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Multivariate Mean Inactivity Time Functions with Reliability Applications

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Abstract. AIn this paper we introduce and study a multivariate notions of mean inactivity time (MIT) functions. Basic properties of these functions are derived and their relationship to the multivariate conditional reversed hazard rate functions is studied. A partial ordering, called MIT ordering, of non-negative random vectors is introduced and its basic properties are presented. Its relationship to reversed hazard rate ordering is pointed out. Finally, using the MIT ordering, a bivariate and multivariate notions of IMIT (increasing mean inactivity time) class is introduced and studied.

Key Words : *Mean inactivity time, reversed hazard rate order, stochastic order, bivariate life classes, DRHR, IMIT*

1. INTRODUCTION

In reliability theory, the concept of inactivity time is extensively used in modeling lifetime data and defining various life classes. In this paper, we provide some new results involving the mean inactivity time function (MIT) in a univariate case and introduce a multivariate notion of it in the context of multivariate reliability theory. Before we go into the details, let us quickly review some common notions of univariate stochastic orderings that will be considered in this paper (see *Shaked* and *Shanthikumar* (1994) for an exhaustive monograph on this topic). We shall denote the distribution function, the survival function and the reversed hazard rate function of a univariate random variable T by F_T, $\overline{F}_T$ and λ_{F_T}, respectively. We use a similar notation for all other random variables.

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A random variable T is said to be stochastically smaller than another random variable S (denoted by $T \leq_{ST} S$) if $\overline{F}_T(t) \leq \overline{F}_S(t)$ for all t.

A stronger notion of stochastic dominance is that of reversed hazard rate ordering: T is said to be smaller than S in reversed hazard rate ordering (denoted by $T \leq_{RHR} S$) if, and only, if

$$F_T(t)/F_S(t) \text{ is non-increasing in } t. \tag{1.1}$$

In the continuous case, another condition equivalent to (1.1) is

$$\lambda_{F_T}(t) \leq \lambda_{F_S}(t), \text{ for all } t \geq 0. \tag{1.2}$$

For any random variable T, let $T_t = [T-t|T > t]$ denote a random variable whose distribution is the same as the conditional distribution of $T - t$ given that $T > t$. When T is the lifetime of a device, T_t can be regarded as the residual lifetime of the device at time t, given that the device has survived up to time t (see, *Ahmad, Kayid* and *Li* (2005)). Another ordering which relates the residual lifetimes is the mean residual life order: T is said to be smaller than S in mean residual order (denoted by $T \leq_{MRL} S$) if, and only, if (*Alzaid,* 1988)

$$\frac{\int_t^\infty \overline{F}_T(x)dx}{\overline{F}_T(t)} \leq \frac{\int_t^\infty \overline{F}_S(x)dx}{\overline{F}_S(t)}, \text{ for all } t \geq 0. \tag{1.3}$$

On the other hand, in many reliability problems it is of interest to consider variables of the kind $T_{(t)} = [t - T \mid T \leq t]$, for fixed $t > 0$, having distribution function $F_{(t)}(s) = P[t - T \leq s \mid T \leq t]$, and known in literature as inactivity time (*Chandra* and *Roy* (2001), *Li* and *Lu* (2003), *Kayid* and *Ahmad* (2004), *Ahmad, Kayid* and *Pellerey* (2005), *Ahmad* and *Kayid* (2005), *Li* and *Xu* (2006) and *Ahmad, Hu* and *Kayid* (2006)).

The MIT function of T is defined by

$$\mu(t) = \begin{cases} E[t - T|T \leq t], & t > 0, \\ 0, & otherwise. \end{cases} \tag{1.4}$$

In the current investigation, we further develop the mean inactivity time (MIT) functions. The construction of this paper is as follows: First, in *Section* 2, we review some basic properties concerning of the MIT functions in the univariate setting and introduce some new results. In *Section* 3, we give the formal definition of the bivariate and multivariate version of the MIT functions and introduce a partial ordering (called the MIT ordering) of vectors of random inactivity time. In that section, using the MIT ordering, a bivariate and multivariate notion of IMIT (increasing mean inactivity time) is introduced and studied. Finally, in *Section* 4 we provide some applications in reliability theory.

2. UNIVARIATE MIT FUNCTIONS

First, we summarize the relationships between the distribution function, reversed hazard rate function and the mean inactivity time function. Its well-known fact that each of the function F, λ and μ uniquely determines the other two. Some of the relationships, which can be easily verified, are listed below:

(a) Relationship between λ and μ:

$$\lambda(t) = \frac{1-\mu^{'}(t)}{\mu(t)}, \quad \mu^{'} \equiv \left(\frac{d}{dt}(\mu)\right), \tag{2.1}$$

$$\mu(t) = \int_0^t \exp\left\{-\int_u^t \lambda(y)dy\right\} du. \tag{2.2}$$

(b) Relationship between F and μ:

$$\mu(t) = \int_0^t \frac{F(x)dx}{F(t)}, \tag{2.3}$$

$$F(t) = \exp\left\{-\int_t^{\infty} \frac{1-\mu^{'}(u)}{\mu(u)} du\right\}, \tag{2.4}$$

$$F(t) = \lim_{x\to\infty} \frac{\mu(x)}{\mu(t)} \exp\left\{-\int_t^x \frac{1}{\mu(u)} du\right\}, \quad for\ all\ t > 0. \tag{2.5}$$

As $\mu(0) = 0$, the corresponding MIT for $t \to 0$ should be equal to 0. The MIT function defines a distribution function, if

1. $\mu(0) = 0$; $\mu(x) > 0$, $x > 0$,
2. $\mu^{'}(x) < 1$,
3. $\int_0^{\infty} \left(\left(1-\mu^{'}(u)\right)/\mu(u)\right) du = \infty$,
4. $\int_t^{\infty} \left(\left(1-\mu^{'}(u)\right)/\mu(u)\right) du < \infty$, for all $t > 0$.

So far we have considered only one random variable T with MIT function μ. Consider another random variable S with an absolutely continuous distribution function F_S and MIT function β. The random variable T is said to be smaller than S in the MIT ordering (denoted by $T \leq_{MIT} S$) if, and only, if

$$\mu(t) \equiv \int_0^t \frac{F_T(x)dx}{F_T(t)} \geq \int_0^t \frac{F_S(x)dx}{F_S(t)} \equiv \beta(t), for\ all\ t > 0. \tag{2.6}$$

This ordering was recently introduced and studied by *Nanda et al.* (2003), *Kayid* and *Ahmad* (2004), *Ahmad* and *Kayid* (2005), *Ahmad, Kayid* and *Pellerey.* (2005). See further work by *Li* and *Xu* (2006). From the definition of $\leq_{MIT}$ order it could be thought that the results one can obtain on the MIT order follow directly from the results on the $\leq_{MRL}$ order and the fact that $X \leq_{MIT} Y$ if, and only, if $-X \geq_{MRL} -Y$. However, this is not generally true since this property can not be useful when one assumes non-negativity of X and Y, which is assumed for most of the known results on $\leq_{MRL}$ order. Such equivalence can be applied only in the case X and Y have a common finality bounded support $[0, l]$, $l \in R^+$, since in that case we can write it as $X \leq_{MIT} Y$ if, and only, if $l - X \geq_{MRL} l - Y$. But it is our intention, in this paper, to consider also the case of unbounded supports (that are more common in real applicative problems).

On the other hand, many non-parametric classes of distributions have been defined in literature to describe reliability properties of random lifetimes. Related to $\leq_{ST}$ order and $\leq_{MIT}$ order, two classes of life distributions have been introduced and studied in the literature. These are the decreasing reversed hazard rate (DRHR) and increasing mean inactivity time (IMIT) classes of life distributions (cf. *Block et al.* (1998), *Nanda et al.* (2003), *Kayid* and *Ahmad* (2004), *Ahmad* and *Kayid* (2005) and *Li* and *Xu* (2006)). Their definitions are also recalled here.

Definition 2.1.

A non-negative random variable T is said to be:

(i) decreasing reversed hazard rate (denoted by $T \in DRHR$) if, and only, if $T_{(s)} \leq_{ST} T_{(t)}, for\ all\ 0 < s < t$,
(ii) increasing mean inactivity time (denoted by $T \in IMIT$) if, and only, if $[T_{(t)}]$ is increasing in $t \geq 0$.

A wide range of distributions happen to be DRHR. These include two-parameter Weibull, gamma, Makeham, Pareto, log-normal and linear failure rate distributions. In addition, the linear failure rate and the Makeham distributions are also included in the IMIT class. The following implications among some of the above mentioned orders and non-parametric classes are well known (see *Nanda et al.* (2003) and *Kayid* and *Ahmad* (2004)):

$$T \leq_{RHR} S \Rightarrow T \leq_{MIT} S, \tag{2.7}$$

and DRHR $\subset IMIT$.

Next, we give some new results related the MIT order in univariate case. To state and prove *Theorem* 2.1 below, we need the following result which is due to *Li* and *Xu* (2006).

Lemma 2.1.

Assume that ϕ is strictly increasing concave function and $\phi(0) = 0$. If $T \leq_{MIT} S$ then $\phi(T) \leq_{MIT} \phi(S)$.

The next theorem states that the IMIT aging notion is preserved under increasing concave transformations. For similar kind of results, see *Al-Wasel et al.* (2006).

Theorem 2.1.

A non-negative random variable T is IMIT if, and only, if $\phi(T_t) \leq_{MIT} \phi(T)$, for all non-negative increasing concave function ϕ with $\phi(0) = 0$ and $t \geq 0$.

Proof.

We give the proof of the necessity only. Suppose that T is IMIT and let ϕ be any non-negative increasing concave function with $\phi(0) = 0$. Hence, we have to prove that

$$[\phi(T_t) \mid \phi(T_t) < s] \leq_{MIT} [\phi(T) \mid \phi(T) < s], \ for\ all\ s < min(\phi(u_{T_t}), \phi(u_T)). \quad (2.8)$$

From the assumption, it follows that $(\mathrm{T}_t)_{\phi^{-1}(s)} \leq_{MIT} (T)_{\phi^{-1}(s)}$or, equivalently,

$$[T_t \mid T_t < \phi^{-1}(s)] \leq_{MIT} [T \mid T < \phi^{-1}(s)], \ for\ each\ s. \quad (2.9)$$

Here the inverse ϕ^{-1} of ϕ is taken to be the right continuous version of it defined by $\phi^{-1}(u) = sup\{x : \phi(x) \leq u\}$ for $u \in R$. From the definition of ϕ^{-1} and the continuity of ϕ, it is easy to check that $u < \phi^{-1}(s)$ if, and only, if $\phi(u) < s$. Thus (2.9) can be rewritten as $[\mathrm{T}_t \mid \phi(T_t) < s] \leq_{MIT} [T \mid \phi(T) < s]$for each$s$ and $t \geq 0$. Appealing to *Lemma* 2.1, we get that $\phi([T_t \mid \phi(T_t) < s]) \leq_{MIT} \phi([T \mid \phi(T) < s])$,implying (2.8). This completes the proof. ∎

Consider now a sequence of independent and identical distributed (*i.i.d.*) random variables $T_1, T_2, ...$and let Θ be a positive integer-valued random variable which is independent of the T_i. Denote by

$$T_{(1:\Theta)} \equiv \min\{T_1, T_2, ..., T_\Theta\}, \quad (2.10)$$

and

$$T_{(\Theta:\Theta)} \equiv \max\{T_1, T_2, ..., T_\Theta\}. \quad (2.11)$$

The random variables $T_{(1:\Theta)}$ and $T_{(\Theta:\Theta)}$ arise naturally in reliability theory as the lifetimes of a series and parallel systems, respectively, with the random number Θ of identical components with lifetimes $T_1, T_2, ..., T_\Theta$. In life-testing, if a random censoring is adopted, then the completely observed data constitute a sample of random size, say $T_1, T_2, ..., T_\Theta$, where $\Theta > 0$ is a random variable of integer value. In actuarial science, the claims received by an insurer in a certain time interval should also be a sample of random size, and, $T_{(\Theta:\Theta)}$, denotes the largest claim amount of the period. Also $T_{(1:\Theta)}$ arises naturally in survival analysis as the minimal survival time of a transplant operation, where Θ of them are defective and hence may cause death.

Recently, many authors have paid their attention to behaviors of aging properties for random minima and maxima (see, *Shaked* and *Wong* (1997), *Li* and *Zuo* (2004) and *Ahmad* and *Kayid* (2006)). Next, we study behavior of the IMIT life distribution for random maxima. To state and prove *Theorem* 2.2 below, we need the following lemma. The proof is trivial and hence is omitted.

Lemma 2.2.

A non-negative random variable T has the $IMIT$ property if, and only, if $T_{(s)} \leq_{MRL} T_{(t)},\ for\ all\ 0 < s \leq t.$

Theorem 2.2.

Let $T_1, T_2, ..., T_\Theta$ be a sequence of *i.i.d.* random variables copies of T and Θ is independent of T_i's. If $\max\{T_1, T_2, ..., T_\Theta\}$ is IMIT then T is also IMIT.

Proof.

First note that, for all $s \geq 0$,

$$(\max\{T_1, ..., T_\Theta\})_{(s)} =_{ST} \min\left\{(T_1)_{(s)}, ..., (T_\Theta)_{(s)}\right\}. \tag{2.12}$$

Since $\max\{T_1, ..., T_\Theta\}$ is IMIT, by *Lemma* 2.2, it holds that,

$$(\max\{T_1, ..., T_\Theta\})_{(s)} \leq_{MRL} (\max\{T_1, ..., T_\Theta\})_{(t)}, for\ all\ 0 < s \leq t. \tag{2.13}$$

Thus, by (2.12), for all $0 < s \leq t$,

$$\min\left\{(T_1)_{(s)}, ..., (T_\Theta)_{(s)}\right\} \leq_{MRL} \min\left\{(T_1)_{(t)}, ..., (T_\Theta)_{(t)}\right\}. \tag{2.14}$$

In view of *Corollary* 3.3. of *Li* and *Xu* (2006) we have

$$\min\{T_1, ..., T_\Theta\} \leq_{MRL} \min\{S_1, ..., S_\Theta\} \Rightarrow T \leq_{MRL} S. \tag{2.15}$$

Now it follow from (2.14) and (2.15) that $T_{(s)} \leq_{MRL} T_{(t)}$, for all $0 < s \leq t$. Again, by *Lemma* 2.2, T is IMIT. ∎

3. MULTIVARIATE CONDITIONAL MIT FUNCTIONS

A complex system usually consists of several components which are working under the same environment and hence their lifetimes are, generally, dependent. In the literature several attempts have been made to extend the concepts of univariate stochastic orders and/ or life classes to the bivariate and multivariate cases. Some important references are *Barlow* and *Spizzichino* (1993), *Bassan* and *Spizzichino* (1999) and *Bassan et al.* (2002), among others.

This section is divided into two main subsections. The first one is concerned with the construction of the bivariate case of the MIT functions. In the second subsection, the multivariate case of the MIT functions is presented and studied.

3.1 Bivariate case

Let $\mathbf{T} = (T_1, T_2)$ be a random vector admitting an absolute continuous distribution function $\psi(t_1, t_2)$ in the support of $R_2^+ = \{(t_1, t_2) \mid t_1, t_2 > 0\}$. Let $\psi(t_1, t_2) = P\{T_1 < t_1, T_2 < t_2\}$ be the distribution function of $\mathbf{T}$. The conditional MIT function of $\mathbf{T}$ is defined by the vector $\mu(t_1, t_2) = (\mu_1(t_1, t_2), \mu_2(t_1, t_2))$,where,

$$\mu_i(t_1, t_2) = E\left[t_i - T_i | T_i < t_i, T_j = t_j\right], t_1, t_2 > 0, i, j = 1, 2, i \neq j. \quad (3.1)$$

For $i = 1$, the equation (3.1) reduce to

$$\mu_1(t_1, t_2) = \frac{1}{\psi_1(t_1, t_2)} \int_0^{t_1} \psi_1(x, t_2) dx, \quad (3.2)$$

where, $\psi_1(t_1, t_2)$ is the distribution function of T_1 given $T_2 = t_2$.
Similarly for $i = 2$,

$$\mu_2(t_1, t_2) = \frac{1}{\psi_2(t_1, t_2)} \int_0^{t_1} \psi_1(t_1, x) dx, \quad (3.3)$$

where, $\psi_2(t_1, t_2)$ is the distribution function of T_2 given $T_1 = t_1$.

In reliability analysis $\mu_1(t_1, t_2)$ measure the expected waiting time of the first component conditioned on the fact that the failure of the first had occurred in $[0, x]$ and the failure time of the second is t_2. From (3.2) and (3.3), it is easy to observe that the bivariate MIT determine the conditional distribution of T_i given $T_j = x_j$, $i, j = 1, 2, i \neq j$. Further, it is easier for an engineer to postulate the behavior of these functions which will be useful in studying the reliability properties of the system.

Consider now another random vector $\mathbf{S} = (S_1, S_2)$ admitting an absolute continuous distribution function $\varphi(t_1, t_2)$ in the support of $R_2^+ = \{(t_1, t_2) \mid t_1, t_2 > 0\}$. The bivariate MIT function of $\mathbf{S}$ is defined by the vector

$$\beta(t_1, t_2) = (\beta_1(t_1, t_2), \beta_2(t_1, t_2)), \quad (3.4)$$

where,

$$\beta_i(t_1, t_2) = E\left[t_i - S_i | S_i < t_i, S_j = t_j\right], t_1, t_2 > 0, i, j = 1, 2, i \neq j. \quad (3.5)$$

We say that a random vector $\mathbf{T}$ is less than $\mathbf{S}$ in the MIT ordering (denoted by $\mathbf{T} \leq_{MIT} \mathbf{S}$) if for every $i = 1, 2$ and $t > 0$ one has $\mu_i(t_1, t_2) \geq \beta_i(t_1, t_2)$, for all $t_1, t_2 > 0$.

Let us now introduce the definition of the IMIT life class in bivariate case.

Definition 3.1.

An exchangeable random vector $\mathbf{T} = (T_1, T_2)$ is said to have bivariate IMIT (BIMIT) distribution if for $t_1 < t_2$

$$E\left[t_1 - T_1 \mid T_1 < t_1, T_2 < t_2\right] \leq E\left[t_2 - T_2 \mid T_1 < t_1, T_2 < t_2\right]$$

or equivalent, $\mathbf{T}$ is BIMIT if, and only, if

$$\int_0^{t_1} F(x,t_2)dx \leq \int_0^{t_2} F(x,t_1)dx$$

for all $t_1 < t_2$. Note that, for a univariate case, if

$$[t - T \mid T < t] \leq_{MIT} T, \quad for\ any\ t \geq 0, \tag{3.6}$$

then T is of IMIT. Therefore, in the case of *i.i.d.* variables T_1, T_2, the condition that they are IMIT is equivalent to $[t - T_1 \mid T_1 < t] \leq_{MIT} T_2, \quad for\ any\ t \geq 0$.

3.2 Multivariate case

Let $\mathbf{T} = (T_1, T_2, ..., T_p)$ be a vector of non-negative random variables. The random variables $T_1, T_2, ..., T_p$ can be thought of as lifetimes of p components. Suppose that $\mathbf{T}$ has an absolutely continuous distribution function $\mathbf{F}$. As in bivariate case, in order to avoid technical complications, we assume that the support of $\mathbf{F}$ is $[0, \infty)^p$.

Let $\mathbf{I} = \{i_1, i_2, ..., i_k\} \subset \{1, 2, ..., p\} \equiv N$, $\mathbf{t} = (t_1, t_2, ..., t_\vartheta)$ and $j \in \bar{\mathbf{I}}$. Then $\mathbf{t_I}$ denotes $(t_{i_1}, t_{i_2}, ..., t_{i_k})$. The vector $(1, 1, ..., 1)$ is denoted by $\mathbf{e}$; its dimension can always be determined from the context. If $\mathbf{I}$ is a subset of N then $\bar{\mathbf{I}}$ denotes its complement with respect to N. For $\mathbf{I}, \mathbf{t_I}, \mathbf{t}$ and j, as above, we define the multivariate conditional MIT of T_j at time t by

$$\mu_j(t \mid \mathbf{T_I} = \mathbf{t_I}) = E[t - T_j \mid \mathbf{T_I} = \mathbf{t_I}, \mathbf{T_I} < \mathbf{t_e}]. \tag{3.7}$$

First, we introduce a notation that will be useful here. Let $\omega_{(t)}$ denote a realization of the reversed failure times, up to time t, associated with the random vector $\mathbf{T}$. That is, $\omega_{(t)}$ is an event of the form $\omega_{(t)} = \{\mathbf{T_I} = \mathbf{t_I}, \mathbf{T_I} < \mathbf{t_e}\}$, where $I \subset N$ and $\mathbf{t_e} \geq 0_\mathbf{e}$. Such an event is called a reversed history; it lists reversed failures that have occurred up to time t (see *Shaked* and *Shantikumar* (1994)). With this notation, we abbreviate $\mu_j(t \mid \mathbf{T_I} = \mathbf{t_I})$ defined in (3.5) by $\mu_j(\omega_{(t)})$.

Consider now another random vector $\mathbf{S}$ and let us denote to the multivariate conditional MIT function of S_j, given a history $\omega_{(t)}$ associated with $\mathbf{S}$, by $\beta_j(\omega_{(t)}) = E[t - S_j \mid \omega_{(t)}]$.

We say that $\mathbf{T}$ is less than $\mathbf{S}$ in the MIT ordering (denoted by $\mathbf{T} \leq_{MIT} \mathbf{S}$) if, and only, if for every $t > 0$

$$\mu_j(\omega'_{(t)}) \geq \beta_j(\omega_{(t)}), \quad whenever \quad \omega_{(t)} \leq \omega'_{(t)}, \tag{3.8}$$

where j denotes a component which has not failed by time t in reversed history $\omega'_{(t)}$.

Condition (3.8) is the multivariate analog of (2.7). It simply state that at any time t the MIT of T_i is smaller than the MIT of S_j. Next, we denote

$$\begin{aligned} RHR_j(\mathbf{t}) &= \lambda_j(T \mid \mathbf{T_I} = \mathbf{t_I}) \\ &= \lim \frac{1}{\Delta t} P\left\{t - \Delta t < t_j \leq t \mid \omega_{(t)}\right\}, \end{aligned}$$

and we call this function the multivariate conditional reversed hazard rate of T_j at time t. It should be pointed out that the following multivariate analog of (2.7) holds: $\mathbf{T} \leq_{RH} \mathbf{S} \Rightarrow \mathbf{T} \leq_{MIT} \mathbf{S}$.

By the multivariate analog of the MIT ordering one can generalize (3.6) to the multivariate case, thus introducing notions of multivariate IMIT distributions. The proof of the following preposition is similar to the proof of *Theorem* 4.1(b) of *Shaked* and *Shanthikumar* (1990), and is omitted.

Proposition 3.1.

A non-negative random vector $\mathbf{T}$ is of IMIT if

$$\left[\mathbf{t_e} - \mathbf{T} \mid \omega_{(t)}\right] \leq_{MIT} \mathbf{T} \tag{3.9}$$

for any history $h_{(t)}, t \geq 0$.

Let us now study multivariate extension of (3.6). This define class of M-IMIT (multivariate IMIT) distributions. Denote by $\vartheta_{(t)}$ to the time shift operator. That is, if $\mathbf{T} = (T_1, T_2, ..., T_p)$ is a non-negative vector of random lifetimes then $\vartheta_{(t)}\mathbf{T} = (\mathbf{t_e} - \mathbf{T})^+$, $t \geq 0$. Let $\omega'_{(t)}$ denote the information contained in $\omega'_{(t')}$ over the interval $[0, t]$. We say that $\omega_{(t)}$ is less severe than $\omega'_{(t')}$ (denoted by $\omega_{(t)} \leq \omega'_{(t')}$) if $\omega_{(t)} \leq \omega'_{(t)}$. Therefore a possible multivariate extension of (3.6) is the following.

Definition 3.2.

For $t \geq 0$ and history $\omega_{(t)}$ if

$$\left[\vartheta_{(t)}\mathbf{T} \mid \omega_{(t)}\right] \leq_{MIT} \mathbf{T}. \tag{3.10}$$

then any random vector $\mathbf{T}$ (or its distribution) is of M-IMIT.

Other situation, in which M-IMIT random vectors arias, can be identified from the following proposition which points out an important closure property.

Proposition 3.2.

If $T_1, T_2, ..., T_p$ are independent IMIT random variables then $\mathbf{T} = (T_1, T_2, ..., T_p)$ is M-IMIT.

This proposition can be easily proven using (3.6) and (3.10).

To introduce the definition of the DRHR and the IMIT classes in the multivariate setting, let use denote by $RHR_i\{t\}$ to the multivariate reversed hazard for $\mathbf{T}_i$ and

$\mu_i\{\mathbf{t}\}$ to the multivariate MIT for $\mathbf{T}_i$, where $\mu_i(t_1, t_2, ..., t_p) = E[t_i - \mathbf{T}_i \mid \chi_{1:i:p}\langle \mathbf{T}_i < t_i\rangle]$,where $\chi_{1:i:p}\langle z\rangle \equiv (x_1, x_2, ..., x_{i-1}, x_{i+1}, ..., x_p)$; z is any value (including null).

Definition 3.3.

A cumulative distribution function $\mathbf{F}$ is said to be:

(*i*) multivariate decreasing reversed failure rate ($\underline{M}DRHR$) if, and only, if $RHR_i\{t\}$ decreasing in t_i for all $\mathbf{t}$ and each i.

(*ii*) multivariate increasing mean inactivity time if, and only ($\underline{M}IMIT$), if $\mu_i\{\mathbf{t}\}$ increasing in t_i for all $\mathbf{t}$ and each i.

Theorem 3.1.

The following chain of implication is true: $\underline{\text{M}}$DRHR$\subset$ $\underline{M}IMIT$.

Proof.

Let $\Omega_{1:i:p} \equiv (X_1 < x_1, ..., X_{i-1} < x_{i-1}, \zeta, X_{i+1} < x_{i+1}, ..., X_p < x_p)$; ζ is any relation (including null). Note that the distribution function F is $\underline{M}DRHR$ if, and only, if

$$RHR_i\{t_{1:i:p}\langle t_i - u\rangle\} \geq RHR_i\{t_{1:i:p}\langle t_i\rangle\}, \tag{3.11}$$

for all $u \geq 0$, and for all x and each i.

On the other hand,

$$\begin{aligned} MIT_i\{t\} &= E\{t_i - T_i \mid \Omega_{1:i:p}\langle T_i < t_i\rangle\} \\ &= \frac{\int_0^\infty cdf\{t_{1:i:p}\langle t_i - u\rangle\}du}{cdf\{t_{1:i:p}\langle t_i\rangle\}}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial MIT_i\{t\}}{\partial\{t_i\}} &= \int_0^\infty \left[\frac{cdf\{t_{1:i:p}\langle t_i - u\rangle\}}{cdf\{t_{1:i:p}\langle t_i\rangle\}}\right] \cdot \left[RHR_i\{t_{1:i:p}\langle t_i\rangle\} - RHR_i\{t_{1:i:p}\langle t_i + u\rangle\}\right] du \\ &\geq 0, \end{aligned}$$

where is the inequality follows from (3.11). Hence $cdf\{x\}$ is $\underline{M}DRHR$. ∎

4. SOME APPLICATIONS

In reliability theory, a multivariate classification system is meaningful and usable for survival only if it has the 4 essential properties:

(1) A p-variate classification system should reduce to the standard univariate classification system when $p = 1$.

(2) If, according to a multivariate classification system, a $\mathbf{T}$ belongs to a particular class, say A, then every subset $(T_{i_1}, T_{i_2}, ..., T_{i_r})$, $1 \leq r \leq p$, of $\mathbf{T}$ must also belong to the same class A.

(3) If, according to a multivariate classification system, 2 $s-$independent $\mathbf{T}_1$ and $\mathbf{T}_2$, belong to a particular class, say class A, then the collection of all $\mathbf{T} = (\mathbf{T_1}, \mathbf{T_2})$ the component lives must also belong to class A.

(4) If, according to a multivariate classification system, a $\mathbf{T}$ belongs to a particular class, say A, then a positive scaling $(c_1 \cdot T_1, c_2 \cdot T_2, ..., c_p \cdot T_p)$ of $\mathbf{T}$ for all $c_i > 0$, must also belong to class A.

In the above properties, we point out that property (2) allows for removal of components from a s-independent system without affecting the multivariate class property while property (3) allows inclusion of additional components to a $s-$independent system retaining the same classification system. In addition, property (4) ensures that a simultaneous change in scales of measurements can be undertaken without disturbing the multivariate class property (For more details, please see, *Roy* (2001)).

While defining a system, *Hawryszkiewycz* (1994) has pointed out that a system should preferably be mad up of subsystems, to handle system complexity and improve the understanding of a system. A good system is made of highly s-independent subsystems with emphasis on their integration. This means expanding a system into s-independent subsystems during design (see, *Lloyd* and *Lipow* (1990)). A special use of property (2) is for developing reliability bounds. By appealing to marginal analysis, univariate reliability bounds and property (2), bounds are easily developed for the multivariate s-independent system in an iterative way. The objective of this section is to show that the $\underline{M}DRHR$ and $\underline{M}IMIT$ possesses all of the above essential properties so that reliability bounds can be easily worked out.

Before state and prove *Theorem* 4.1 below let us denote by

$$\mathbf{T}_1 = (T_1, T_2, ..., T_p), \quad \mathbf{T}_2 = (T_{p+1}, T_{p+2}, ..., T_{p+q}). \tag{4.1}$$

Theorem 4.1.

If two s-independent vector variables defined in (4.1) belong to a $\underline{M}DRHR$ ($\underline{M}IMIT$) class, then $\mathbf{T} = (\mathbf{T}_1, \mathbf{T}_2)$ also belong to the same $\underline{M}DRHR$ ($\underline{M}IMIT$) class.

Proof.

First, note that the distribution function of $\mathbf{T}$ is F(t)= $\prod k = 1^2 \mathbf{F}_k(\mathbf{t}_k)$,where $\mathbf{F}_k(\mathbf{t}_k)$ is the distribution function of $\mathbf{T}_k$, $k = 1, 2$. Thus, RH(t)=RH(t_1) +

$RH(\mathbf{t}_2)$and

$$\begin{aligned} RH_i(\mathbf{t}) &= rh_{1,i}(\mathbf{t}_1), \quad for i = 1,2,...,p, \\ RH_i(\mathbf{t}) &= rh_{2,i}(\mathbf{t}_2), \quad for i = p+1, p+2, ..., p+q, \end{aligned}$$

where, $RH(\mathbf{t})$ and $RH_i(\mathbf{t})$ are the multivariate reversed hazard at $\mathbf{t}$ and the multivariate reversed hazard for component i at $\mathbf{t}$, respectively. Then, following *Definition* 3.3 (i), the $\underline{M}DRHR$ property ensured for $\mathbf{T}$ from the $\underline{M}DRHR$ property of $\mathbf{T}_1$ and $\mathbf{T}_2$.

For the $\underline{M}IMIT$ class, observe that

$$\mu_i(\mathbf{t}) = \mu_{1,i}(\mathbf{t}_1), \quad for i = 1,2,...,p$$

$$\mu_i(\mathbf{t}) = \mu_{2,i}(\mathbf{t}_2), \quad for i = p+1, ..., p+q.$$

Hence, from the $\underline{M}IMIT$ property of $\mathbf{T}_1$ and $\mathbf{T}_2$, and from *Definition* 3.3 (ii), this is the $\underline{M}IMIT$ property of $\mathbf{T}$. ∎

Theorem 4.2.

If a vector life variable $\mathbf{T} = (T_1, T_2, ..., T_p)$ belong to a $\underline{M}DRHR$ ($\underline{M}IMIT$) class, then $\mathbf{T}_c = (c \cdot T_1, c \cdot T_2, ..., c \cdot T_p)$ also belong to the same $\underline{M}DRHR$ ($\underline{M}IMIT$) class.

Proof.

Let $\mathbf{F}_c(\mathbf{t})$, $RHR_{c,i}(\mathbf{t})$ and $\mu_{c,i}(t)$ are the distribution function, the multivariate reversed hazard of component i and the multivariate MIT of component i of $\mathbf{T}_c$, respectively. Observe that $\mathrm{F}_c(\mathbf{t}) = \mathbf{F}\left(\frac{t_1}{c_1}, \frac{t_2}{c_2}, ..., \frac{t_p}{c_p}\right)$;hence for each $i = 1,2,...,p$

$$RHR_{c,i}(\mathbf{t}) = \frac{1}{c_i} \cdot RHR_i(\frac{t_1}{c_1}, \frac{t_2}{c_2}, ..., \frac{t_p}{c_p}), \tag{4.2}$$

and

$$\mu_{c,i}(t) = \frac{1}{c_i} \cdot \mu_i(\frac{t_1}{c_1}, \frac{t_2}{c_2}, ..., \frac{t_p}{c_p}). \tag{4.3}$$

Then if $\mathbf{T}$ is $\underline{M}DRHR$, the $RHR_{c,i}(\mathbf{t})$ given in (4.2) ensures the $\underline{M}DRHR$ class property of $\mathbf{T}_c$ for all $c_i > 0$. The remaining part of the theorem can be easily deduced by using (4.3). ∎

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