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Application of Fractal Theory to Various Surfaces

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Abstract: In this study, the general theory of fractality is discussed to provide a fundamental understanding of fractal geometry applied to heterogeneous material surfaces like pavement surface and rock surface. It is well known that many physical phenomena and systems are chaotic, random and that the features of roughness are found at a wide spectrum of length scales from the length of the sample to the atomic scales. Studying the mechanics of these physical phenomena, it is absolutely necessary to characterize such multiscaled rough surfaces and to know the structural property of such surfaces at all length scales relevant to the phenomenon. This study emphasizes the role of fractal geometry to characterize the roughness of various surfaces. Pavement roughness and rock surface roughness were examined to correlate their roughness property to fractality.

Keywords: fractal, skid number, joint roughness coefficient, fractal dimension.

1. Introduction

Fractal comes from the Latin adjective "fractus," which has the same root as fraction and fragmented; it is related to "frangere," which means "to break." Mandelbrot[1] introduced the concept of fractal "dimensions," a concept developed much earlier in specialized areas of physics and mathematics, to general sciences. The term Mandelbrot called "fractal" was a relatively new topic, but individual fractals have been known to the mathematicians for quite a long time. The basic fractal arose roughly between 1875 and 1922 and remains associated with names like Weierstrass, Cantor, Peano, Lebesque, Hausdorff, and Besicovitch.

Topology is a the well-defined branch of mathematics that addresses the measurement, properties, and relationships of points, lines, angles, surfaces and solids. This branch of mathematics studies the properties of given elements which remain invariant under specified transformations. Topology also represents an arrangement of objects or parts, which suggest a particular shape or figure. Obviously, this particular aspect of the notion of a shape is very useful in the study of natural curves such as coastlines, fracture surface profiles, rock surfaces, pavement surface etc., because it simply indicates that they are all topologically identical to each other. This identity is underlined by the fact that the topological dimension of all curves is equal to 1.

Considering an isotropic and homogeneous rough surface of dimension D_s, the property of isotropy relates to the invariance of the probability distribution under the rotation of the coordinate axes and reflection on any plane. The homogeneity of a surface implies that the probability distribution of the heights is independent of the location on the surface. The profile, $z(x)$, of such a surface along a straight line and in any arbitrary direction is of dimension $D = D_s - 1$[1] and is a statistically valid representation of the surface. Such a profile is typically obtained by a mechanical measurement or by an optical technique.

In this study, fractal properties are extended to the roughness of such surfaces as pavement surface and rock surface. And fractal dimensions for each different surface are qualitatively expressed in terms of conventional properties.

2. Application of fractal theory to pavement surface

2.1 Skid number

Skid resistance is by definition[2] "the retarding force generated by the interaction between a pavement and a tire under locked nonrotating conditions." Skid resistance measurements, conducted in accordance with ASTM Standard E247, are reported as Skid Number (SN) values.[3] SN values are usually measured at 40 mph and are equal to the force required to slide the locked test tire on a wet pavement, multiplied by 100 and divided by the effective wheel load.[4] Numerous studies on the relationship of accident risk with frictional coefficients, as measured on the road, have concluded that accident risk decreases with better skid resistance.[5,6] Usually, skid resistance is a very difficult parameter to estimate. The tire-pavement friction is affected by a large number of parameters, which include the type and condition of tire, braking system, speed, temperature, etc.

An attempt to approximate the pavement surface texture with fractals is made by Panagouli.[7] In order to take into account its accurate geometry, the fractal concept is a proper mathematical tool for pavement surface. Using fractal interpolation function, Panagouli generated pavement surface texture in terms of fractal dimension range of 1.043 to 1.589.

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2.2 Mathematical model for the approximation

In this section the fractal geometry by means of the notion of the fractal interpolation functions (FI) is will be defined. Generally, when an experimental curve given by a finite number of points $\{(x_i, y_i); i = 0, 1, K, N\}$ is approximated, a type of mathematical model that approximate this curve has to be chosen in advance. Thus, a transformation from a discrete set of data to a continuous model is made. Such a transformation can be done in our case by using a fractal interpolation function $g: [x_o, x_N] \rightarrow \Re$, i.e. a function g such that $g(x_i) = y_i$; $i = 0, K, N$. More specifically, if C^o is the set of continuous functions $g: [x_o, x_N] \rightarrow \Re$, then the sequence of functions $g_{m+1}(x) = (Tg_m)(x)$, where the operator $T: C^o \rightarrow C^o$ is defined by:

$$(Tg)(a_i x + b_i) = c_i x + d_i g(x) + f_i \qquad i = 1, 2, K, N. \tag{1}$$

And it converges to a curve ϕ as $m \rightarrow \infty$. The convergence of T to a curve $\phi \in C^o$ is due to the fact T that is a contraction mapping[8] Thus, T possesses a unique fixed point in C^o. This curve is also the unique attractor of the 'hyperbolic' iterated function system (IFS) $\{\Re^2; w_i, i = 1, 2, K, N\}$ defined by the 'shear' transformation.

$$\{x, y\} \rightarrow w_i \begin{Bmatrix} x \\ \overline{y} \end{Bmatrix} = \begin{bmatrix} a_i & 0 \\ c_i & d_i \end{bmatrix} \begin{Bmatrix} x \\ \overline{y} \end{Bmatrix} + \begin{Bmatrix} b_i \\ \overline{f_i} \end{Bmatrix}; i = 1, K, N \tag{2}$$

Every transformation w_i which is specified by the five real numbers a_i, c_i, d_i, and f_i is constrained by the given set of data according to:

$$w_i \begin{Bmatrix} x_o \\ \overline{y_o} \end{Bmatrix} = \begin{Bmatrix} x_{i-1} \\ \overline{y_{i-1}} \end{Bmatrix} \text{ and } w_i \begin{Bmatrix} x_N \\ \overline{y_N} \end{Bmatrix} = \begin{Bmatrix} x_i \\ \overline{y_i} \end{Bmatrix}; i = 1, K, N \tag{3}$$

From the above restrictions, it follows that there is one free parameter in each transformation. We choose this parameter to be d_i which must obey the boundary condition of $0 \le d_i < 1$. Moreover the remaining coefficients are given by the formulas:

$$a_i = \frac{(x_i - x_{i-1})}{(x_N - x_o)}, \quad \frac{(x_N x_{i-1} - x_o x_i)}{(x_N - x_o)}$$

$$c_i = \frac{(y_i - y_{i-1})}{(x_N - x_o)} - d_i \cdot \frac{(y_N - y_o)}{(x_N - x_o)} \tag{4}$$

$$f_i = \frac{(x_N y_{i-1} - x_o y_i)}{(x_N - x_o)} - d_i \cdot \frac{(x_N y_o - y_N x_o)}{(x_N - x_o)}$$

The parameters, which characterize this interpolation function, are given in Table 1.

It is important to mention here that the Hausdorff dimension has a restricted application to the study of fractal curves resulting in such science fields as physics, biology or engineering. For these reasons, the fractal dimension D of the fractal interpolation function is used. For the definition of this dimension D let us assume that $\varepsilon > 0$ and $N(\varepsilon)$ denotes the number of square boxes of side length ε which intersect the graph of the fractal interpolation function. Then the fractal dimension D is given by the relation $N(\varepsilon) \approx \text{constant} \times \varepsilon^{-D}$ as $\varepsilon \rightarrow 0$. It has been shown that if $\sum_{i=1}^{N} |d_i| > 1$ and the interpolation points

Table 1 Parameters characterizing pavement surfaces.

a_i	c_i	d_i	b_i	f_i
0.1111	0.0046	0.90	0.0	0.00
0.0010	−0.0043	0.75	5.0	0.20
0.1111	0.0013	0.65	10.0	0.00
0.1111	0.0014	0.50	15.0	0.05
0.1111	0.0010	0.50	20.0	0.11
0.1111	0.0014	0.40	25.0	0.15
0.1111	−0.0013	0.35	30.0	0.21
0.1111	−0.0013	0.95	35.0	0.15
0.1111	−0.0020	0.20	40.0	0.08

do not lie on a single straight line, then the fractal dimension of F is the unique solution D to the following Equation:

$$\sum_{i=1}^{N} |d_i| a_i^{D-1} = 1 \tag{5}$$

It is remarkable that the fractal dimension does not depend on the values $\{y_i; i = 0, 1, K, N\}$, aside from the constraint that the interpolation points be noncollinear. Hence it is easy to explore a collection of fractal interpolation functions, having the same fractal dimension, by imposing the following simple constraint on the vertical scaling factors d_i:

$$\sum_{i=1}^{N} |d_i| = N^{D-1} \tag{6}$$

In the sequel fractal interpolation functions are generalized by assuming that the given set of points has the form $\{(x_i, F_i) \in \Re \times Y; i = 0, 1, K, N\}$ with $x_o < x_1 < \Lambda < x_N$. The interpolation function corresponding to this set of data is an continuous function $f: [x_o, x_N] \rightarrow Y$ such that $f(x_1) = F_i$; $i = 0, 1, K, N$. If Z denotes the Cartesian product space $\Re \times Y$, θ denotes a positive number, and d is a metric defined on Z by the relation:

$$d(z_1, z_2) = |x_1 - x_2| + \theta d_y(y_1, y_2) \tag{7}$$

for all points $z_1 = (x_1, y_1)$ and $z_2 = (x_2, y_2)$ in Z, then (Z, d) is a complete metric space. Based on the above data the following transformations are defined:

$$L_i: I \rightarrow I_i = [x_{i-1}, x_i], L_i(x) = a_i x + b \tag{8}$$

where a_i, b_i are given by the relation above $M_i: Z \rightarrow Y$ and such that:

$$d_y(M_i(a, y), M_i(b, y)) \le c|a - b| \tag{9a}$$

for all $a, b \in \Re$ and $y \in Y$

$$d_y(M_i(x, a), M_i(x, b)) \le s d_y(a, b) \tag{9b}$$

for all $x \in \Re$ and $a, b \in Y$

These two transformations (L_i, M_i) define an IFS $\{Z| w_i; i = 1, 2, K, N\}$, where $w_i(x, y) = (L_i(x), M_i(x, y))$.

It is also assumed that every transformation M_i is constrained by the given set of data according to:

$$M_i(x_o, F_o) = F_{i-1} \tag{10}$$
$$M_i(x_N, F_N) = F_i$$

From Eqs. 4, 8, and 10, it is easily obtained that:

$$w_i(x_o, F_o) = (x_{i-1}, F_{i-1}) \text{ and} \quad (11)$$
$$w_i(x_N, F_N) = (x_i, F_i)$$

The IFS $\{Z| w_i; i = 1, 2, K, N\}$ defined above has a unique attractor, which is the graph of a continuous function $f: [x_o, x_N] \rightarrow Y$. It satisfies the relations $f(x_i) F_i$, i.e. is the fractal interpolation function of the data. It is important to mention here that the previous results can be used to obtain flexible interpolation functions. This has already been achieved at the beginning of this section where fractal interpolation functions $g: [x_o, x_N] \rightarrow \Re$ were obtained.

2.3 Simulation of the pavement surface

In order to describe the pavement surface texture, information about the depth of micro-texture and macro-texture is needed. Pavement irregularities are continuous from pavement defaults (mega-texture) to through the micro scale. The pattern of pavement surface irregularities remains similar and is reproduced irrespective of the scale of the surface texture irregularity considered texture. The fractal nature of the pavement surface is, therefore, inherent. As it has been mentioned, fractal concepts have already been used in evaluating certain subjects in pavement engineering.

The range of surface irregularities which is of interest is skid resistance, comprising what has been described as micro/macro texture. It has been mentioned that micro-textures are classified as irregularities between 0.005 and 0.3 *mm*. A harsh surface pavement has an average micro-texture depth of 0.05 mm (50 μm). Macro-textures are considered as irregularities between 0.3 and 4.0 *mm*. A pavement surface is considered rough if the average depth of macro-texture is more than 1.0 *mm*. Apart from the depth of surface irregularities, their density (wavelength) is also necessary for the geometric determination of the pavement surface profile. According to the literature, this scale of surface irregularities the wavelength (λ) is about 3 to 20 times greater than the depth of the irregularities themselves.[9]

From the above, it is obvious that the fractal approximation offers an ideal framework for the description of the pavement surface. This approximation covers all types of pavement surface, as classified with respect to their micro/macro texture.

In Fig. 1 shows all four characteristics and a typical profile of the pavement surfaces, which are approximated by fractal interpolation functions. For every profile, a different set of free parameters $d_i; i = 1, 2, K, 9$ has been taken.

In Table 2 we present free parameters for every fractal interpolation function, which corresponds to a different type of pavement surface.

As it has already been mentioned, there is a gives set of points for every fractal interpolation function. In our case, where four different fractal interpolation functions are given, these sets of data are as follows.

For a rough and harsh pavement surface texture:

$\{(x_i^A, y_i^A; i = 0, 1, 2, K, 9\} =$
$\{(0.0, 0.0), (5.0, 0.8), (10.0, 0.0), (15.0, 0.005),$
$(20.0, 0.01), (25.0, -0.7), (30.0, -0.07),$
$(35.0\ -0.04), (40.0, -0.01), (45.0, 0.0)\}$

For a rough and polished pavement surface texture:
$\{(x_i^B, y_i^B; i = 0, 1, 2, K, 9\} =$
$\{(0.0, 0.0), (5.0, 0.8), (10.0, 1.0), (15.0, 0.005),$

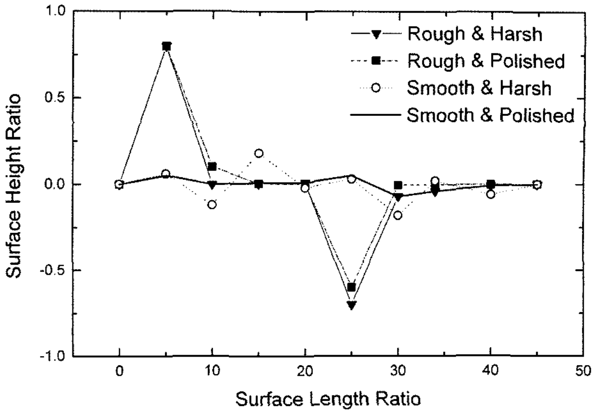

Fig. 1 Coefficients of fractal interpolation functions.

$(20.0, 0.003), (25.0, -0.6), (30.0, -0.005),$
$(35.0\ -0.002), (40.0, -0.001), (45.0, 0.0)\}$

For a smooth and harsh pavement surface texture:

$\{(x_i^C, y_i^C; i = 0, 1, 2, K, 9\} =$
$\{(0.0, 0.0), (5.0, 0.06), (10.0, -0.12), (15.0, 0.18),$
$(20.0, -0.02), (25.0, 0.03), (30.0, -0.18),$
$(35.0, 0.002), (40.0, -0.06), (45.0, 0.0)\}$

For a smooth and polished pavement surface texture:

$\{(x_i^A, y_i^A; i = 0, 1, 2, K, 9\} =$
$\{(0.0, 0.0), (5.0, 0.05), (10.0, 0.00), (15.0, 0.01),$
$(20.0, 0.01), (25.0, 0.05), (30.0, -0.07),$
$(35.0, -0.04), (40.0, -0.01), (45.0, 0.0)\}$

And the relationship between pavement length and irregularities is depicted in Figs. 2 and 3 for rough and smooth surfaces, respectively.

2.4 Correlation between skid number and fractal dimension

As it is shown in Fig. 4, every type of pavement surface approximated by a fractal interpolation function is characterized by a value of the fractal dimension D. It should be noted here that D has its maximum value in the case of a pavement surface with both micro texture and macro texture. Moreover, it is observed that D is greater for these cases, where the surface texture irregularities predominate at the micro scale, than for the cases, where these irregularities predominate at the macro scale. According to the literature, the average values of skid resistance (SN) for the characteristic pavement surface types are presented in Table 3. Based on Figs. 2 and 3, where every road profile is approximated by a fractal interpolation function with a fractal dimension D, and on Table 3, a relation between the skid number and the fractal dimension can be identified. In Fig. 4 this relation is presented for the characteristic pavement surfaces. Fractal dimension and skid number seem to vary in a proportional way.

Fractal dimension and SN values seem to be proportional to each other. After regression analysis, following relationship has been found:

$$D = 0.01148 \cdot (SN) + 0.8233 \quad (12)$$

By using the fractal interpolation function, pavement surface

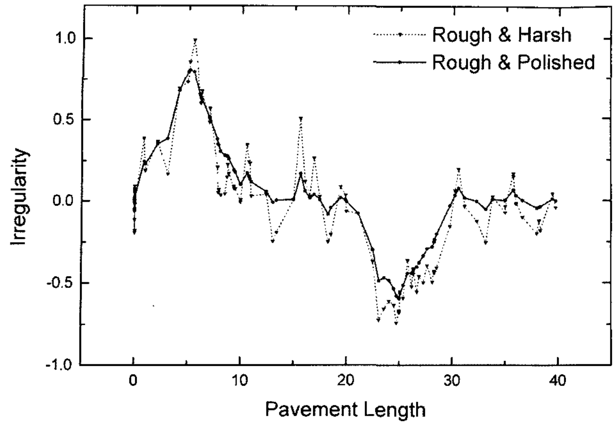

Fig. 2 Pavement surface texture–rough surfaces.

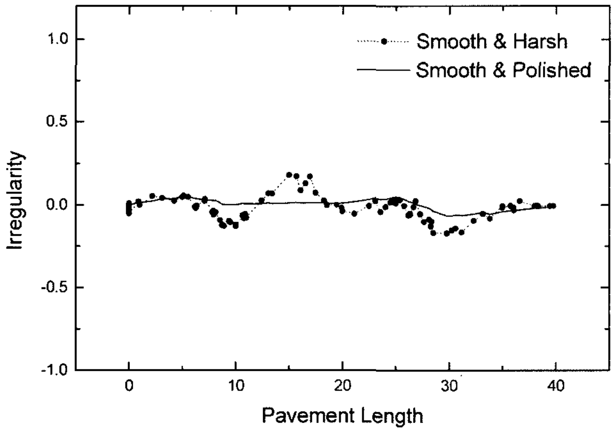

Fig. 3 Pavement surface texture–smooth surfaces.

Table 2 The free parameters d_1 of every fractal interpolation function.

Profile	d_1	d_2	d_3	d_4	d_5	d_6	d_7	d_8	d_9
A	0.45	0.50	0.40	0.40	0.50	0.40	0.35	0.35	0.30
B	0.20	0.20	0.10	0.10	0.20	0.20	0.10	0.10	0.10
C	0.40	0.35	0.30	0.30	0.40	0.35	0.40	0.30	0.35
D	0.15	0.10	0.15	0.15	0.10	0.15	0.10	0.15	0.05

can be simulated in terms of fractal dimension. Although fractal dimension from interpolation function depends on the coefficients of fractal interpolation function, it can be substituted by skid number in the mathematical framework. It is apparent that the parameters of interpolation function need significant further investigations.

3. Application of fractal theory to rock surface

Surface profiling in rock can be performed using either the actual rough surface or exact negative replicas often made with plaster of silicone rubber. Silicon rubber is well suited for injection into discontinuities that are hard to reach with mapping instruments, as their injection is followed by chemical curing rather than air dry curing. Surface roughness of rock can be classified in the three orders of roughness. First order roughness is the large wave of the surface, second order roughness is at a finer level that considers the small scale asperities, and the fine asperities or grain size level of asperities are considered to be at the third order of roughness. The suitable measuring device should be chosen according to the required resolution for the mapping of the rock surfaces.

Table 3 Average values of skid number for different pavement profiles.

Profile	Characteristic pavement profiles	Skid Number
A	Pavements having both micro and macro texture	SN = 65 – 70
B	Polished but with macro texture pavements	SN = 25
C	Smooth but with micro texture pavements	SN = 60
D	Mirror pavements; lack of micro macro texture	SN = 20
E	Typical average pavements	SN = 40

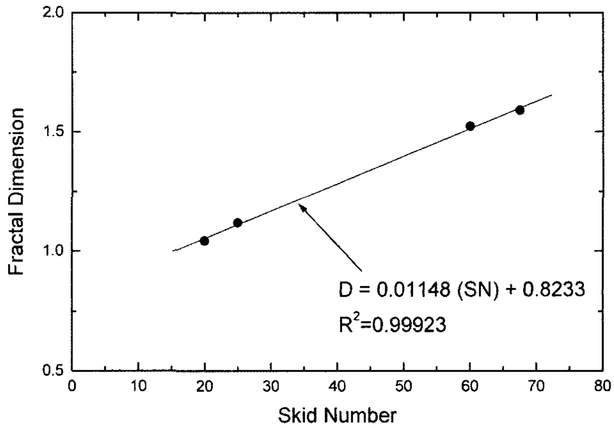

Fig. 4 Relationship between fractal dimension and skid numbe.

3.1 Joint roughness coefficient (JRC)

The engineering properties of rock masses are influenced by two important factors: (1) the physical properties of intact blocks of rock, and (2) the properties of discontinuities such as joints, faults, foliation, and bedding planes, which bound the individual blocks.

Structures built in or on a rock mass are affected by the shear strength of discontinuities within the mass. If a rock mass contains unfavorably oriented discontinuities, its strength for engineering purposes may be greatly reduced. Rock mass behavior is also influenced by rock joint properties such as surface roughness, weathering, and the presence of infilling. The surface roughness of a rock joint controls the shear strength and dilatancy of the rock joint. The strength of the whole mass of rock (the intact blocks plus the inherent discontinuities) must be considered in the design of structure.[10] Many researches have been conduced to investigate some of the factors, which influence the strength along the natural discontinuities. Of particular notice may be the nature and magnitude of the surface roughness on the rock joint surfaces.

Joint surfaces can be digitized with a mechanical profilometer like the one used in this study. These linear profiles form the basis for analytical measures of roughness. Fractal dimension can be obtained by surface roughness. Once the characteristics of the fractal dimension and joint roughness coefficient are determined, an empirical relationship relating shear strength and fractal dimension can be established.

The joint roughness coefficient is determined by comparing the rock surface roughness with 10 sets of standardized rock roughness profiles ranging from 0 to 20 as shown in Fig. 5. This method of roughness determination has been supported by the Intenational Society of Rock Mechanics.[11]

3.2 Correlation between JRC and fractal dimension

Berry and Lewis[12] determined the fractal dimension for joint and bedding planes using elevation data measured along a line traversing these surfaces. They made field measurements of rock surface roughness on a joint surface exposed by a two-plane wedge type rock slide at the Libby Dam, Montana. They described an empirical relationship between JRC and fractal dimension (D) for 15 rock joint profiles. The empirical relationship between JRC and fractal dimension is a linear relationship to each other like the following:

$$JCR = -1022.55 + 1023.92 \cdot D \quad (13)$$

where, JRC is the joint roughness coefficient, D is the fractal dimension.

Previously[13] also found the relationship between joint roughness and fractal dimension to be:

$$JCR = -1001.56 + 1003.11 \cdot D \quad (14)$$

The approximated relationship between joint roughness coefficient and fractal dimension can be of exponential relation rather than linear relationship (as it can be seen in Fig. 6). Its suggested relation is presented in the following:

$$D = 0.00128^{*} \exp(JRC/7.75507) \quad (15)$$

Therefore, joint roughness coefficient can be determined more precisely by using Eq. 15.

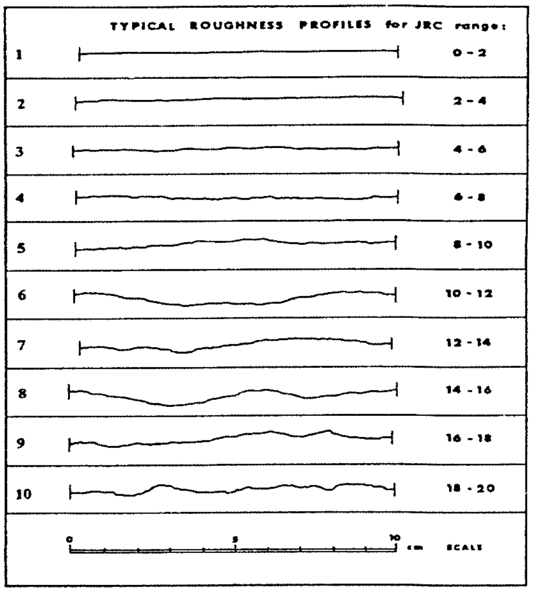

Fig. 5 Rock joint roughness coefficient.[10]

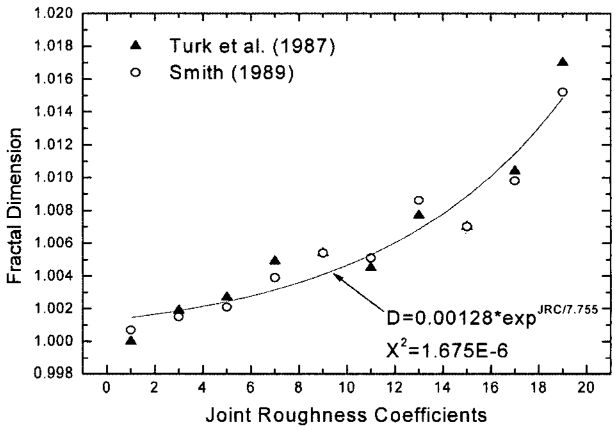

Fig. 6 Correlation between JRC and fractal dimension.

4. Conclusions and discussion

It is well known that many physical phenomena and systems are chaotic, random and that various features of roughness are found at a wide spectrum of length scales from the length of the sample to the atomic scales. This study emphasizes the role of fractal geometry to characterize the roughness of various surfaces. Pavement roughness and rock surface roughness were examined in order to correlate their roughness property to fractality. Using the fractality concept, following conclusions can be drawn;

1) The fractal geometry by means of the notion of the fractal interpolation function is defined to characterize the roughness of the pavement surface.

2) Fractal dimension and skid number (SN) have a proportional relationship expressed as,

$D = 0.01148 \cdot (SN) + 0.8233$

3) Correlation between joint roughness coefficient (JRC) of rock surface and the fractal dimension is expressed as,

$D = 0.00128^{*} \exp(JRC/7.75507)$

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