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A CHARACTERIZATION OF LOCAL RESOLVENT SETS

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ABSTRACT. Let T be a bounded linear operator on a Banach space X. And let $\rho_T(x)$ be the local resolvent set of T at $x \in X$. Then we prove that a complex number λ belongs to $\rho_T(x)$ if and only if there is a sequence $\{x_n\}$ in X such that $x_n = (T-\lambda)x_{n+1}$ for $n = 0, 1, 2, \ldots$, $x_0 = x$ and $\{\|x_n\|^{\frac{1}{n}}\}$ is bounded.

1. Introduction

Let X be a Banach space over the complex plane $\mathbb{C}$. Let $\mathcal{L}(X)$ denote the Banach algebra of all bounded linear operators on X. For a given $T \in \mathcal{L}(X)$, let $\sigma(T)$ and $\rho(T)$ denote the spectrum and the resolvent set of T, respectively. The *local resolvent set* $\rho_T(x)$ of T at the point $x \in X$ is defined as the union of all open subsets U of $\mathbb{C}$ for which there is an analytic function $f : U \to X$ which satisfies

$$(T-\lambda)f(\lambda) = x \quad \text{for all} \quad \lambda \in U.$$

The *local spectrum* $\sigma_T(x)$ of T at x is then defined as

$$\sigma_T(x) = \mathbb{C} \setminus \rho_T(x).$$

Clearly, the local resolvent set $\rho_T(x)$ is open, and the local spectrum $\sigma_T(x)$ is closed. For each $x \in X$, the function $f(\lambda) : \rho(T) \to X$ defined by

$$f(\lambda) = (T-\lambda)^{-1}x$$

is analytic on $\rho(T)$ and satisfies

$$(T-\lambda)f(\lambda) = x \quad \text{for all} \quad \lambda \in \rho(T).$$

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Hence the resolvent set $\rho(T)$ is always subset of $\rho_T(x)$ and hence $\sigma_T(x)$ is always subset of $\sigma(T)$.

The analytic solutions occurring in the definition of the local resolvent set may be thought of as local extensions of the function $(T-\lambda)^{-1}x : \rho(T) \to X$. There is no uniqueness implied. Thus we need the following definition.

An operator $T \in L(X)$ is said to have the *single-valued extension property*, abbreviated SVEP, if for every open set $U \subseteq \mathbb{C}$, the only analytic solution $f : U \to X$ of the equation

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in U$$

is the zero function on U. Hence if T has the SVEP, then for each $x \in X$ there is the maximal analytic extension of $(T-\lambda)^{-1}x$ on $\rho_T(x)$.

2. The single valued extension property and local resolvent sets

Let $\sigma_p(T)$ denote the point spectrum of $T \in L(X)$.

PROPOSITION 1. *Let $T \in L(X)$ be an operator which has empty interior of its point spectrum. Then T has the SVEP.*

PROOF. Let $T \in L(X)$ be an operator which has empty interior of its point spectrum $\sigma_p(T)$. Suppose that T has not the SVEP. Then there is an analytic function $f : U \to X$ on an open set $U \subseteq \mathbb{C}$ with

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in U,$$

but f is not identically zero on U. Thus there is a $\lambda \in U$ such that $f(\lambda) \neq 0$. Then by the identity theorem there is an open neighborhood W of λ with $W \subseteq U$ and $f(\mu) \neq 0$ for all $\mu \in W$. This implies that

$$W \subseteq \sigma_p(T).$$

This is a contradiction. This completes the proof. □

Many operators on a Banach space X have the empty interior of its point spectrum. For example, if T is a compact operator or Volterra operator or unilateral shift operator or operator which has real spectrum, then $\sigma_p(T)^0 = \emptyset$. Hence these operators have the SVEP.

Let H be a Hilbert space over the complex plane $\mathbb{C}$ with the inner product $\langle\cdot,\cdot\rangle$. And $\mathcal{L}(H)$ denotes the C^*-algebra of bounded linear operators on a Hilbert space H. And let T^* denote the adjoint of T. The operator $T \in \mathcal{L}(H)$ is said to be *hyponormal* if its self commutator $[T^*,T] = T^*T - TT^*$ is positive, that is

$$\langle\, [T^*,T]x, x\,\rangle \;\geq\; 0,$$

or equivalently

$$\|T^*x\| \leq \|Tx\|$$

for every $x \in H$.

Define the *joint point spectrum* of T as follow:

$$\sigma_{jp}(T) = \{\lambda \in \mathbb{C} : \text{there is a non zero } x \in H \text{ such that } (T-\lambda)x = 0 \text{ and } (T^* - \overline{\lambda})x = 0\}.$$

Then clearly $\sigma_{jp}(T) \subseteq \sigma_p(T)$. In the case of some important operator T, $\sigma_p(T) = \sigma_{jp}(T)$. In fact, if T is a hyponormal operator, then $(T-\lambda)$ is also hyponormal and hence $\|(T-\lambda)^*(x)\| \leq \|(T-\lambda)(x)\|$ for all $x \in H$. Thus if T is hyponormal, then $\sigma_p(T) = \sigma_{jp}(T)$.

PROPOSITION 2. *Let T be a bounded linear operator on a Hilbert space H with $\sigma_p(T) = \sigma_{jp}(T)$. Then T has the SVEP.*

PROOF. Let $f : U \to H$ be an analytic function on an open set $U \subseteq \mathbb{C}$ with

$$(T-\lambda)f(\lambda) = 0 \quad \text{for all} \quad \lambda \in U.$$

Let λ be a fixed element in U. Define the map $g : U \to \mathbb{C}$ by

$$g(\zeta) = \langle f(\lambda), f(\zeta)\rangle \quad \text{for all} \quad \zeta \in U.$$

Then $g : U \to \mathbb{C}$ is continuous on U. Then for every $\mu \in U$ such that $\mu \neq \lambda$, $Tf(\mu) = \mu f(\mu)$. Since $\sigma_p(T) = \sigma_{jp}(T)$, we have

$$T^*f(\mu) = \overline{\mu}f(\mu) \quad \text{for all} \quad \mu \in \mathbb{C}\setminus\{\lambda\}.$$

Therefore, we have

$$\begin{aligned} \lambda g(\mu) &= \langle \lambda f(\lambda), f(\mu)\rangle \\ &= \langle Tf(\lambda), f(\mu)\rangle \\ &= \langle f(\lambda), \overline{\mu} g(\mu)\rangle \\ &= \mu g(\mu). \end{aligned}$$

Since $\lambda \neq \mu$, $g(\mu) = 0$. Since $\mu \in U$ and $\mu \neq \lambda$, we have

$$g(\mu) = 0 \quad \text{for all} \quad \mu \in U \setminus \{\lambda\}.$$

But then, by the continuity of g, $g(\lambda) = 0$. This implies that $f(\lambda) = 0$. Since λ is arbitrary, f is the zero function on U. This completes the proof. □

COROLLARY 3. *Let T be a hyponormal operator on a Hilbert space H. Then T has the SVEP.*

PROPOSITION 4. *Let T be a bounded linear operator on a Banach space X and let $x \in X$. Then a complex number λ belongs to $\rho_T(x)$ if and only if there is a sequence $\{x_n\}$ in X such that $x_n = (T - \lambda)x_{n+1}$ for $n = 0, 1, 2, \ldots$, $x_0 = x$ and $\{\|x_n\|^{\frac{1}{n}}\}$ is bounded.*

PROOF. ($\Rightarrow$) Let $\lambda \in \rho_T(x)$. And let $x_0 = x$. Then there is an open neighborhood U of λ and there is an analytic function $f : U \to X$ such that

$$(T - \mu)f(\mu) = x_0 \quad \text{for all} \quad \mu \in U.$$

By differentiation n-times of the analytic constant function $(T - \lambda)f(\mu) = x_0$ on U, we have

$$(T - \mu)f^{(n)}(\mu) = nf^{(n-1)}(\mu)$$

for all $\mu \in U$ and $n = 1, 2, \ldots$. Let

$$x_n = \frac{1}{(n-1)!}f^{(n-1)}(\lambda), \qquad n = 1, 2, \ldots .$$

Then we have

$$\begin{aligned}(T - \lambda)x_{n+1} &= \frac{1}{n!}(T - \lambda)f^{(n)}(\lambda) \\ &= \frac{1}{n!}nf^{(n-1)}(\lambda) \\ &= \frac{1}{(n-1)!}f^{(n-1)}(\lambda) \\ &= x_n\end{aligned}$$

for all $n = 0, 1, \ldots$. Since $f : U \to X$ is analytic,

$$\limsup_{n\to\infty} \|x_n\|^{\frac{1}{n}} = \limsup_{n\to\infty} \|\frac{1}{(n-1)!}f^{(n-1)}(\lambda)\|^{\frac{1}{n}} < \infty.$$

Hence $\{\|x_n\|^{\frac{1}{n}}\}$ is bounded.

($\Leftarrow$) Let $x = x_0$. Define the function

$$g(\mu) = \sum_{n=1}^{\infty} x_n(\mu - \lambda)^{n-1}.$$

Since $\{\|x_n\|^{\frac{1}{n}}\}$ is bounded, g is analytic on $\{\mu \in \mathbb{C} : |\mu - \lambda| < M\}$, where $M = \dfrac{1}{\limsup_{n\to\infty}\|x_n\|^{\frac{1}{n}}}$. Then we have

$$\begin{aligned}(T-\mu)g(\mu) &= (T-\mu)(\sum_{n=1}^{\infty} x_n(\mu-\lambda)^{n-1}) \\ &= \sum_{n=1}^{\infty}(T-\lambda-(\mu-\lambda))x_n(\mu-\lambda)^{n-1} \\ &= \sum_{n=1}^{\infty}(T-\lambda)x_n(\mu-\lambda)^{n-1} - \sum_{n=1}^{\infty} x_n(\mu-\lambda)^n \\ &= \sum_{n=1}^{\infty} x_{n-1}(\mu-\lambda)^{n-1} - \sum_{n=1}^{\infty} x_n(\mu-\lambda)^n \\ &= x_0.\end{aligned}$$

Hence $\lambda \in \rho_T(x)$. □

EXAMPLE. Let $C(\Omega)$ be the Banach algebra of all continuous complex valued functions on a compact Hausdorff space Ω endowed with pointwise operations and the supremum norm. For a given $g \in C(\Omega)$, let T be the operator of multiplication on $C(\Omega)$ by $g \in C(\Omega)$. That is,

$$(Tf)(\lambda) = g(\lambda)f(\lambda) \quad \text{for all} \quad f \in C(\Omega).$$

Clearly $\sigma(T) = g(\Omega)$. We claim that $\sigma_T(f) = g(\mathrm{supp} f)$ for all $f \in C(\Omega)$, where

$$\mathrm{supp} f = \overline{\{\lambda \in \Omega : f(\lambda) \neq 0\}}$$

denotes, as usual, the *support* of the function $f \in C(\Omega)$.

To verify this, let $f \in C(\Omega)$ be given. And let $\omega \in \Omega$ with $f(\omega) \neq 0$. Then for each $k \in C(\Omega)$,

$$(T - g(\omega))k(\omega) = 0.$$

Hence we have

$$(T - g(\omega))k \neq f \quad \text{for all} \quad k \in C(\Omega).$$

Hence by Proposition 4, $g(\omega) \notin \rho_T(f)$. That is $g(\omega) \in \sigma_T(f)$. Therefore, we have

$$g(\{\omega \in \Omega : f(\omega) \neq 0\} \subseteq \sigma_T(f).$$

By the continuity of g, we have

$$g(\text{supp} f) \subseteq \sigma_T(f).$$

Conversely, let $\lambda \in \mathbb{C} \backslash g(\text{supp} f)$. Then there is an open neighborhood U of λ with $U \cap g(\text{supp} f) = \emptyset$. Define the sequence $\{f_n\}$ by

$$f_n(\omega) = \begin{cases} \dfrac{f(\omega)}{(g(\omega) - \lambda)^n}, & \text{if} \quad g(\omega) \notin U \\ 0, & \text{if} \quad g(\omega) \in U. \end{cases}$$

Then clearly f_n is well defined and continuous for each $n = 1, 2, \ldots$. Let $f_0 = f$. Then we have

$$(T - \lambda) f_n = f_{n-1} \quad \text{for all} \quad n = 1, 2, \ldots .$$

And for each $n = 1, 2, \ldots,$

$$\|f_n\| \leq \frac{\|f\|}{\text{dist}(\lambda, g(\text{supp} f))^n},$$

where $\text{dist}(\lambda, g(\text{supp} f))$ is the distance from λ to the compact set $g(\text{supp} f)$. Hence the sequence $\{\|f_n\|^{\frac{1}{n}}\}$ is bounded. By Proposition 4, $\lambda \in \rho_T(f)$. Hence we have

$$\sigma_T(f) \subseteq g(\text{supp} f).$$

Therefore, $\sigma_T(f) = g(\text{supp} f)$ is proved.

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