [17], [19]).

In this section we give some inclusion relations between $S_\lambda(\Delta^r,E,u)$ and $[V,\lambda,p]_1(\Delta^r,E,u)$.

DEFINITION 3.1. A sequence $x=(x_k)$ is said to be $\lambda^r_{E,u}$-statistical convergent to a number L if for every $\epsilon>0$,

$$\lim_n \frac{1}{\lambda_n} \mid \{k\in I_n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\} \mid = 0.$$

In this case we write $S_\lambda(\Delta^r,E,u)-\lim x = L$ or $x_k \to LS_\lambda(\Delta^r,E,u)$. In the case $\lambda_n = n$, and $L=0$, we shall write $S(\Delta^r,E,u)$ and $S_{\lambda_0}(\Delta^r,E,u)$ instead of $S_\lambda(\Delta^r,E,u)$.

THEOREM 3.2. *Let $\lambda=(\lambda_n)$ be the same as in Section 1. Then*

(i) If $x_k \to L\ [V,\lambda]_1(\Delta^r,E,u)$, then $x_k \to LS_\lambda(\Delta^r,E,u)$,

(ii) If $x\in l_\infty(\Delta^r,E,u)$ and $x_k \to LS_\lambda(\Delta^r,E,u)$, then $x_k \to L[V,\lambda]_1(\Delta^r,E,u)$,

(iii) $S_\lambda(\Delta^r,E,u)\cap l_\infty(\Delta^r,E,u) = [V,\lambda]_1(\Delta^r,E,u)\cap l_\infty(\Delta^r,E,u)$, where $l_\infty(\Delta^r,E,u)=\{x\in\omega(E) : \sup_k \| \Delta^r u_k x_k \| < \infty\}$.

PROOF. (i) Let $\epsilon>0$ and $x_k \to L\ [V,\lambda]_1(\Delta^r,E,u)$. Then we have

$$\sum_{k\in I_n} \| \Delta^r u_k x_k - Le \| \geq \epsilon \mid \{k\in I_n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\} \mid .$$

Hence $x_k \to LS_\lambda(\Delta^r,E,u)$.

In fact the set $[V,\lambda]_1(\Delta^r, E, u)$ is a proper subset of $S_\lambda(\Delta^r, E, u)$. To show this, let $E = \mathbb{C}$ and define $x = (x_k)$ such that:

$$\Delta^r u_k x_k = \begin{cases} k & \text{for } n - [\sqrt{n}] + 1 \leq k \leq n \\ 0; & \text{otherwise.} \end{cases}$$

Then $x \notin l_\infty(\Delta^r, E, u)$, $x_k \to 0 S_\lambda(\Delta^r, E, u)$, and $x \notin [V,\lambda]_1(\Delta^r, E, u)$.

(ii) Suppose that $x_k \to L S_\lambda(\Delta^r, E, u)$,and $x \in l_\infty(\Delta^r, E, u)$, say $\| \Delta^r u_k x_k - Le \| \leq M$. Given $\epsilon > 0$, we have

$$\begin{aligned} & \frac{1}{\lambda_n} \sum_{k \in I_n} \| \Delta^r u_k x_k - Le \| \\ = & \frac{1}{\lambda_n} \sum k \in I_n, \\ & \| \Delta^r u_k x_k - Le \| \geq \epsilon \| \Delta^r u_k x_k - Le \| + \frac{1}{\lambda_n} \sum k \in I_n, \\ & \| \Delta^r u_k x_k - Le \| < \epsilon \| \Delta^r u_k x_k - Le \| \\ \leq & \frac{M}{\lambda_n} \{k \in I_n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\} + \epsilon. \end{aligned}$$

Hence x is $\lambda^r_{E,u}$-statistical convergent to a number L.

(iii) This immediately follows from (i) and (ii). □

THEOREM 3.3. *If* $\liminf \frac{\lambda_n}{n} > 0$, *then* $S(\Delta^r, E, u) \subseteq S_\lambda(\Delta^r, E, u)$.

PROOF. For given $\epsilon > 0$,we get

$$\{k \leq n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\} \supset \{k \in I_n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\}.$$

Hence

$$\begin{aligned} & \frac{1}{n} \mid \{k \leq n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\} \mid \\ \geq & \frac{1}{n} \mid \{k \leq n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\} \mid \\ \geq & \frac{\lambda_n}{n} \frac{1}{\lambda_n} \mid \{k \in I_n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\} \mid . \end{aligned}$$

Therefore $x \in S_\lambda(\Delta^r, E, u)$. □

THEOREM 3.4. *Let f be a modulus function and* $\sup p_k = H$. *Then* $[V,\lambda, f, p]_1(\Delta^r, E, u) \subset S_\lambda(\Delta^r, E, u)$.

PROOF. Let $f \in [V,\lambda, f, p]_1(\Delta^r, E, u)$ and $\epsilon > 0$ be given. Let $\sum_1$ denote the sum over $k \leq n$ such that $\| \Delta^r u_k x_k - Le \| \geq \epsilon\}$ and $\sum_2$

denote the sum over $k \leq n$ such that $\| \Delta^r u_k x_k - Le \| < \epsilon\}$. Then

$$\begin{aligned}
& \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\| \Delta^r u_k x_k - Le \|)]^{p_k} \\
= & \frac{1}{\lambda_n} \sum_1 [f(\| \Delta^r u_k x_k - Le \|)]^{p_k} + \frac{1}{\lambda_n} \sum_2 [f(\| \Delta^r u_k x_k - Le \|)]^{p_k} \\
\geq & \frac{1}{\lambda_n} \sum_1 [f(\| \Delta^r u_k x_k - Le \|)]^{p_k} \\
\geq & \frac{1}{\lambda_n} \sum_1 [f(\epsilon)]^{p_k} \\
\geq & \frac{1}{\lambda_n} \sum_1 \min([f(\epsilon)]^{\inf p_k}, [f(\epsilon)]^H) \\
\geq & \frac{1}{\lambda_n} \mid \{k \in I_n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\} \mid \min([f(\epsilon)]^{\inf p_k}, [f(\epsilon)]^H).
\end{aligned}$$

Hence $x \in S_\lambda(\Delta^r, E, u)$. □

THEOREM 3.5. *Let f be bounded and $0 < h = \inf_k p_k \leq p_k \leq \sup_k p_k = H < \infty$. Then $S_\lambda(\Delta^r, E, u) \subset [V, \lambda, f, p]_1(\Delta^r, E, u)$.*

PROOF. Suppose that f is bounded. Let $\epsilon > 0$ be given and $\sum_1$ and $\sum_2$ be as in the previous theorem. Since f is bounded, there exists an integer K such that $f(x) < K$, for all $x \geq 0$. Then

$$\begin{aligned}
& \frac{1}{\lambda_n} \sum_{k \in I_n} [f(\| \Delta^r u_k x_k - Le \|)]^{p_k} \\
= & \frac{1}{\lambda_n} \sum_1 [f(\| \Delta^r u_k x_k - Le \|)]^{p_k} + \frac{1}{\lambda_n} \sum_2 [f(\| \Delta^r u_k x_k - Le \|)]^{p_k} \\
\leq & \frac{1}{\lambda_n} \sum_1 \max(K^h, K^H) + \frac{1}{\lambda_n} \sum_2 [f(\epsilon)]^{p_k} \\
\leq & \max(K^h, K^H) \frac{1}{\lambda_n} \mid \{k \in I_n : \| \Delta^r u_k x_k - Le \| \geq \epsilon\} \mid \\
& + \max(f(\epsilon)^h, f(\epsilon)^H).
\end{aligned}$$

Hence $x \in [V, \lambda, f, p]_1(\Delta^r, E, u)$. □

THEOREM 3.6. *$S_\lambda(\Delta^r, E, u) = [V, \lambda, f, p]_1(\Delta^r, E, u)$ if and only if f is bounded.*

PROOF. Let f be bounded. By Theorems 3.4 and 3.5, we have $S_\lambda(\Delta^r, E, u) = [V, \lambda, f, p]_1(\Delta^r, E, u)$.

Conversely suppose that f is bounded. Then there exists a sequence (t_k) of positive numbers with $f(t_k) = k^2$, for $k = 1, 2, \ldots$. If we choose

$$\Delta^r u_i x_i = \begin{cases} t_k; & i = k^2, i = 1, 2, \ldots \\ 0; & \text{otherwise.} \end{cases}$$

Then we have

$$\frac{1}{\lambda_n} \mid \{k \in I_n : \| \Delta^r u_k x_k \| \geq \epsilon\} \mid \leq \frac{\sqrt{\lambda_n - 1}}{\lambda_n}$$

for all n and so $x \in S_\lambda(\Delta^r, E, u)$, but $x \notin [V, \lambda, f, p]_1(\Delta^r, E, u)$ for $E = \mathbb{C}$. This contradicts to $S_\lambda(\Delta^r, E, u) = [V, \lambda, f, p]_1(\Delta^r, E, u)$.

THEOREM 3.7. *The sequence spaces $[V, \lambda, f, p]_0(\Delta^r, E, u)$, $[V, \lambda, f, p]_1(\Delta^r, E, u)$, $[V, \lambda, f, p]_\infty(\Delta^r, E, u)$, $S_\lambda(\Delta^r, E, u)$ and $S_{\lambda 0}(\Delta^r, E, u)$ are not solid for $r \geq 1$.*

PROOF. Let $E = \mathbb{C}$, $p_k = 1$ for all k, $f(x) = x$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $(x_k) = (x^r) \in [V, \lambda, f, p]_\infty(\Delta^r, E, u)$ when $\alpha_k = k$ for all $k \in \mathbb{N}$. Hence $[V, \lambda, f, p]_\infty(\Delta^r, E, u)$ is not solid. The other cases can be proved on considering similar examples. □

From the above theorems, we may give the following corollary:

COROLLARY 3.8. *The sequence spaces $[V, \lambda, f, p]_0(\Delta^r, E, u)$, $[V, \lambda, f, p]_1(\Delta^r, E, u)$, and $[V, \lambda, f, p]_\infty(\Delta^r, E, u)$, are not perfect for $r \geq 1$.*

THEOREM 3.9. *The sequence spaces $[V, \lambda, f, p]_1(\Delta^r, E, u)$, $S_\lambda(\Delta^r, E, u)$ and $S_{\lambda 0}(\Delta^r, E, u)$ are not symmetric for $r \geq 1$.*

PROOF. Let $E = \mathbb{C}$, $p_k = 1$ for all k, $f(x) = x$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $(x_k) = (x^r) \in [V, \lambda, f, p]_\infty(\Delta^r, E, u)$. Let (y_k) be a rearrangement of (x_k), which is defined as follows

$$(y_k) = \{x_1, x_2, x_4, x_3, x_9, x_5, x_{16}, x_6, x_{25}, x_7, x_{36}, x_8, x_{49}, x_{10}, \ldots\}.$$

Then $(y_k) \notin [V, \lambda, f, p]_\infty(\Delta^r, E, u)$.

For the space $S_{\lambda 0}(\Delta^r, E, u)$, consider the sequence $x = (x_k)$ defined by

$$x_k = \begin{cases} 1; & \text{if } (2i-1)^2 \leq k < (2i)^2, \ i = 1, 2, \ldots \\ 4; & \text{otherwise.} \end{cases}$$

Then $(x_k) \in S_0(\Delta, u)$. Let (y_k) be the same as above, then $(y_k) \notin S_0(\Delta, u)$. □

REMARK 3.10. The space $[V, \lambda, f, p]_\infty(\Delta^r, E, u)$ is not symmetric for $r \geq 2$.

THEOREM 3.11. *The sequence spaces* $[V, \lambda, f, p]_Z(\Delta^r, E, u), S_\lambda(\Delta^r, E, u)$, *and* $S_{\lambda 0}(\Delta^r, E, u)$ *are not sequence algebras.*

PROOF. Let $E = \mathbb{C}$, $p_k = 1$ for all $k \in \mathbb{N}$, $f(x) = x$ and $\lambda_n = n$ for all $n \in \mathbb{N}$. Then $x = (k^{r-2}), y = (k^{r-2}) \in [V, \lambda, f, p]_Z(\Delta^r, E, u)$, but $x \cdot y \notin [V, \lambda, f, p]_Z(\Delta^r, E, u)$. The other cases can be proved on considering similar examples. □

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