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ON THE AXIOM OF CHOICE OF WEAK TOPOS $\mathcal{F}uz$

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ABSTRACT. Topos is a set-like category. In topos, the axiom of choice can be expressed as (AC1), (AC2) and (AC3). Category $\mathcal{F}uz$ of fuzzy sets has a similar function to the topos *Set* and it forms weak topos. But $\mathcal{F}uz$ does not satisfy (AC1), (AC2) and (AC3). So we define (WAC1), (WAC2) and (WAC3) in weak topos $\mathcal{F}uz$. And we show that they are equivalent in $\mathcal{F}uz$.

1. Introduction

In a topos, the axiom of choice can be expressed as following.

(AC1) Every epimorphism is a retraction.

(AC2) For any noninitial object A and $f : A \to B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.

(AC3) For any noninitial object A, there exists $\sigma : \Omega^A \to A$ such that for all $f : 1 \to \Omega^A$, we have $\sigma \circ f \in f'$ where $f' : A' \to A$ is a monomorphism, provided that $ev \circ (f \times i_A)$ is not the characteristic morphism of $0 \to A$.

They are not necessarily so related ([1], [4]). But in the topos *Set* they are equivalent.

Category $\mathcal{F}uz$ of fuzzy sets has a similar function to the topos *Set*. $\mathcal{F}uz$ has finite products, middle object, equalizers, exponentials and weak subobject classifier. So $\mathcal{F}uz$ forms a weak topos ([5], [6]). But $\mathcal{F}uz$ does not satisfy (AC1), (AC2) and (AC3).

In this paper, we define (WAC1), (WAC2), (WAC3) as following.

(WAC1) Every epimorphism $f : (A, \alpha_A) \to (B, \alpha_B)$, where $\alpha_B(b) = Max\{\alpha_A[f[A']] \mid b = f[A'], A' \subseteq A\}$ and there exists an element $a' \in A$ such that $\alpha_A(a') \geqq \alpha_B(b)$ for all $b \in B$, is a retraction.

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(WAC2) For any noninitial object A and $f : A \to B$, where $\alpha_B(b) = Max\{\alpha_A[f[A']] \mid b = f[A'], A' \subseteq A\}$ and there exists an element $a' \in A$ such that $\alpha_A(a') \geqq \alpha_B(b)$ for all $b \in B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.

(WAC3) For any noninitial object A, there exists $\sigma : \Omega^A \to A$ such that for all $f : A \to \Omega$, we have $\alpha_A(\sigma(f)) = 1$ where $\alpha_A : A \to I$.
And we show that (WAC1), (WAC2), (WAC3) are equivalent in weak topos $\mathcal{F}uz$.

2. Preliminaries

In this section, we state some definitions and properties which will serve as the basic tools for the arguments used to prove our results.

DEFINITION 2.1. An *elementary topos* is a category $\mathcal{E}$ such that
(T1) $\mathcal{E}$ is finitely complete,
(T2) $\mathcal{E}$ has exponentiation,
(T3) $\mathcal{E}$ has a subobject classifier.

(T2) means that for every object A in $\mathcal{E}$, endofunctor $(-) \times A$ has its right adjoint $(-)^A$. Hence for every object A in $\mathcal{E}$, there exists an object B^A, and a morphism $ev_A : B^A \times A \to B$, called the evaluation map of A, such that for any Y and $f : Y \times A \to B$ in $\mathcal{E}$, there exists a unique morphism g such that $ev_A \circ (g \times i_A) = f$;

$$\begin{array}{ccc} Y \times A & \xrightarrow{f} & B \\ {\scriptstyle g\times i_A}\downarrow & & \downarrow{\scriptstyle i_B} \\ B^A \times A & \xrightarrow{ev_A} & B \end{array}$$

And subobject classifier in (T3) is an $\mathcal{E}$-object Ω, together with a morphism $\top : 1 \to \Omega$ such that for any monomorphism $h : D \to C$, there is unique morphism $\chi_h : C \to \Omega$, called the character of $h : D \to C$ that makes the following diagram a pull-back;

$$\begin{array}{ccc} D & \xrightarrow{!} & 1 \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \top} \\ C & \xrightarrow{\chi_h} & \Omega \end{array}$$

EXAMPLE 2.2. Category *Set* is a topos. The terminal object is the one-element sets $\{*\}$. The subobject classifier is $\top : \{*\} \to \Omega$ with $\Omega = \{0,1\}$ defined by $\top(*) = 1$. If we define

$$\begin{cases} \chi_h = 1 & \text{if} \quad c = h(d) \quad \text{for some} \quad d \in D, \\ \chi_h = 0 & \text{otherwise,} \end{cases}$$

then χ_h is a characteristic function of D .

Category $\mathcal{F}uz$ of fuzzy sets is a category whose object is (A, α_A) where A is an *Set*-object and $\alpha_A : A \to I$ is a *Set*-morphism with $I = (0,1]$ and morphism from (A, α_A) to (B, α_B) is a *Set*-morphism $f : A \to B$ such that $\alpha_A(a) \leqq \alpha_B \circ f(a)$.

DEFINITION 2.3. We say that an object (I, α_I) is a middle object of $\mathcal{F}uz$ if there exists a unique morphism $f : A \to I$ such that $\alpha_A(a) = \alpha_I \circ f(a)$ for all (A, α_A) and $a \in A$.

DEFINITION 2.4. We say that an object (J, α_J) is a weak subobject classifier of $\mathcal{F}uz$ if there exists a unique morphism $\alpha_f : (A, \alpha_A) \to (J, \alpha_J)$ for all monomorphism $f : (B, \alpha_B) \to (A, \alpha_A)$ where $J = [0,1]$ and $\alpha_J(j) = 1$ for all $j \in J$ such that $\alpha_f(a) \leqq \alpha_A(a)$ and the following diagram is a pull-back.

$$\begin{array}{ccc} (B, \alpha_B) & \xrightarrow{\alpha_B} & (I, \alpha_I) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle i} \\ (A, \alpha_A) & \xrightarrow[\alpha_f]{} & (J, \alpha_J) \end{array}$$

DEFINITION 2.5. A weak topos is a category $\mathcal{W}$ such that
(WT1) $\mathcal{W}$ has equalizer, finite product and exponentiation,
(WT2) $\mathcal{W}$ has a middle object,
(WT3) $\mathcal{W}$ has a weak subobject classifier.

3. Main parts

THEOREM 3.1. *In a weak topos $\mathcal{F}uz$ the following statements are equivalent*:

(WAC1) *Every epimorphism $f : (A, \alpha_A) \to (B, \alpha_B)$, where $\alpha_B(b) = Max\{\alpha_A[f[A']] \mid b = f[A'], A' \subseteq A\}$ and there exists an element $a' \in A$ such that $\alpha_A(a') \geqq \alpha_B(b)$ for all $b \in B$, is a retraction.*

(WAC2) *For any noninitial object A and $f : A \to B$, where $\alpha_B(b) = Max\{\alpha_A[f[A']] \mid b = f[A'], A' \subseteq A\}$ and there exists an element $a' \in A$ such that $\alpha_A(a') \geqq \alpha_B(b)$ for all $b \in B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.*

PROOF. (WAC1) $\Rightarrow$ (WAC2) Since $f : (A, \alpha_A) \to (B, \alpha_B)$ is factored by $(f[A], \alpha_{f[A]})$, we get $f = m \circ e$ where e is an epimorphism and m is a monomorphism. By hypothesis there exists a morphism $s : f[A] \to A$ such that $e \circ s = i_{f[A]}$. Since A is a disjoint union of $f[A]$ and $B - f[A]$, we can construct a morphism $h : (B - f[A]) \to A$ defined by $h(b') = a'$ for all $b' \in B - f[A]$, where $\alpha_A(a') \geqq \alpha_B(b)$ for all $b \in B$. So we have that $\alpha_A \circ h \leqq \alpha_{B-f[A]}$. By the property of coproduct, there exists a morphism $g : B \to A$ such that $g \circ m = s$. That is, the following diagram commute.

$$
\begin{array}{ccc}
A & \xleftarrow{\;h\;} & B - f(A) \\
{\scriptstyle i_A}\downarrow & & \downarrow \\
A & \xleftarrow[\;g\;]{} & B \\
{\scriptstyle i_A}\uparrow & & \uparrow{\scriptstyle m} \\
A & \xleftarrow[\;s\;]{} & f[A]
\end{array}
$$

We only claim that $f \circ g \circ f = f$.

$$
\begin{aligned}
f \circ g \circ f &= (m \circ e) \circ g \circ (m \circ e) \\
&= (m \circ e) \circ (g \circ m) \circ e \\
&= (m \circ e) \circ s \circ e \\
&= m \circ (e \circ s) \circ e \\
&= m \circ e = f.
\end{aligned}
$$

(WAC2) $\Rightarrow$ (WAC1) Let $f : (A, \alpha_A) \to (B, \alpha_B)$ be an epimorphism such that $\alpha_A \leqq \alpha_B \circ f$. By hypothesis there exists a morphism $g : (B, \alpha_B) \to (A, \alpha_A)$ such that $\alpha_B \leqq \alpha_A \circ g$ and $e \circ s \circ e = e$. Since e is an epimorphism, we have $f \circ g = i_B$. Hence f is a retraction. □

THEOREM 3.2. *In a weak topos $\mathcal{F}uz$ which is normal, the following statements are equivalent:*

(WAC1) *Every epimorphism $f : (A, \alpha_A) \to (B, \alpha_B)$, where $\alpha_B(b) = Max\{\alpha_A[f[A']] \mid b = f[A'], A' \subseteq A\}$ and there exists an element $a' \in A$ such that $\alpha_A(a') \geqq \alpha_B(b)$ for all $b \in B$, is a retraction.*

(WAC3) *For any noninitial object* A*, there exists* $\sigma : \Omega^A \to A$ *such that for all* $f : A \to \Omega$*, we have* $\alpha_A(\sigma(f)) = 1$ *where* $\alpha_A : A \to I$*.*

PROOF. (WAC1) $\Rightarrow$ (WAC3) Consider a morphism $ev : A \times \Omega^A \to \Omega$ defined by $ev(a,s) = s(a)$. By the property of product, for any two morphisms $ev : A \times \Omega^A \to \Omega$ and $p_2 : A \times \Omega^A \to \Omega^A$ there exists a morphism $(ev, p_2) : A \times \Omega^A \to \Omega \times \Omega^A$ such that $p'_2 \circ (ev, p_2) = p_2$ and $p'_1 \circ (ev, p_2) = ev$ where $p'_1 : \Omega \times \Omega^A \to \Omega$ and $p'_2 : \Omega \times \Omega^A \to \Omega^A$. That is, the following diagram commute.

$$\begin{array}{ccc} A \times \Omega^A & \xrightarrow{ev} & \Omega \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle p'_1} \\ A \times \Omega^A & \xrightarrow[(ev,p_2)]{} & \Omega \times \Omega^A \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p'_2} \\ A \times \Omega^A & \xrightarrow[p_2]{} & \Omega^A \end{array}$$

Since (ev, p_2) is an epimorphism, there exists a morphism $h : \Omega \times \Omega^A \to A \times \Omega^A$ such that $(ev, p_2) \circ h = i_{\Omega \times \Omega^A}$. Also for a morphism $g : \Omega^A \to \Omega$ where $g(s) = 1$ for all $s \in \Omega^A$ and a morphism $i_{\Omega^A} : \Omega^A \to \Omega^A$ there exists a morphism $(g, i_{\Omega^A}) : \Omega^A \to \Omega \times \Omega^A$ such that $p'_1 \circ (g, i_{\Omega^A}) = g$, $p'_2 \circ (g, i_{\Omega^A}) = i_{\Omega^A}$. That is, the following diagram commute.

$$\begin{array}{ccc} \Omega & \xleftarrow{g} & \Omega^A \\ {\scriptstyle p'_1}\uparrow & & \uparrow{\scriptstyle i} \\ \Omega \times \Omega^A & \xleftarrow[(g,i_{\Omega^A})]{} & \Omega^A \\ {\scriptstyle p'_2}\downarrow & & \downarrow{\scriptstyle i} \\ \Omega^A & \xleftarrow[i_{\Omega^A}]{} & \Omega^A \end{array}$$

We get $h \circ (g, i_{\Omega^A})(s) = h(1, s) = (a, u)$ for some $a \in A$ and $u \in \Omega^A$. Also $u = p_2(a, u) = p_2 \circ h(1, s) = p'_2 \circ (ev.p_2) \circ h(1, s) = p'_2(1, s) = s$. That is, $h \circ (g, i_{\Omega^A})(s) = (a, s)$. By $(ev, p_2) \circ h \circ (g, i_{\Omega^A})(s) = (ev.p_2)(a, s)$ and $g(s) = 1$ for all $s \in \Omega^A$, we get $s(a) = 1$. So $p_1 \circ h \circ (g, i_{\Omega^A})(s) = p_1(a, s) = a$. Let $\sigma = p_1 \circ h \circ (g, i_{\Omega^A})$, then $\sigma(s) = a$ and $s(a) = 1$.

(WAC3) $\Rightarrow$ (WAC1) For an epimorphism $f : A \to B$, we construct a morphism $\Omega^f : \Omega^B \to \Omega^A$ defined by $\Omega^f(s) = s \circ f$ where $s : B \to \Omega$. We only claim that $f \circ \sigma \circ \Omega^f \circ j = i_B$ where $j : B \to \Omega^B$ defined by $j_b(b) = 1$ and fixed for otherwise.

$$\begin{array}{ccc} \Omega^A & \xleftarrow{\ \Omega^f\ } & \Omega^B \\ \sigma \downarrow & & \uparrow j \\ A & \xrightarrow[\ f\ ]{} & B \end{array}$$

Then $f \circ \sigma \circ \Omega^f \circ j(b) = f \circ \sigma(j_b \circ f)$. Let $\sigma(j_b \circ f) = a$ then, by definition of σ, we get that $j_b(f(a)) = 1$ and $b = f(a)$. Therefore $f \circ \sigma \circ \Omega^f \circ j(b) = b$. □

COROLLARY 3.3. *In a weak topos $\mathcal{F}uz$ which is normal, the following statements are equivalent:*

(WAC1) *Every epimorphism $f : (A, \alpha_A) \to (B, \alpha_B)$, where $\alpha_B(b) = Max\{\alpha_A[f[A']] \mid b = f[A'], A' \subseteq A\}$ and there exists an element $a' \in A$ such that $\alpha_A(a') \geqq \alpha_B(b)$ for all $b \in B$, is a retraction.*

(WAC2) *For any noninitial object A and $f : A \to B$, where $\alpha_B(b) = Max\{\alpha_A[f[A']] \mid b = f[A'], A' \subseteq A\}$ and there exists an element $a' \in A$ such that $\alpha_A(a') \geqq \alpha_B(b)$ for all $b \in B$, there exists a morphism $g : B \to A$ such that $f \circ g \circ f = f$.*

(WAC3) *For any noninitial object A, there exists $\sigma : \Omega^A \to A$ such that for all $f : A \to \Omega$, we have $\alpha_A(\sigma(f)) = 1$ where $\alpha_A : A \to I$.*

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