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# Quasiconcave Bilevel Programming Problem

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## ABSTRACT

Bilevel programming problem is a two-stage optimization problem where the constraint region of the first level problem is implicitly determined by another optimization problem. In this paper we consider the bilevel quadratic/linear fractional programming problem in which the objective function of the first level is quasiconcave, the objective function of the second level is linear fractional and the feasible region is a convex polyhedron. Considering the relationship between feasible solutions to the problem and bases of the coefficient submatrix associated to variables of the second level, an enumerative algorithm is proposed which finds a global optimum to the problem.

Keywords: Bilevel Programming, Non-convex Optimization, Quasiconcave Functions, Quadratic Programming , Enumerative Method

## 1. INTRODUCTION

In this paper we consider the Bilevel Quadratic/Linear Fractional Programming Problem (BQLFP) defined as

$$\textbf{(BQLFP)}: \min_{X_1} f_1(x) = Z_1(X)Z_2(X) = (C^T X + \alpha_1)(D^T X + \beta_1)$$
$$= (c_1 X_1 + c_2 X_2 + \alpha_1)(d_1 X_1 + d_2 X_2 + \beta_1)$$

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where $X_2$ solves

$$\min_{X_2} f_2(X) = \frac{\alpha_2 + c^{11}X_1 + c^{12}X_2}{\beta_2 + c^{21}X_1 + c^{22}X_2}$$

subject to

$$X = (X_1, X_2) \in S,$$

where $X_1 \in \mathbb{R}^{n_1}$ and $X_2 \in \mathbb{R}^{n_2}$ are the variables controlled by the first level and the second level decision maker resp.; $c_1, c_2, d_1, d_2, c^{11}, c^{12}, c^{21}$ and $c^{22}$ are vectors of conformal dimension; $\alpha_1, \alpha_2, \beta_1$ and $\beta_2$ are scalars and the common constraint region to both levels is a polyhedron i.e.

$$S = \{(X_1,\ X_2):\ A^1X_1 + A^2X_2 = b,\ X_1 \geq 0,\ X_2 \geq 0\}$$

where $A^1$ is an $m \times n_1$ matrix $A^2$ is an $m \times n_2$ matrix and $b$ is an $m$-vector.

Based upon the fact that concave functions have an extreme point optimal solution, a number of algorithms have been proposed to minimize concave functions. The most notable among them are cutting plane method [1, 10, 16], branch and bound methods [7, 13, 14] and by ranking extreme points [3, 12]. These algorithms have been applied to a number of special problems, such as the optimization of concave quadratic functions and bilinear programming problems. H. Konno and *T.* Kuno [9] have proposed an algorithm for solving a linear, multiplicative programming problem by the combination of the parametric simplex method and the standard convex minimization problem. A bibiliography of reference on bilevel and multilevel programming, which is updated biannually, can be found in [15]. Candler and Townsley [6] provided a setting of the bilevel problem as a generalisation of standard mathematical programming problems. A bibliography of references on bilevel and multilevel programming problems in both linear and non-linear cases can be found in [14]. A Quadratic Fractional Programming Problem (QFP) has been studied by many authors in 1994. R. Gupta and M.C. Puri [8] developed an algorithm for ranking the extreme points of QFP. Mathur and Puri [11] in 1995 stated that the optimal solution of the bilevel linear fractional programming problem might occur at a non-extreme point. But Calvete and Gale [4] in 1998 proved that, under some assumptions, the optimal solution of the bilevel programming problem , in which the objective functions are quasi-concave and the constraint region common to both levels is a polyhedron, can be found at an extreme point of the polyhedron. Later in 1999 [5], they proposed an enumerative algorithm that finds a global optimal solution to the bilevel linear/linear fractional programming problem. In this paper, considering the relationship between points of the inducible region and bases of the coefficient submatrix associated to variables of the second level, an algorithm is proposed which finds a global optimum solution to the BQLFP problem in a finite number of steps

## 2. THEORETICAL DEVELOPMENT

We assume that $\beta_2 + c^{21}X_1 + c^{22}X_2 > 0 \quad \forall (X_1, X_2) \in S$ is assumed to be non-empty and compact. $A^2$ has full row-rank and $m < n_2$. The projection of $S$ on to $\mathbb{R}^{n_1}$ is denoted by

$$S_1 = \{X_1 \in \mathbb{R}^{n_1} : (X_1, X_2) \in S\}$$

Let $V_1$ and $V_2$ be respectively the sets of indices of first level and second level controlled variables.

For each $X_1 \in S_1$, the feasible region of the second level decision maker is

$$S(X_1) = \{X_2 \in \mathbb{R}^{n_2} : A^2 X_2 = b - A^1 X_1, X_2 \geq 0\}.$$

It is also a non-empty compact polyhedron. The inducible region or feasible region of the first level decision maker is denoted by

$$IR = \left\{(X_1, X_2) \ : \ X_1 \geq 0 \text{ where } X_2 \text{ minimizes} \right.$$
$$\left. \left\{ \frac{\alpha_2 + c^{11}X_1 + c^{12}X_2}{\beta_2 + c^{21}X_1 + c^{22}X_2};\ A^1X_1 + A^2X_2 = b,\ X_2 \geq 0 \ \ over\ X_2 \right\} \right\}$$

It is also assumed that for each value of $X_1 \in S_1$, there will be a unique solution to the followers' problem.
The following lemma and theorem are stated and proved in [2].

**Lemma 1:** The inducible region of the quasiconcave bilevel programming problem is piecewise linear.

**Theorem 1:** There is an extreme point of the feasible region $S$ which is an optimal solution to the quasiconcave bilevel programming problem.

In the BQLFP defined above, $f_1$ and $f_2$ are quasiconcave ($f_1$: indefinite quadratic, $f_2$: linear fractional i.e. ratio of two affine functions). Hence in accordance to the above lemma, the inducible region IR of BQLFP is comprised of the union of connected faces of $S$ and the optimal solution is at an extreme point of the feasible region $S$. This suggests the use of extreme point search method to develop an algorithm for solving the BQLFP problem.

## 3. ALGORITHMIC DEVELOPMENT

The optimal solution of BQLFP being the point of IR, the points of IR and their corresponding basis are analyzed to get a better solution, if possible.

For each $X_1 \in S_1$, a point of IR is obtained by solving the linear fractional programming problem.

**FP(X₁):**

$$\min_{X_2} \frac{c^{12}X_2 + \bar{\alpha}_2}{c^{22}X_2 + \bar{\beta}_2}$$

subject to

$$X_2 \in S(X_1)$$

where $\bar{\alpha}_2 = c^{11}X_1 + \alpha_2$, $\bar{\beta}_2 = c^{21}X_1 + \beta_2$.

Hence an extreme point $X_2$ of the polyhedron $S(X_1)$ can be found which solves FP($X_1$) and the point $(X_1, X_2)$, so obtained, belongs to the inducible region.

Since a basis $B$ of $A^2$ is associated to $X_2$, we can associate a basis $B$ of $A^2$ to each point of IR. Therefore we need only to consider these basis. To solve FP($X_1$), we consider the parametric approach. In this case, it is known that an optimal solution to the following linear parametric problem LP($X_1$) is an optimal solution to FP($X_1$) verifying the condition that $F(\lambda) = 0$:

**LP(X₁):**

$$F(\lambda) = \min_{X_2}\{(c^{12}X_2 + \bar{\alpha}_2) - \lambda(c^{22}X_2 + \bar{\beta}_2)\}$$

subject to

$$X_2 \in S(X_1)$$

Hence an extreme point $X_2$ of $S(X_1)$ can be found which solves FP($X_1$) and the point so obtained $(X_1, X_2) \in \text{IR}$. Since a basis $B$ of $A^2$ is associated to $X_2$, we can associate a basis of $A^2$ to each point of IR. Therefore only these basis need to be considered.

Consider the basis $B$ of $A^2$.

In order to obtain points of IR, there must exist $X_1 \in S_1$ such that $B$ is a feasible basis to LP(X1). Also, for some $\lambda$, $B$ should verify the optimality conditions of problem LP(X1) along with the condition that $F(\lambda) = 0$ for at least one of these values of $\lambda$.
During the verification of optimality conditions, we will get a lower bound $\lambda_\ell$ and an upper bound $\lambda_u$ for $\lambda$.
Hence for at least one $\lambda$, $F(\lambda) = 0$ implies

$$\lambda_\ell \leq \frac{c^{12}X_{2B} + \bar{\alpha}_2}{c^{22}X_{2B} + \bar{\beta}_2} \leq \lambda_u \qquad (1)$$

where $X_{2B}$ stands for the value of $X_2$ associated with the basis $B$.

i.e. $$X_{2B} = B^{-1}(b - A^1X_1),\ X_1 \geq 0,\ B^{-1}(b - A^1X_1) \geq 0$$

Since a basis $B$ should verify the optimality conditions of problem $\mathrm{LP}(X_1)$, it suffices to show that the condition

**(OC):** $$(c_j^{12} - \lambda c_j^{22}) - (c_B^{12} - \lambda c_B^{22})B^{-1}A_j^2 \geq 0 \qquad \forall\ j \in V_2$$

where $c_j^{12}$ and $c_j^{22}$ are the $j$th components of vectors $c^{12}$ and $c^{22}$ respectively; $c_B^{12}$ and $c_B^{22}$ are the m-row vectors of $c^{12}$ and $c^{22}$ associated to the basic variables of $B$ and $A_j^2$ is the $j$th column of $A^2$.
While we check this condition, we obtain the interval $[\lambda_\ell, \lambda_u]$ for the parameter $\lambda$. If $\lambda_\ell = -\infty$ or $\lambda_u = \infty$, the interval $[\lambda_\ell, \lambda_u]$ will be open in that extreme. If no such $\lambda$ exists so that basis $B$ verifies condition (1), then this base is of no interest because it is impossible to obtain a point of the inducible region corresponding to it.
Now, points of IR corresponding to the basis $B$ is obtained by solving the quadratic programming problem:

**(QPP(B)):** $$\min_{X_1} f_1(X) = (c_1X_1 + c_2X_{2B} + \alpha_1)(d_1X_1 + d_2X_{2B} + \beta_1)$$

subject to

$$A^1X_1 + A^2X_{2B} = b$$
$$X_1,\ X_{2B} \geq 0$$

Note that while $B$ is analyzed, variables of the second level not associated to $B$ are equal to zero.

Note that $f_1$ agrees with the objective functions of QPP(B) because while $B$ is being analyzed, the variables of the second level not associated to it are equal to zero.

Suppose QPP(B) is feasible and (1) is verified, then the optimal solution $\bar{X} = (\bar{X}_1, \bar{X}_2)$ is the best point of IR. Then we search for a new basis which can improve the value of $f_1$ obtained earlier.

Let $T$ be the set of indices of variables associated to the basis $B$.

**Lemma 2:** Any basis from $A^2$ capable of providing a point of IR better than $\bar{X}$ must include atleast one vector whose index belongs to the set

$$C_1 = \{j \in V_2 \text{-} T :\ L_j > 0\}$$

where $L_j = Z_1(z_2^j \text{-} d_j) + Z_2(z_1^j \text{-} c_j) - \theta(z_1^j - c_1)(z_2^j - c_j)$ is the $j$th reduced cost coefficient in the optimal solution of QPP(B).

**Proof :** Let $f_1(\bar{X})$ denote the value of the first level objective function at $\bar{X}$. According to $\bar{X}$ the matrix $[A_1, A_2]$ is decomposed into $[B, N]$, where $B$ is an $m \times m$ basis matrix associated to basic variables of $\bar{X}$. For each $X \in \text{IR}$, we can write

$$\begin{aligned} f_1(X) &= Z_1(X)Z_2(X) \\ &= (c^TX + \alpha_1)(D^TX + \beta_1) \\ &= (c_1X_1 + c_2X_2 + \alpha_1)(d_1X_1 + d_2X_2 + \beta_1) \end{aligned}$$

Let $\bar{X}_B$ be a b.f.s. obtained by phase I of the simplex method. If we enter $a_j$ into the basis and depart $b_r$ let $\hat{X}_B$ be the new b.f.s. s.t.

$$\hat{X}_{B_i} = \overline{X}_{\mathbf{B}_i} - \overline{X}_{B_r}\frac{Y_{ij}}{Y_{rj}} \text{ and } \hat{X}_{B_r} = \frac{\overline{X}_{B_r}}{Y_{r_j}} > 0$$

i.e. $\hat{X}_{B_i} = \overline{X}_{B_i} - \theta Y_{ij}$ and $\hat{X}_{B_r} = \theta$

Given $\overline{Z}_1 = C_B^T \overline{X}_B + \alpha_1$ and $\overline{Z}_2 = D_B^T \overline{X}_B + \beta_1$, new value of objective function is

$$\begin{aligned}
f_1(\hat{X}) &= Z_1(\hat{X})Z_2(\hat{X}) \\
&= \left(\sum_{i=1}^{m} \hat{c}_{B_i}\hat{X}_{B_i} + \alpha_1\right)\left(\sum_{i=1}^{m} \hat{d}_{B_i}\hat{X}_{B_i} + \beta_1\right) \\
&= \left(\sum_{\substack{i=1\\ i\neq r}}^{m} c_{B_i}(\overline{X}_{B_i} - \theta Y_{ij}) + \hat{c}_{B_r}\theta + \alpha_1\right)\left(\sum_{\substack{i=1\\ i\neq r}}^{m} d_{B_i}(\overline{X}_{B_i} - \theta Y_{ij}) + \hat{d}_{B_r}\theta + \beta_1\right) \\
&= \left(\sum_{\substack{i=1\\ i\neq r}}^{m} c_{B_i}(\overline{X}_{B_i} - \theta Y_{ij}) + c_j\theta + \alpha_1\right)\left(\sum_{\substack{i=1\\ i\neq r}}^{m} d_{B_i}(\overline{X}_{B_i} - \theta Y_{ij}) + d_j\theta + \beta_1\right) \\
&\qquad\qquad (\textit{Note } \hat{c}_{B_r} = c_j,\ \hat{d}_{B_r} = d_j) \\
&= \left(\sum_{i=1}^{m} c_{B_i}\overline{X}_{B_i} - \theta\sum_{i=1}^{m} c_{B_i}Y_{ij} + c_j\theta + \alpha_1\right)\left(\sum_{i=1}^{m} d_{B_i}\overline{X}_{B_i} - \theta\sum_{i=1}^{m} d_{B_i}Y_{ij} + d_j\theta + \beta_1\right) \\
&= (\overline{Z}_1 - \theta z_j^1 + c_j\theta)(\overline{Z}_2 - \theta z_j^2 + d_j\theta) \\
&= (\overline{Z}_1 - \theta(z_j^1 - c_j))(\overline{Z}_2 - \theta(z_j^2 - d_j)) \\
&= (\overline{Z}_1\overline{Z}_2 - \theta(\overline{Z}_1(z_j^2 - d_j) + \overline{Z}_2(z_j^1 - c_j) - \theta(z_j^1 - c_j)(z_j^2 - d_j)) \\
&= \overline{Z}_1\overline{Z}_2 - \theta L_j \\
&< \overline{Z}_1\overline{Z}_2 = f_1(\overline{X}) \qquad \text{(because } \theta > 0 \text{ and } L_j > 0)
\end{aligned}$$

So, in order to improve the first level, we must consider the variables with indices j such that $L_i > 0$. If $\overline{X}$ solves problem QPP(B), then $L_i \leq 0 \ \ \forall j \in V_1$ and $\forall j \in T$.

If $C_1 = \phi$, $f_1$ cannot be improved and the current best point is optimum to BQLFP. If we have previously built sets $C_1^1, C_1^2, \ldots, C_1^i$, the new basis should include at least one index from each sets $C_1^1, C_1^2, \ldots, C_1^i$.

Let $E_1 = \bigcup_i \{C_1^i\}$.

Let, if possible, QPP(B) be infeasible or QPP(B) be feasible, but its optimal solution does not verify (1). Since this basis cannot be optimal to the second level problem, this basis is no longer we are interested in. If $C_2$ denotes the set of indices associated to B, then the new basis should not include all vectors with indices in the set $C_2$.

If we have previously built sets $(C_2^1, C_2^2, \ldots, C_2^i)$ the new basis should not include all vectors with indices in each of these sets.

Let $E_2 = \bigcup_i \{C_2^i\}$.

**Remark 1:** To select indices which form the new basis it is suggested to solve the following system for $w_i$:

**(P1):**

$$\sum \omega_j \delta_j \geq 1, \qquad \delta_j = \begin{cases} 1, & if\ j \in C_1,\ C_1 \in E_1 \\ 0 & Otherwise \end{cases}$$

$$\sum_J \omega_j \delta_j \leq \sum_J \delta_j - 1, \qquad \delta_j = \begin{cases} 1, & if\ j \in C_2,\ C_2 \in E_2 \\ 0 & Otherwise \end{cases}$$

$$\sum_j \omega_j = m$$

$$\omega_j \in \{0,1\},\ j \in V_2$$

The required indices correspond to $j$ where $\omega_j = 1$.

2. If the basis $B$ so formed has rank $K < m$, then $B = [\hat{B}, N]$ where $\hat{B}$ is a matrix of independent vectors of $B$ and $N$ is a matrix of $(m-k)$ vectors of $A^2$, is a basis from $A^2$.

## 3.1 Algorithm for Solving the Bilevel Programming Problem

**Step 0:** Solve the problem (QPP):

**(QPP):** $\min_{X_1} f_1 = (c_1 X_1 + c_2 X_2 + \alpha_1)(d_1 X_1 + d_2 X_2 + \beta_1)$

s.t. $(X_1, X_2) \in S$

0.1 If it is not feasible, stop. BQLFP is not feasible.

0.2 If the optimal solution exists, let $\bar{X} = (\bar{X}_1, \bar{X}_2)$ be an optimal solution.

0.3 Put $X_1 = \bar{X}_1$, solve. $\text{FP}(\bar{X}_1)$ Let $\hat{X}_2$ be the optimal solution.

0.4 If $\hat{X}_2 = \bar{X}_2$, stop. $(\bar{X}_1, \bar{X}_2)$ is the current best point of IR.

Set $E_1 = \phi_1$, $E_2 = \phi$.

**Step 1:** Solve QPP(B)

1.1 If QPP(B) is not feasible or if optimal solution of QPP(B) does not verify (1), then compute $C_2$

Set $E_2 = E_2 \cup \{C_2\}$. Go to step 3.

1.2 If QPP(B) is feasible and (1) is verified, then compare the current best point of IR with this solution and update if necessary.

**Step 2:** Compute $C_1$. If $C_1 = \phi$, stop. The current best point of IR is the optimal solution to (BQLFP).

Otherwise set $E_1 = E_1 \cup \{C_1\}$.

**Step 3:** Solve $P_1$. If $P_1$ is not feasible, stop. The current best point of IR is optimal to (BQLFP).

**Step 4:** Construct $B$ and compute $[\lambda_\ell, \lambda_u]$ by solving (1). If no such $\lambda$ exists for (1), compute $C_2$. Set $E_2 = E_2 \cup \{C_2\}$ and go to step 3.

Otherwise go to step 1.

## 4. ILLUSTRATIVE EXAMPLE

For illustration we consider the following example:

$$\min_{x_1 + x_2} f_1 = (-x_1 - 3x_2 - x_3 - x_4 - 1)(x_1 + 2x_3 + 1).$$

where $(x_3, x_4, x_5, x_6)$ solves

$$\min_{x_3, x_4, x_5, x_6} f_2 = \frac{2 + x_1 + 2x_2 - 2x_3 - x_4}{1 + 3x_1 + x_2 + 3x_3 + x_4}$$

subject to

$$\begin{aligned} x_1 - x_2 + x_3 + x_5 \qquad &= 4 \\ x_1 + x_2 + \quad 2x_4 + x_6 &= 6 \\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0 \end{aligned}$$

Solve the Leader's problem (QPP) given by

**(QPP):**
$$\min_{x_1+x_2} f_1 = (-x_1 - 3x_2 - x_3 - x_4 - 1)(x_1 + 2x_3 + 1)$$

subject to

$$x_1 - x_2 + x_3 + x_5 = 4$$
$$x_1 + x_2 + 2x_4 + x_6 = 6$$
$$x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$$

The initial basic feasible solution is as shown in Table 1.

Table 1.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | $d_j \rightarrow$ | 1 | 0 | 2 | 0 | 0 | 0 |
| | | | $c_j \rightarrow$ | −1 | −3 | −1 | −1 | 0 | 0 |
| $D_B$ | $C_B$ | $X_B$ | Values of $X_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
| 0 | 0 | $x_5$ | 4 | 1 | −1 | 1 | 0 | 1 | 0 |
| 0 | 0 | $x_6$ | 6 | 1 | 1 | 0 | 2 | 0 | 1 |
| $Z_1 = -1$ | | | $z_j^2 - c_j \rightarrow$ | 1 | 3 | 1 | 1 | 0 | 0 |
| $Z_2 = 1$ | | | $z_j^2 - d_j \rightarrow$ | −1 | 0 | −2 | 0 | 0 | 0 |
| $f_1 = -1$ | | | $L_j \rightarrow$ | $2+\theta$ | 3 | $3+2\theta$ | 1 | 0 | 0 |

$L_3 = (3 + 2\theta)$ is most positive. Note that

$$\theta = \min_j \frac{X_{B_i}}{Y_{ij}}$$

Hence $x_3$ enters and $x_5$ leaves the basis. By continuing in the same manner we obtain Table 2.

Table 2.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | | $d_j \rightarrow$ | 1 | 0 | 2 | 0 | 0 | 0 |
| | | | $c_j \rightarrow$ | −1 | −3 | −1 | −1 | 0 | 0 |
| $D_B$ | $C_B$ | $X_B$ | Values of $X_B$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
| 2 | −1 | $x_3$ | 5 | 2 | 0 | 1 | 2 | 1 | 1 |
| 0 | −3 | $x_2$ | 1 | 1 | 1 | 0 | 2 | 0 | 1 |
| $Z_1 = -9$ | | | $z_j^2 - c_j \rightarrow$ | −4 | 0 | 0 | −7 | −1 | −4 |
| $Z_2 = 11$ | | | $z_j^2 - d_j \rightarrow$ | 3 | 0 | 0 | 5 | 2 | 2 |
| $f_1 = -99$ | | | $L_j \rightarrow$ | $-71+12\theta$ | 0 | 0 | $-122+35\theta$ | $-29+2\theta$ | $-62+8\theta$ |

Since $L_i \leq 0 \quad \forall\, j$, Table 2 is an optimal one. Therefore,

$$\begin{aligned}\bar{X} &= (\bar{X}_1, \bar{X}_2)\\ &= (\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5, \bar{x}_6)\\ &= (0,\ 1,\ 5,\ 0,\ 0,\ 0,)\end{aligned}$$

is the optimal solution to the Leader's problem (QPP). Put $x_1 = 0$, $x_2 = 1$ and solve the follower's problem by the parametric approach i.e. solve

**LP($\bar{\mathbf{X}}_1$):** $F(\lambda) = \min_{x_1 + x_2} f_2 = (-2 - 3\lambda)x_3 + (-1 - \lambda)x_4 + (4 - 2\lambda)$

subject to

$$\begin{aligned} x_3 + x_5 &= 5\\ 2x_4 + x_6 &= 5\\ x_3, x_4, x_5, x_6 &\geq 5\end{aligned}$$

The optimal solution of $\mathrm{FP}(\bar{X}_1)$ is $\tilde{\mathrm{X}}_2 = (5,\ \frac{5}{2}, 0, 0)$ with $[\lambda^{\ell}, \lambda^{u}] = \left[-\frac{2}{3}, \infty\right]$ and $F(\lambda) = 0$ for $\lambda = -\frac{17}{39}$.

As $\tilde{X}_2 \neq \bar{X}_2$, the current best point of IR is (0, 1, 5, 5/2, 0, 0) and $f_1 = -116.5$.

$$\text{Set } E_1 = \phi,\ E_2 = \phi.$$

Basis $B_1$ given by vectors with indices 3 and 4 is that first base to analyze. Solve QPP($B_1$) given by

**QPP(B₁):** $\min f_1 = (-x_1 - 3x_2 - x_3 - x_4 - 1)(x_1 + 2x_3 + 1)$

subject to

$$\begin{aligned} x_1 - x_2 + x_3 &= 4\\ x_1 + x_2 + 2x_4 &= 4\\ x_1, x_2, x_3, x_4 &\geq 0\end{aligned}$$

The optimal solution is $\bar{X} = (0,\ 6,\ 10,\ 0)$, $f_1 = -609$.

Since $f_2(\bar{X}) = -\frac{6}{37} \in \left[-\frac{2}{3}, \infty\right]$, condition (1) is verified.

Because $f_1(\bar{X}) = -609 < -116.5$, update the current best point of IR as (0, 6, 10, 0, 0, 0) for which $f_1 = -609$.

Note that variables of the second level not associated to $B_1$, and therefore not to be considered when solving problem QPP($B_1$), are also included in order to construct the corresponding set $C_1$.

Because reduced cost $L_j$ for none of the $j$'s is positive, therefore, $C_1 = \phi$.

Hence, the algorithm terminates here and the current best point of IR i.e. $\bar{X}$ = (0, 6, 10, 0, 0, 0) is the global optimum to QPP. Its objective function value is $f_1 = -609$.

## 5. CONCLUSION

In this paper, the bilevel quadratic/linear fractional programming problem (BQLFP) has been considered. This problem assumes that objective functions of both levels are, respectively, quadratic and linear fractional and the feasible region is a convex polyhedron. The problem is solved by converting the quadratic objective into a linear objective. Considering the relationship between feasible solutions to the problem and bases of the coefficient submatrix associated to variables of the second level, an enumerative algorithm is proposed which finds a global optimum to the problem and an illustration is demonstrated.

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