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Applying Novel Mean Residual Life Confidence Intervals

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Abstract. Typical confidence intervals for a mean or mean residual life (MRL) are centered about the mean or mean residual life. We discuss novel confidence intervals that produce statements like "we are 95% confident that the MRL function, e(t), is greater than a prespecified μ_O for all t in the interval [0, $\hat{\theta}$)" where $\hat{\theta}$ is determined from the sample data, confidence level, and μ_O. Also, we can have statements like "we are 95% confident that the MRL of population 1, namely $e_1(t)$, is greater than the MRL of population 2, $e_2(t)$, for all t in the interval [0, $\hat{\theta}$)" where $\hat{\theta}$ is determined from the sample data and confidence level. We illustrate these one and two sample confidence intervals on internal bonds (tensile strengths) for an important modern engineered wood product, called medium density fiberboard (MDF), used internationally.

Keywords : *mean residual life, expected remaining life, reliability, novel confidence intervals, internal bond, tensile strength, strength of materials, medium density fiberboard.*

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1.1. INTRODUCTION

Mean remaining (or residual) life (MRL) functions and tables have been studied and commented on by many individuals over the years. Guess and Proschan (1988), Chiang (1968), and Deevey (1947), cite the use of the mean residual life for annuities via expected life tables in ancient Roman culture. More recently, a wide host of papers covers many other aspects of MRL, for example, Guess, Zhang, Young, and Leon (2005), Zhao and Elsayed (2005). Anis, Basu, and Mitra (2004), Bradley and Gupta (2003), Asadi and Ebrahimi (2000), Na and Kim, (2000), Lim and Park (1998), Guess, Nam, and Park (1998), Guess, Walker, and Gallant (1992), Abouammoh (1988), Oakes and Dasu (1990), Berger, Boos, and Guess (1988), Guess and Park (1988), Guess, Hollander and Proschan (1984). These citations are a brief list of the many excellent papers written on MRL. We plan later an extensive survey of MRL works.

Note that for a random lifetime X, the MRL is defined as the conditional expectation E(X-t | X > t). This can be further represented and simplified using the reliability function R(t) = P(X>t) = 1 – F(t) as

$$e(t)=E(X-t\,|\,X>t)=\left(\int_t^{\infty}R(x)dx\right)/R(t)$$

where we assume R(t) > 0 for e(t) to be well defined. Also note, the empirical MRL is easily calculated by substituting the standard empirical reliability function into the formula of e(t) for R(t). Compare Guess and Proschan (1988). In this paper we will calculate the empirical MRL at the order statistics, and then linearize it in between the order statistics.

Recall the MRL function can exist, while the failure rate function might not exist, or, vice a versa, the failure rate function can exist without the MRL function existing. The reliability function can be represented as a function of the MRL, as

$$R(t)=\frac{e(0)}{e(t)}\exp\left(-\int_0^t\left[\frac{1}{e(x)}\right]dx\right)$$

Compare, for example, Guess and Proschan (1988) for additional comments and insights on MRL. For more information on special classes of distributions connected with MRL that have been widely studied and tested in a variety of situations see the helpful paper by Hollander and Proschan (1984), plus the classic book by Barlow and Proschan (1981).

Typical confidence intervals for a mean or mean residual life (MRL) are centered about the mean or mean residual life. We discuss novel confidence intervals that allow making statements like "we are 95% confident that the MRL e(t) is greater than a prespecified μ_O for all t in the interval [0, $\hat{\theta}$)" where $\hat{\theta}$ is determined from the sample data, confidence level, and μ_O. Also, we can have statements like "we are 95% confident that the MRL of population 1, namely $e_1(t)$, is greater than the MRL of population 2, $e_2(t)$, for all t in the interval [0, $\hat{\theta}$)" where again $\hat{\theta}$ is determined from the sample data and confidence level. Other types of confidence statements are also possible.

We illustrate these sample confidence intervals on internal bonds (tensile strengths) of an important modern engineered wood product, called medium density fiberboard (MDF), used internationally. See Berger, Boos, and Guess (1988) and Balgopal (1989) for more on these types of MRL intervals. For more on MDF see Guess, León, Chen, and Young, (2004), Guess, Edwards, Pickrell, and Young (2003), and Young and Guess (2002).

In Section 2, we discuss the helpful two sample case of confidence intervals using mean residual life functions (MRL). We apply these confidence intervals to real data from tests of tensile strength of MDF. We emphasize that these intervals can be used even more broadly, not just for regular life data. The intervals can be used for any time or stress to response data, plus financial data, etc.

Recall that Weibull's original reliability function was developed by him studying and fitting strengths for various materials (see Weibull 1939, 1951). Product "life" for MDF can be measured in terms of the strength to failure, as opposed to the time to failure. The strength (e.g., internal bond) or pounds per square inch (p.s.i.) to failure is a crucial reliability parameter of the product. It naturally allows the producer to make assurances to customers about the quality, safety, and useful "life" range of the product. We will write MRL where we understand it is actually mean remaining pressure (measured in p.s.i.) until failure.

In Section 3 we present the one sample case and applications of it to MDF data. In Section 4 we have concluding comments and recommendations on these confidence intervals and future work.

2. TWO SAMPLE MEAN RESIDUAL LIFE CONFIDENCE INTERVALS ON MODERN ENGINEERED WOOD

Compare Young and Guess (2002) for how MDF data is stored and used in a real time database with regression modeling to predict strength. This provides quick feedback to the manufacturer in order to minimize process inputs and maximize product quality within specified limits. One key metric used by manufacturers of the quality or reliability of MDF is internal bond (IB). Samples from a cross section of the MDF are tested by being pulled apart. The IB at failure is then measured in pounds per square inch (p.s.i.) or the corresponding metric units (kilograms per cubic meter).

We use $\hat{e}_{46}(t)$ to denote the empirical MRL of an MDF product with density of 46 pounds per cubic foot (lbs/ft^3), thickness of 0.625 inches, and width of 61 inches. We employ $\hat{e}_{48}(t)$, to denote the empirical MRL of an MDF product with density of 48 pounds per cubic foot (lbs/ft^3), thickness of 0.625 inches, and width of 61 inches. Naturally, the corresponding population MRL's are written $e_{46}(t)$ and $e_{48}(t)$, respectively. Our sample size for density of 48 is $n_{48} = 108$

units, while the sample size for density of 46 is $n_{46} = 975$ units. A priori, MDF workers would conjecture that a higher density of 48 would yield a greater average IB and MRL. This turned out to be mostly true, but surprisingly was not always the case.

We discuss three figures that provide different insights into the MRL's, into the corresponding novel confidence intervals, and into the specific statistical functions that are used to create these confidence intervals. The empirical MRL is plotted at each unique failure then linearized between points as seen in the figures. In Figure 2.1 we graph the empirical MRL's for both $\hat{e}_{46}(t)$ and $\hat{e}_{48}(t)$. It is natural to conjecture for these products a decreasing empirical MRL. Recall this suggests the classical DMRL (Decreasing Mean Residual Life) class tested in Hollander and Proschan (1975), plus other helpful DMRL tests by other authors. Note for density 48 there is graphical evidence of a DMRL, but this is not the case for density 46 (Figure 2.1).

As expected the MRL for $\hat{e}_{48}(t)$ is much higher, starting with a sample mean of $\hat{e}_{48}(0) =$ 185.7 p.s.i., corresponding to mean residual pressure to failure in p.s.i. (which again will be understood when we use the standard abbreviation of MRL). This is higher, as would be conjectured, than a lower density MRL's sample mean of $\hat{e}_{46}(0) = 122.7$ p.s.i. Also as conjectured, $\hat{e}_{48}(t)$ dominates $\hat{e}_{46}(t)$ for all $t > 0$ until around $t = 150$ p.s.i., when they surprisingly switch roles after a crossing. Also, note that $\hat{e}_{46}(t)$ has a later peak at 151.4 with the MRL being $\hat{e}_{46}(151.4) = 49.3$, while the other MRL (which workers expected to be higher) is actually lower instead with $\hat{e}_{48}(151.4) = 34.3$.

Recall that a priori one would not conjecture this switching. This suggests that for a density of 46 lbs/ft^3 some units may be produced with unnecessarily high raw material set points, i.e., slow production transition from a density of 46 to 48, producing an intermediate type of product misclassified as having a density of 46 lbs/ft^3. The upper turning in the MRL of $\hat{e}_{46}(t)$ is unusual and may yield higher product costs for the density of 46 lbs/ft^3 product that does not require the stronger IB. The MRL provides an interesting rubric for product classification and continuous improvement.

The increase in MRL above 135 p.s.i., for the product with a density of 46 lbs/ft^3, was a surprise. This may show a setup change by the manufacturer to a higher targeted strength product, i.e., the manufacturer produces a higher strength product with higher resin and wood at a slower line speed, and is unable to instantaneously meet target specifications during setup change from the nominal strength to higher strength product.

It is obvious to practitioners from the MRL graph in Figure 2.2 that a hybrid product, or medium-strength product, is likely being produced. The MRL graph in Figure 2.1 reveals an opportunity cost for the manufacturer, e.g., improve setup change time to minimize resin usage, optimize line speed targets during product change, etc. The MRL graph in Figure 2.1 may have, also, identified a new product opportunity for the manufacturer. These empirical MRL behaviors can be powerful diagnostic tools to facilitate training, continuous improvement, and ultimately cost savings.

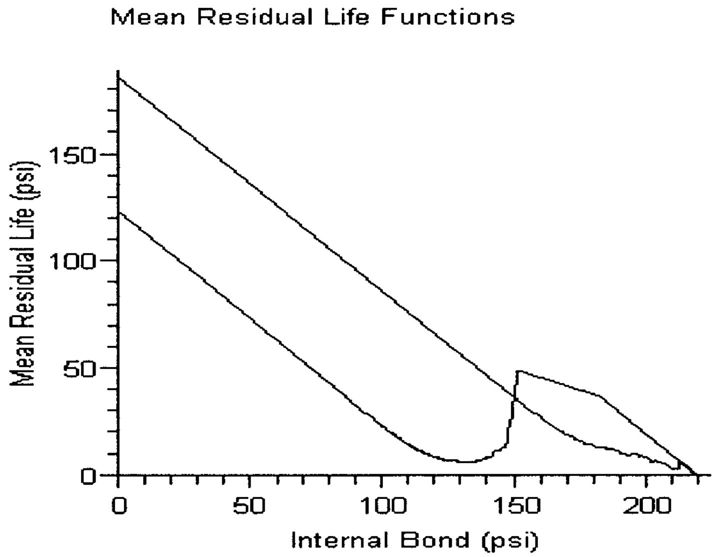

Figure 2.1. Empirical mean residual life functions of expected remaining pressure till failure. Top graph is the MRL $\hat{e}_{48}(t)$ for MDF with density 48, bottom is $\hat{e}_{46}(t)$ the empirical MRL for a density 46. Note $\hat{e}_{48}(0) = 185.7$ and $\hat{e}_{46}(0) = 122.7$, crossing and later peak for $\hat{e}_{46}(t)$.

Figure 2.2 has the two empirical MRL functions plus the statistical function $Z_{mn}(t)$ described in Berger, Boos, and Guess (1988) and the critical threshold z value straight line of $z = 2.33$ for determining a 99% confidence band. Note that $Z_{mn}(t)$ is essentially a two sample statistic on the difference between two population means, but here it is for the MRL functions at time t. Also, note that m and n are the respectively sample sizes $m = n_{48} = 108$ and $n = n_{46} = 975$. The thicker line is the statistical test function $Z_{mn}(t)$. Note $Z_{mn}(t)$ crosses from above the critical threshold line of $z = 2.33$ at the point $\hat{\theta} = 147.98$. This implies, "we have 99% confidence that the population MRL for density 48, $e_{48}(t)$, dominates as statistically significant the population MRL for density 46, $e_{46}(t)$, for the entire interval [0, 147.98)." These can be thought of as lower confidence bounds of the form $C = 1-\alpha$, with $e_{48}(t) > e_{46}(t)$ for all t in $[0, \hat{\theta})$ where the $\hat{\theta} = \inf \{t \geq 0: Z_{mn}(t) \leq z_{\alpha}\}$ where z_{α} is the standard normal upper α quantile, $z_{\alpha} = 2.33$, for the one sided lower 99% confidence interval. For a 95% we would use $z_{\alpha} = 1.645$, while for 90% confidence we would use $z_\alpha = 1.28$, etc. See Berger, Boos, and Guess (1988) for more comments and other types of these novel confidences intervals.

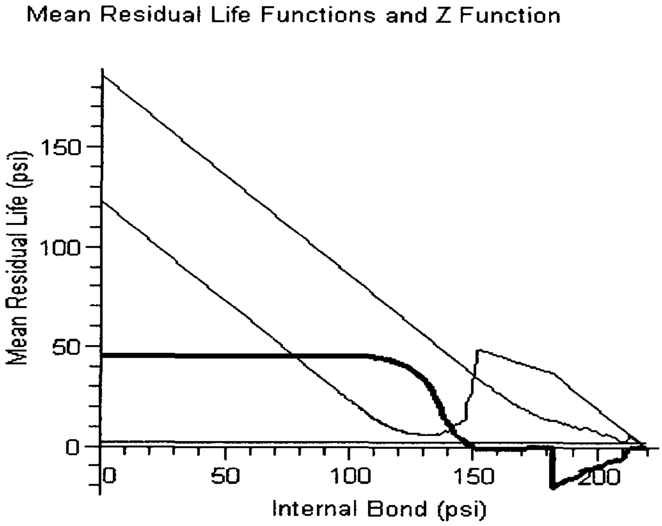

Figure 2.2. Same as Figure 2.1, additional discerning statistical difference test function $Z_{mn}(t)$ as the darker line. Yields a crossing from above on the critical threshold line $z_\alpha = 2.33$ at $\hat{\theta} = \inf \{t \geq 0: Z_{mn}(t) \leq z_\alpha\} = 147.98$, producing a 99% one sided confidence interval.

Lastly we give insight into the need to adjust a two sample procedure when the remaining units (i.e., remaining sample sizes) are small (Figure 2.3). Figure 2.3 has all of the Figure 2.2 functions, and also has the "t" value adjusted for use in a two sample procedure using the adjusted "t" and adjusted degrees of freedom. We replace the z percentile with the appropriate adjusted "t" percentile. Recall our initial large sample sizes $n_{48} = 108$ and $n_{46} = 975$. Typically, MRL is a large sample approach, but we need to stress the need for care in the later tails of the MRL when the remaining samples might be small. Note that the t values jump above 2.33 or below -2.33 to the "t" heights in Figure 2.3. Also, there is more need to prespecify particular aspects there. For additional specific details see Berger, Boos, and Guess (1988).

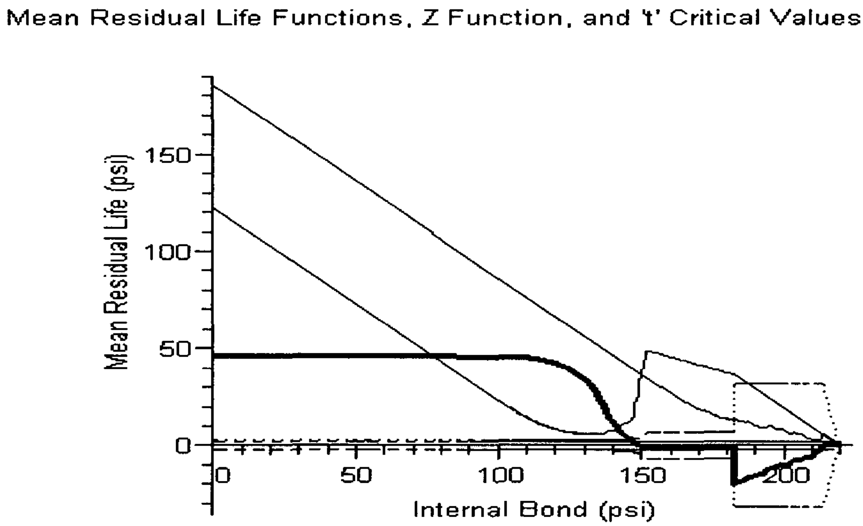

Figure 2.3. Same as Figure 2.2, added another critical z = -2.33 and the adjusted "t" values for when the remaining sample sizes are small. Note the jump in the "t" values beyond 150.

Obviously other techniques such as TTT plots (see, Klefsjö 1991), box plots, histograms, as well as MRL plots, are helpful for process improvement and training. We stress the helpfulness of the graphs, but especially these novel confidence intervals for comparisons that are statistically valid in the two sample case and, as seen next, the one sample case.

3. SAMPLE MEAN RESIDUAL LIFE CONFIDENCE INTERVALS ON MODERN ENGINEERED WOOD

We now illustrate the one sample versions of these confidence intervals. For the population MRL $e_{48}(t)$ of density 48, a one sided 99% confidence interval for $e_{48}(t)$ to be, for example, above 80 p.s.i. would yield $\hat{\theta}$ = 102.57. This implies we can say with 99% confidence that the population MRL $e_{48}(t)$ is larger than 80 p.s.i. for all t in the entire interval [0, 102.57).

For the population MRL $e_{46}(t)$ of density 46, a one sided 99% confidence interval for $e_{46}(t)$ to be above 80 p.s.i. would yield instead $\hat{\theta}$ = 41.97. This implies we can say with 99% confidence that the population MRL $e_{46}(t)$ is larger than 80 p.s.i. for all t in the entire interval [0, 41.97).

Note the MRL may help in determining actual minimum safety standards and extra safety thresholds that are statistically valid. MRL may also lead to economic benefits for the manufacturer from reduced rework and improved efficiency.

4.4. SUMMARY AND CLOSING REMARKS

We recommend using these novel confidence intervals for both two sample and one sample settings. They provide additional insights that can be used to quantify aspects suggested by graphical comparisons, which are also useful. These statistical intervals can be used to explore the data and find key thresholds. Note how unique, striking behaviors can be identified by comparative MRL plots and these novel confidence intervals on real word data sets on tensile strength measured by IB. This may facilitate training, process improvement and cost savings.

The calculations and graphs were done in Maple, version 10. Code from Maple can be outputted in Matlab and C++. Other languages naturally are also available, for example R or S+. Copies of our code in Maple are available by emailing Research Associate Professor Timothy M. Young at tmyoung1@utk.edu. We plan later an extensive survey of MRL works.

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