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Reliability Equivalences of a Series System Consists of n Independent and Non-identical Components

A. M. Sarhan* †
Department of Mathematics
Mansoura University, Mansoura 35516, Egypt

A. Mustafa
Department of Mathematics
Mansoura University, Mansoura 35516, Egypt

Abstract. This paper introduces different vectors of the reliability equivalence factors of a series system consists of n independent and non-identical components. The failure rates of the system components are assumed to be constant. The reliability function and mean time to failure are used as performances to derive the reliability equivalences of the system. The results presented here generalize those available in the literatures. Numerical study is given to explain how one can utilize the theoretical results obtained.

Key Words : *Coherent systems, exponential distribution, redundancy method, reduction method, fractiles*

1. INTRODUCTION

The concept of reliability equivalence factors has been introduced by Rade (1993a). Rade (1993a, 1993b) has applied such concept to simple series and parallel systems. He used three different methods to improve the reliability of the proposed system. Sarhan (2000, 2002, 2004, 2005) and Sarhan et al. (2004) applied this concept on more general systems.

Generally, there are two basic methods to improve a given system, see Sarhan (2000):

*Corresponding Author.
E-mail address: asarhan0@yahoo.com
†Current address: Department of Statistics and O.R., Faculty of Science, King Saud University, P.O. Box 2455 Riyadh 11451, Saudi Arabia

1. Reduction method, *RM*.
2. Redundancy method: this method includes the following three possible methods:
   (*2.1*) Hot duplication method, *HDM*.
   (*2.2*) Cold duplication method, *CDM*.
   (*2.3*) Imperfect switch duplication method, *ISDM*.

Using redundancy method may not be an appropriate method for a system in which the minimum size and weight are overriding considerations: for example, in spacecraft, satellites or other space applications, in well-logging equipment, and in pacemakers and similar biomedical applications, see Lewis (1996). In this applications, space or weight limitations may dictate an increase in component reliability rather than redundancy. Therefore, more emphasis must be placed on robust design, manufacturing quality control, and on controlling the operating environment. Thus, the concept of reliability equivalent takes place. In this concept, the improved design of the system, which obtained by following the reduction method, should equivalent to that improved design of the system which obtained by using one of the redundancy methods.

The system reliability (survival) function and system mean time to failure will be used as references of the system performance. For this reason, we will derive the reliability functions and mean time to failures of the original system and the improved systems using each of the methods mentioned above.

In the previous studies, it is assumed that the system can be improved by reducing the failure rates some of the system components by the same factor ρ, $0 < \rho < 1$. In this paper, we consider a more general assumption. In this assumption, we assume that the failure rates of the components, belong to a set of system components to be improved, will be reduced by a vector of factors $\vec{\rho} = (\rho_1, \cdots, \rho_m)$, where m is the number of components belong to this set. That is, the failure rate of a component $i \in B$, B is a subset of system components to be improved, will be reduced by a respective factor ρ_i. We will deduce two vectors of reliability equivalence factors of n components series system based on the above assumption. The current study generalizes the results given in Sarhan (2000), since Sarhan's results can be obtained from the results presented her by setting $\rho_1 = \rho_2 = \cdots = \rho_m = \rho$.

In what follows, we introduce the required definitions of the two different vectors of reliability equivalence factors.

Definition 1. The vector of survival reliability equivalence factors, say VSREF, $\vec{\rho}^{D}_{A,B}(\alpha) = \left(\rho^{D}_{A,i_1}, \ldots, \rho^{D}_{A,i_m}\right)$, is defined as the solution $\vec{\rho}_B = \vec{\rho}^{D}_{A,B}(\alpha)$ of the following equations

$$R_{B,\vec{\rho}_B}(t) = \alpha \text{ and } R_A^D(t) = \alpha \tag{1.1}$$

where $R_{B,\vec{\rho}_B}(t)$ denotes the reliability function of the system which is improved by reducing the failure rate of each component $i \in B = \{i_1, \ldots, i_m\} \subseteq \{1, 2, \ldots, n\}$ by the factor $\rho_i = \rho_{A,i}^D$, $0 < \rho_i < 1$, $i \in B$. While $R_A^D(t)$ is the reliability function of the system improved by improving all components belong to the set $A \subset \{1, 2, \ldots, n\}$ according to hot (D=H), cold (D=C), and imperfect switch (D=I) duplication methods.

Definition 2. The vector of mean reliability equivalence factors, say VMREF, $\vec{\rho}_B = \vec{\xi}_{A,B}^D = \left(\xi_{A,i_1}^D, \ldots, \xi_{A,i_m}^D\right)$, is defined as the solution $\vec{\rho}_B$ of the following equation

$$\text{MTTF}_{B,\vec{\rho}_B} = \text{MTTF}_A^D \tag{1.2}$$

where $\text{MTTF}_{B,\vec{\rho}_B}$ and MTTF_A^D are given by

$$\text{MTTF}_{B,\vec{\rho}_B} = \int_0^\infty R_{B,\vec{\rho}_B}(t)dt \text{ and } \text{MTTF}_A^D = \int_0^\infty R_A^D(t)dt, D = H, C, I$$

This paper is organized as follows. Section 2 presents the survival function and mean time to failure to the underlying system. Section 3 gives the survival functions and mean time to failures of the improved designs of the system. The α-fractiles of the original and improved systems are calculated in Section 4. The vectors of reliability equivalence factors are obtained in Section 5. Finally, numerical results and conclusions are given in Section 6.

2. THE ORIGINAL SYSTEM

We consider a system that consists of n independent and non-identical components connected in series. In what follows, we give the reliability function and mean time to failure of this system. Let T_i be the lifetime of the component i, $i = 1, 2, \ldots, n$. It is assumed that T_i is exponentially distributed random variable with parameter λ_i.

The reliability function of the system $R(t)$ is given by

$$R(t) = \prod_{i=1}^{n} \exp\{-\lambda_i t\} = \exp\{-\Lambda t\}, \tag{2.1}$$

where $\Lambda = \sum_{i=1}^n \lambda_i$.

Let MTTF be the system mean time to failure, which is given by

$$\text{MTTF} = \int_0^\infty R(t)\, dt = \frac{1}{\Lambda}. \tag{2.2}$$

3. THE IMPROVED SYSTEMS

The system reliability can be improved according to one of the following four different methods:

(1) **RM**: Reducing the failure rates of some of the system components that belong to the set $B = \{i_1, \cdots, i_m\} \subseteq \{1, 2, \cdots, n\}$, $m \leq n$, each by its own, that is the failure rate of component $i \in B$, will be reduced by a factor ρ_i, $0 < \rho_i < 1$.

(2) **HDM**: Duplicating each component belongs to the set $A = \{i_1, \cdots, i_m\} \subseteq \{1, 2, \cdots, n\}$, $m \leq n$. Namely, each component belongs to the set A is duplicated by a hot redundant standby component.

(3) **CDM**: Assuming cold duplication of each component belongs to the set $A = \{i_1, \cdots, i_m\} \subseteq \{1, 2, \ldots, n\}$, $m \leq n$. That is, each component belongs to the set A is duplicated by a cold redundant standby components.

(4) **ISDM**: Assuming cold redundant standby component connected by random switch to the each component belongs to the set $A = \{i_1, \cdots, i_m\} \subseteq \{1, 2, \ldots, n\}$, $m \leq n$. This means that, each component in the set A is connected by a cold redundant standby component via an imperfect switch. Note that, different components have different switches.

To derive the vectors of reliability equivalence factors of the underlying system, we make equivalence between the improved systems which obtained by using the RM and the redundancy method (HDM, CDM and ISDM).

This process can be made by following the following steps:

(i) Derive the reliability functions and mean time to failures of the original and improved systems.

(ii) Equating the reliability function of the improved system that obtained by using the RM with the reliability function of the improved system that obtained by using one of the rest improved methods at a specified level.

(iii) Equating the mean time to failure of the improved system that obtained by using the RM with the mean time to failure of the improved system that obtained by using one of the rest improved methods.

It seems from the above description that the reliability function and mean time to failure will be used as performances measures of the improved systems.

3.1 Reduction method

We denoted by $R_{B,\vec{\rho}_B}(t)$ to the reliability function of the system improved when reducing the failure rates of set of components $B = \{i_1, i_2, \cdots, i_m\} \subseteq \{1, 2, \cdots, n\}$, $m \leq n$, by the vector $\vec{\rho} = (\rho_{i_1}, \cdots, \rho_{i_m})$. That is, the failure rate of component $i \in B$ is reduced by the factor ρ_i, the i-th element of vector $\vec{\rho}$. One can obtain the function $R_{B,\vec{\rho}_B}(t)$ as follows

$$\begin{aligned} R_{B,\vec{\rho}_B}(t) &= \Big[\prod_{i\in B} \exp\{-\rho_i\lambda_i t\}\Big]\Big[\prod_{i\in \overline{B}} \exp\{-\lambda_i t\}\Big] \\ &= \exp\left\{-t\sum_{i\in B}\rho_i\lambda_i\right\}\exp\left\{-t\sum_{i\in\overline{B}}\lambda_i\right\} \\ &= \exp\left\{-\left[\Lambda_{\vec{\rho}_B}+\Lambda_{\bar{B}}\right]t\right\} \end{aligned} \tag{3.1}$$

where

$$\Lambda_{\bar{B}} = \sum_{i\in\bar{B}}\lambda_i, \quad \Lambda_{\vec{\rho}_B} = \sum_{i\in B}\rho_i\,\lambda_i, \quad \bar{B} = \{1, 2, \cdots n\}\backslash B.$$

From equation (5), the mean time to failure of the improved system say $\text{MTTF}_{B,\vec{\rho}_B}$, becomes

$$\text{MTTF}_{B,\vec{\rho}_B} = \int_0^\infty R_{B,\vec{\rho}_B}(t)\,dt = \frac{1}{\Lambda_{\bar{B}}+\Lambda_{\vec{\rho}_B}}. \tag{3.2}$$

That is, reducing the failure rates of the set B components by the vector $\vec{\rho}_B$ increases the system mean time to failure by the amount $\frac{\Lambda_B-\Lambda_{\vec{\rho}_B}}{\Lambda[\Lambda_{\bar{B}}+\Lambda_{\vec{\rho}_B}]}$.

3.2 Hot duplication method

Let $R_A^H(t)$ be the reliability function of the improved system assuming hot duplications of a set $A = \{i_1, \cdots, i_m\}$ components, $A \subseteq N = \{1, 2, \ldots, n\}$. The function $R_A^H(t)$ takes the following form, see Sarhan (2000),

$$R_A^H(t) = 2^m \exp\{-\Lambda t\} \prod_{i\in A}\left[1 - \frac{1}{2}\exp\{-\lambda_i t\}\right]. \tag{3.3}$$

One can write (3.3) as in the following form

$$R_A^H(t) = 2^m \exp\{-\Lambda t\} \sum_{\ell=0}^{m}\left[(-1)^\ell 2^{-\ell}\sum_{i=1}^{\binom{m}{\ell}}\exp\{-\gamma_{i(\ell)}^{(m)} t\}\right], \tag{3.4}$$

where

$$\gamma_{i(l)}^{(m)} = \lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_l},\ i_1 < i_2 < \ldots < i_l \in H, \gamma_{i(0)}^{(m)} = 0,$$

$$\gamma_{i(l)}^{(m)} \neq \gamma_{j(l)}^{(m)} \text{ for } i \neq j, \text{ and } 1 \leq i, j \leq \tbinom{m}{l}, m = |A|.$$

Let MTTF_A^H denote the mean time to failure of the system improved in this case. Using equation (3.4), one can obtain MTTF_A^H as

$$\mathrm{MTTF}_A^H = 2^m \sum_{\ell=0}^{m} \left[(-1)^\ell 2^{-\ell} \sum_{i=1}^{\binom{m}{\ell}} \left\{ \Lambda + \gamma_{i(\ell)}^{(m)} \right\}^{-1} \right]. \tag{3.5}$$

Therefore, hot duplications of the set A components increases the system mean time to failure by the following amount

$$\frac{2^m - 1}{\Lambda} + 2^m \sum_{\ell=1}^{m} \left[(-1)^\ell 2^{-\ell} \sum_{i=1}^{\binom{m}{\ell}} \left\{ \Lambda + \gamma_{i(\ell)}^{(m)} \right\}^{-1} \right].$$

3.3 Cold duplication method

Let $R_A^C(t)$ be the reliability function of the system improved by assuming cold duplications of the components belong to the set $A = \{i_1, \cdots, i_m\} \subseteq \{1, 2, \cdots, n\}$. The component $i \in A$ is said to be cold duplicated when a similar component is connected in standby with it, where the standby component replaces immediately the original component once it fails. Sarhan (2000) derived the function $R_A^C(t)$ as in the following form

$$R_A^C(t) = \left[\prod_{i \in A} (1 + \lambda_i t) \right] \exp\{-\Lambda t\}. \tag{3.6}$$

This function can rewritten as

$$R_A^C(t) = \left[\sum_{\ell=0}^{m} a_\ell t^\ell \right] \exp\{-\Lambda t\}, \tag{3.7}$$

where

$$a_\ell = \sum_{i_1 < i_2 < \ldots < i_\ell \in A} \lambda_{i_1} \lambda_{i_2} \ldots \lambda_{i_\ell}, \; a_0 = 1, \; m = |A|. \tag{3.8}$$

From equation (3.7), one can recognize that the mean time to failure of the system improved by CDM is

$$\mathrm{MTTF}_A^C = \sum_{\ell=0}^{m} \frac{a_\ell \, \Gamma(\ell + 1)}{\Lambda^{\ell+1}}.$$

Namely, cold duplications of the set A components increases the system mean time to failure by the amount

$$\sum_{\ell=1}^{m} \frac{a_\ell \, \Gamma(\ell + 1)}{\Lambda^{\ell+1}}.$$

3.4 Imperfect switch duplication method

Now, let us use the ISDM to improve the underlying system. It is assumed in this method that each component in a set $B = \{i_1, \cdots, i_m\} \subseteq A$, say i, is improved by connecting it in parallel with an identical component via an imperfect switch with a constant failure rate, say β_i. Let $R_A^I(t)$ and MTTF_A^I denote, respectively, to the reliability function and the corresponding mean time to failure of the system improved, by improving the set A components, according to ISDM. Sarhan (2000), obtained $R_A^I(t)$ as in the following form

$$R_A^I(t) = \exp\{-\Lambda t\} \prod_{i\in A} \left[\frac{1}{\phi_i}\Big(1+\phi_i - \exp\{-\beta_i t\}\Big)\right], \phi_i = \frac{\beta_i}{\lambda_i}. \tag{3.9}$$

This function can be rewritten as

$$R_A^I(t) = \frac{1}{\prod_{i\in A}\phi_i} \sum_{\ell=0}^{m} \left[(-1)^\ell \sum_{i=1}^{\binom{m}{i}} \Psi_{i(m-\ell)}^{(m)} \exp\left\{-\Big(\Lambda + \beta_{(m)} - \beta_{i(m-\ell)}^{(m)}\Big)t\right\}\right], \tag{3.10}$$

where

$$\psi_i = \phi_i + 1,\ \Psi_{i(\ell)}^{(m)} = \psi_{i_1}\psi_{i_2}\ldots\psi_{i_\ell},\ i_1 < i_2 < \ldots < i_\ell \in A,$$

$$\Psi_{i(\ell)}^{(m)} \neq \Psi_{j(\ell)}^{(m)} \text{ for } i \neq j,\ \Psi_{i(0)}^{(m)} = 1,\ \beta_{(m)} = \sum_{i\in A} \beta_i,$$

$$\beta_{i(\ell)}^{(m)} = \beta_{i_1} + \beta_{i_2} + \ldots + \beta_{i_\ell},\ \beta_{i(\ell)}^{(m)} \neq \beta_{j(\ell)}^{(m)} \text{ for } i \neq j,\ \beta_{i(0)}^{(m)} = 0,\ 1 \le i, j \le \tbinom{m}{i}.$$

From equation (3.10), one can obtain MTTF_A^I as

$$\mathrm{MTTF}_A^I = \frac{1}{\prod_{i\in A}\phi_i} \sum_{\ell=0}^{m} \left[(-1)^\ell \sum_{i=1}^{\binom{m}{\ell}} \frac{\Psi_{i(m-\ell)}^{(m)}}{\Lambda + \beta_{(m)} - \beta_{i(m-\ell)}^{(m)}}\right], \tag{3.11}$$

which can be rewritten as

$$\begin{aligned}\mathrm{MTTF}_A^I &= \mathrm{MTTF} + \frac{1}{\Lambda}\left[\prod_{i\in A}\psi_i - \prod_{i\in A}\phi_i\right] + \\ &\quad \frac{1}{\prod_{i\in A}\phi_i} \sum_{\ell=0}^{m} \left[(-1)^\ell \sum_{i=1}^{\binom{m}{\ell}} \frac{\Psi_{i(m-\ell)}^{(m)}}{\Lambda + \beta_{(m)} - \beta_{i(m-\ell)}^{(m)}}\right],\end{aligned} \tag{3.12}$$

This means that improving the set A of components according to the ISDM, increases the system mean time to failure by the following amount

$$\frac{1}{\Lambda}\left[\prod_{i\in A}\psi_i - \prod_{i\in A}\phi_i\right] + \frac{1}{\prod_{i\in A}\phi_i} \sum_{\ell=0}^{m} \left[(-1)^\ell \sum_{i=1}^{\binom{m}{\ell}} \frac{\Psi_{i(m-\ell)}^{(m)}}{\Lambda + \beta_{(m)} - \beta_{i(m-\ell)}^{(m)}}\right].$$

4. THE α-FRACTILES

This section presents the α-fractiles of the original and improved systems. Let $L(\alpha)$ be the α-fractiles of the original system and $L_A^D(\alpha)$, $D = H, C$, and I, $A \subseteq \{1, 2, \ldots, n\}$, the α-fractiles of the improved systems.

The α-fractiles $L(\alpha)$ and $L_A^D(\alpha)$, are defined as the solution of the following equations, respectively,

$$R\left(\frac{L(\alpha)}{\Lambda}\right) = \alpha, \quad R_A^D\left(\frac{L(\alpha)}{\Lambda}\right) = \alpha. \tag{4.1}$$

It follows from equation (2.1) and the first equation of (4.1) that

$$L(\alpha) = -\ln(\alpha). \tag{4.2}$$

From the second equation of (4.1), when $D = H$, and equation (3.3), one can verify that $L = L_A^H(\alpha)$ satisfies the following equation

$$L + \ln(\alpha) - \left[0.693\,m + \sum_{i \in A} \ln\left(1 - \frac{1}{2}\exp\left\{-\frac{\lambda_i}{\Lambda}L\right\}\right)\right] = 0 \tag{4.3}$$

Similarly, from the second equation of (4.1), when $D = C$, and equation (3.6), $L = L_A^C(\alpha)$ can be obtained by solving the following equation

$$L + \ln(\alpha) - \sum_{i \in A} \ln\left(1 + \frac{\lambda_i}{\Lambda}L\right) = 0 \tag{4.4}$$

Finally, from equation (3.9) and the second equation of (4.1), when $D = I$, $L = L_A^I(\alpha)$ satisfies the following equation

$$L + \ln(\alpha) - \sum_{i \in A}\left[\ln\left(1 + \phi_i - \exp\left\{-\frac{\beta_i}{\Lambda}L\right\}\right) - \ln(\phi_i)\right] = 0. \tag{4.5}$$

Equations (4.3)-(4.5) have no closed form solutions and can be solved using numerical method technique.

5. RELIABILITY EVALUATION FACTORS

Now we are ready to derive the vectors of reliability equivalence factors of the system. In the following subsections, we will deduce VSREF and VMREF of the underlying system.

5.1 The VSREF

Here we will derive three possible VSREF, which will be referred as hot VSREF, cold VSREF and imperfect switch VSREF, which can be obtained respectively by following HDM, CDM and ISDM. We will denote to these vectors respectively by VHSREF, VCSREF and VISREF.

VHSREF: Using equation (1.1), when $D = H$, together with equation (3.1) and (3.3), one can find out the vector of hot survival reliability equivalence factors at a given α, $\vec{\rho}(\alpha) = \vec{\rho}^{H}_{A,B}(\alpha)$, by following the following two steps: firstly, solving the following equation with respect to $\Lambda_{\vec{\rho}}$

$$0.693\, m + \left[\frac{\Lambda_B - \Lambda_\rho}{\Lambda_{\vec{\rho}} + \Lambda_{\bar{B}}}\right] \ln(\alpha) + \sum_{i \in A} \ln\left[1 - \frac{1}{2}\,\alpha^{\frac{\lambda_i}{\Lambda_{\vec{\rho}}+\Lambda_{\bar{B}}}}\right] = 0\,, \tag{5.1}$$

Then using the relation $\Lambda_{\vec{\rho}} = \sum_{i\in B} \lambda_i\, \rho_i$, we can find out $\vec{\rho} = \vec{\rho}^{H}_{A,B}(\alpha) = (\rho_{i_1}(\alpha), \cdots, \rho_{i_m}(\alpha))$. As it seems, when $m \geq 2$, there exists indefinite number of vectors $\vec{\rho}$ satisfy $\Lambda_{\vec{\rho}} = \sum_{i\in B} \lambda_i\, \rho_i$. For example, if $B = \{1, 2\}$, then we have $\Lambda_{\vec{\rho}} = \lambda_1\, \rho_1 + \lambda_2\, \rho_2$, which implies that the solution $\vec{\rho} = (\rho_1, \rho_2)$ is any point lies on the line $\Lambda_{\vec{\rho}} = \rho_1\, \lambda_1 + \rho_2\, \lambda_2$.

Note that, if we assume that all elements of the vector of SREF $\vec{\rho}$ are equal, that is $\rho_{i_1} = \cdots = \rho_{i_m} = \rho_1$, then we will get the survival reliability equivalence factor $\rho^{H}_{A,B} = \frac{\Lambda_{\vec{\rho}}}{\Lambda_B}$, given in Sarhan (2000).

VCSREF: The vector of cold reliability equivalence factors can derived as follows. Using equation (1.1), when $D = C$, together with equation (3.1) and (3.6), we get the following equation

$$\left[\frac{\Lambda_B - \Lambda_{\vec{\rho}}}{\Lambda_{\vec{\rho}} + \Lambda_{\bar{B}}}\right] \ln(\alpha) + \sum_{i \in A} \ln\left[1 - \frac{\lambda_i\, \ln(\alpha)}{\Lambda_{\vec{\rho}} + \Lambda_{\bar{B}}}\right] = 0\,, \tag{5.2}$$

Solving the above equation at a specific α with respect to $\Lambda_{\vec{\rho}}$, we get the value of $\Lambda_{\vec{\rho}}$. Then using the relation $\Lambda_{\vec{\rho}} = \sum_{i\in B} \lambda_i\, \rho_i$, we get the VCSREF $\vec{\rho}^{C}_{A,B}(\alpha) = (\rho_{i_1}(\alpha), \cdots, \rho_{i_m}(\alpha))$. Similar to the case of VCSREF, there exists an infinite number of such vector. Also, under the assumption that all elements of the VCSREF $\vec{\rho}$ are equal, we will get the cold survival reliability equivalence factor given in Sarhan (2000) as a special case.

IVSREF: Finally, one can derive IVSREF $\vec{\rho} = \vec{\rho}^{I}_{A,B}(\alpha)$ as follows. Using equation (1.1), when $D = I$, with the help of (3.1) and (3.9), one gets

$$\left[\frac{\Lambda_B - \Lambda_{\vec{\rho}}}{\Lambda_{\vec{\rho}} + \Lambda_{\bar{B}}}\right] \ln(\alpha) + \sum_{i \in A} \left[\ln\left(1 + \phi_i - \alpha^{\frac{\beta_i}{\Lambda_{\vec{\rho}}+\Lambda_{\bar{B}}}}\right) - \ln(\phi_i)\right] = 0\,, \tag{5.3}$$

Similar to the previous two cases, we have to solve the above equation to get the value of $\Lambda_{\vec{\rho}}$, then we use the relation $\Lambda_{\vec{\rho}} = \sum_{i\in B} \lambda_i\, \rho_i$, to get the IVSREF $\vec{\rho} = \vec{\rho}^{I}_{A,B}(\alpha)$. Also, the cold survival reliability equivalence factor presented in Sarhan (2000), will be obtained under the assumption that the vector $\vec{\rho}$ has equal elements.

Equations (5.1)-(5.3) have no closed form solutions and numerical technique method is required.

5.2 The VMREF

Let us now explain how one can deduce the second type of vectors of reliability equivalence factors of the n-components series system. This type is VMREF say $\vec{\xi}^D_{A,B}$, $D = H, C$ and I.
The vector $\vec{\xi}^D_{A,B}$ can be obtained by equating the mean time of the improved system that obtained by imroving the set B components according to RM with the mean time to failure of the system improved by improving the set A components according to one of the rest method, see Definition 2. It means that, the vector $\vec{\xi}^D_{A,B}$ can be derived as follows. Firstly, solve the following equation with respect to Λ_ρ

$$\Lambda_{\vec{\rho}} = -\Lambda_{\bar{B}} + \frac{1}{\mathrm{MTTF}^D_A}, \tag{5.4}$$

Then using the relation $\Lambda_{\vec{\rho}} = \sum_{i \in B} \lambda_i \, \xi_i$, we can deduce the required vector $\vec{\xi}^D_{A,B} = (\xi_{i_1}, \cdots, \xi_{i_m})$, $D = H, C$ and I.

6. NUMERICAL RESULTS AND CONCLUSIONS

To explain how one can apply theoretical results obtained in the previous sections, we introduce a numerical example. In this example, we assume a series system with three independent components. The failure rates of the first, second and third components are $\lambda_1 = 0.07$, $\lambda_2 = 0.06$ and $\lambda_3 = 0.08$, respectively. This system can be improved by improving two or three components according to all methods previously mentioned. In the ISDM case, we assume that the failure rates of the switches used for components 1, 2, and 3 are $\beta_1 = 0.01$, $\beta_2 = 0.02$ and $\beta_3 = 0.03$, respectively. The mean time to failure of the original system is MTTF=4.7619. Table 6.1 shows the mean time to failure of the improved systems.

Table 6.1 The values of MTTF^D_A.

A	MTTF^H_A	MTTF^I_A	MTTF^C_A
{1,2}	7.4385	8.2902	8.6168
{2,3}	7.6008	8.3393	8.9731
{1,3}	7.7859	8.7956	9.3726
{1,2,3}	9.9728	12.9485	13.7139

From the above Table 6.1, one can conclude that, hot (cold, imperfect switch) duplication of the set $A = \{1, 2\}$ components, increases the MTTF from 4.7619 to

7.4385(8.6168, 8.29021). In same manner we can read the rest results shown in Table 6.1. Further, one can recognize for all studied cases that:

$$\text{MTTF} < \text{MTTF}_A^H < \text{MTTF}_A^I < \text{MTTF}_A^C, \forall A \subseteq \{1,2,3\}.$$

Table 6.2 The α-fractiles of the original system.

α	0.1	0.3	0.5	0.7	0.9
$L(\alpha)$	2.303	1.231	0.693	0.357	0.105

Table 6.3 The α-fractiles $L_A^D(\alpha)$ of the improved system.

	$A=\{1,2\}$			$A=\{1,3\}$		
α	$L^H(\alpha)$	$L^I(\alpha)$	$L^C(\alpha)$	$L^H(\alpha)$	$L^I(\alpha)$	$L^C(\alpha)$
0.1	2.591	3.695	3.878	2.632	3.829	4.107
0.3	1.367	2.173	2.266	1.393	2.311	2.467
0.5	0.791	1.373	1.422	0.807	1.503	1.594
0.7	0.408	0.777	0.798	0.417	0.886	0.929
0.9	0.121	0.257	0.260	0.124	0.316	0.325
	$A=\{2,3\}$			$A=\{1,2,3\}$		
α	$L^H(\alpha)$	$L^I(\alpha)$	$L^C(\alpha)$	$L^H(\alpha)$	$L^I(\alpha)$	$L^C(\alpha)$
0.1	2.611	3.674	3.987	3.943	4.768	5.412
0.3	1.379	2.193	2.361	2.559	3.128	3.572
0.5	0.799	1.409	1.503	1.810	2.226	2.556
0.7	0.413	0.817	0.859	1.218	1.506	1.743
0.9	0.122	0.282	0.289	0.613	0.764	0.904

Table 6.4(a) The values of $\Lambda_{\vec{\rho}} = \Lambda^D_{A,\vec{\rho}_B}(\alpha)$.

		$B=\{1,2\}$			$B=\{1,3\}$		
α	A	Λ_ρ^H	Λ_ρ^I	Λ_ρ^C	Λ_ρ^H	Λ_ρ^I	Λ_ρ^C
0.1	{1,2}	0.06703	0.05087	0.04469	0.08703	0.07087	0.06469
	{1,3}	0.06386	0.04629	0.03774	0.08386	0.06629	0.05774
	{2,3}	0.06560	0.05163	0.04129	0.08557	0.07163	0.06129
0.3	{1,2}	0.04963	0.03638	0.03159	0.06963	0.05638	0.05159
	{1,3}	0.04417	0.02939	0.02251	0.06418	0.04939	0.04251
	{2,3}	0.04702	0.03529	0.02708	0.06702	0.05529	0.04708
0.5	{1,2}	0.03658	0.02602	0.02235	0.05659	0.04602	0.04235
	{1,3}	0.02897	0.01684	0.01134	0.04897	0.03684	0.03134
	{2,3}	0.03285	0.02329	0.01683	0.05285	0.04329	0.03683
0.7	{1,2}	0.02388	0.01638	0.01388	0.04388	0.03638	0.03388
	{1,3}	0.01357	0.00454	0.00046	0.03360	0.02454	0.02054
	{2,3}	0.01873	0.01171	0.00714	0.03873	0.03171	0.02714
0.9	{1,2}	0.00931	0.00603	0.00499	0.02930	0.02603	0.02499
	{1,3}	NA	NA	NA	0.01446	0.00999	0.00811
	{2,3}	0.00179	NA	NA	0.02179	NA	0.01646

The α-fractiles $L(\alpha)$, $L_A^D(\alpha)$, and the values of $\Lambda^D_{\Lambda_B,A}(\alpha) = \sum_{i\in B}\lambda_i\rho_i$, $D = H$, I and C and A, B $\subset \{1,2,3\}$ are calculated using Mathematica Program system according to the previous theoretical formulae. In these calculations the level α is chosen to be 0.1, 0.3, ..., 0.9. The α-fractile of the original system is given in Table 6.1 at some values of α. Table 6.3 represents the α-fractiles of the improved systems which obtained by improving two or three components according to the methods previously mentioned.

Based on the results presented in Tables 6.2 and 6.3, one can recognize that $L(\alpha) < L_A^H(\alpha) < L_A^I(\alpha) < L_A^C(\alpha)$ for all studied cases, which confirms the results obtained for MTTF_A^D.

Table 6.4 shows the values of $\Lambda_{\vec{\rho}} = \Lambda^D_{A,\vec{\rho}_B}(\alpha)$ which we need to compute VSREF $\vec{\rho}^{\,D}_{A,B}(\alpha)$.

According to the results presented in Tables 6.2 to 6.4, one can conclude that:

1. Hot duplications of the components 1 and 2, $A = \{1,2\}$, increase $L(0.1)$ from $\frac{2.303}{\Lambda}$ to $\frac{2.5905}{\Lambda}$, see Table 6.2. The same effect on $L(0.1)$ can occur by reducing the failure rates of:

   (i) the components 1 and 2, $B = \{1,2\}$, by the vector $\vec{\rho} = (\rho_1, \rho_2)$, which satisfies the linear relation $0.07\rho_1 + 0.06\rho_2 = 0.06703$. That is, by reducing the failure rate of component 2 by the factor ρ_2, that taking any value in the interval $(0,1)$, and reducing the failure rate of component 1 by the factor $\rho_1 = \frac{1}{0.07}(0.06703 - 0.06\rho_2)$.

   (ii) the components 1 and 3, $B = \{1,3\}$, by the vector $\vec{\rho} = (\rho_1, \rho_3)$, which satisfies the linear relation $0.07\rho_1 + 0.08\rho_3 = 0.08703$. That is, by reducing the failure rate of component 3 by the factor ρ_3, that taking any value in the interval $(0,1)$, and reducing the failure rate of component 1 by the factor $\rho_1 = \frac{(}{1})0.07(0.08703 - 0.08\rho_3)$.

   (iii) the components 2 and 3, $B = \{2,3\}$, by the vector $\vec{\rho} = (\rho_2, \rho_3)$, which satisfies the linear relation $0.06\rho_2 + 0.08\rho_3 = 0.07703$. That is, by reducing the failure rate of component 3 by the factor ρ_3, that taking any value in the interval $(0,1)$, and reducing the failure rate of component 2 by the factor $\rho_2 = \frac{1}{0.06}(0.07703 - 0.08\rho_3)$.

   (iv) the components 1,2 and 3, $B = \{1,2,3\}$, by the vector $\vec{\rho} = (\rho_1, \rho_2, \rho_3)$, which satisfies the linear relation $0.07\rho_1 + 0.06\rho_2 + 0.08\rho_3 = 0.14703$. That is, by reducing the failure rates of component 2 by the factor $\rho_2 \in (0,1)$, component 3 by factor $\rho_3 \in (0,1)$, and component 1 by the factor ρ_1, that satisfies $\rho_1 = \frac{1}{0.07}(0.14703 - 0.06\rho_2 - 0.08\rho_3)$.

2. Hot duplications of the components 1, 2 and 3, $A = \{1,2,3\}$, increase $L(0.1)$ from $\frac{2.303}{\Lambda}$ to $\frac{3.9429}{\Lambda}$, see Table 6.3. The same effect on $L(0.1)$ can occur by reducing the failure rates of the components 1,2 and 3, $B = \{1,2,3\}$, by the vector $\vec{\rho} = (\rho_1, \rho_2, \rho_3)$ which satisfies $0.07\rho_1 + 0.06\rho_2 + 0.08\rho_3 = 0.1156$. That is, the failure rate of component 2 should be reduced by a factor $\rho_2 \in (0,1)$, the failure rate of component 3 should be reduced by factor a factor $\rho_3 \in (0,1)$ and the failure rate of component 1 should be reduced by the factor $\rho_1 = \frac{1}{0.07}(0.1156 - 0.06\rho_2 - 0.08\rho_3)$.

3. In the same manner one can read the rest results obtained by assuming hot, cold and imperfect switch duplication methods.

4. The notation NA in Table 6.5, means that the value of $\Lambda_{A,B}$ is not available and therefore there is possible equivalence between the system improved by RM and that system improved by using the redundancy method.

Table 6.4(b). The values of $\Lambda_{\vec{\rho}} = \Lambda^{D}_{A,\vec{\rho}_B}(\alpha)$.

		$B = \{2,3\}$			$B = \{1,2,3\}$		
α	A	Λ^H_ρ	Λ^I_ρ	Λ^C_ρ	Λ^H_ρ	Λ^I_ρ	Λ^C_ρ
0.1	{1,2}	0.07703	0.06087	0.05469	0.14703	0.13087	0.12470
	{1,3}	0.07386	0.05629	0.04774	0.14386	0.12629	0.11774
	{2,3}	0.07557	0.06163	0.05129	0.14560	0.13160	0.12129
	{1,2,3}	NA	NA	NA	0.11557	0.10145	0.09499
0.3	{1,2}	0.05963	0.04638	0.04159	0.12890	0.11638	0.11159
	{1,3}	0.05418	0.03939	0.03251	0.12420	0.10940	0.10251
	{2,3}	0.05702	0.04529	0.03708	0.12700	0.11529	0.10708
	{1,2,3}	NA	NA	NA	0.08430	0.08086	0.07773
0.4	{1,2}	0.05290	0.04099	0.03678	0.12293	0.11099	0.10678
	{1,3}	0.04642	0.03294	0.02675	0.11642	0.10294	0.09675
	{2,3}	0.04977	0.03911	0.03177	0.11977	0.10911	0.10177
	{1,2,3}	NA	NA	NA	0.07165	0.07297	0.07118
0.5	{1,2}	0.04659	0.03602	0.03235	0.11659	0.10602	0.10235
	{1,3}	0.03897	0.02680	0.02134	0.10897	0.09684	0.09134
	{2,3}	0.04285	0.03329	0.02683	0.11290	0.10329	0.09683
	{1,2,3}	NA	NA	NA	0.05960	0.06544	0.06497
0.7	{1,2}	0.03390	0.02637	0.02388	0.10390	0.09637	0.09380
	{1,3}	0.02358	0.01454	0.01054	0.09357	0.08454	0.08054
	{2,3}	0.02873	0.02171	0.01714	0.09870	0.09171	0.08714
	{1,2,3}	NA	NA	NA	0.03613	0.04982	0.05222
0.9	{1,2}	0.01931	0.01603	0.01499	0.08931	0.08603	0.08499
	{1,3}	0.00446	NA	NA	0.07446	0.06999	0.06811
	{2,3}	0.01179	0.00848	0.00646	0.81790	0.07847	0.07646
	{1,2,3}	NA	NA	NA	0.01230	0.02909	0.03583

Table 6.5 shows the values of $\Lambda_{\vec{\rho}} = \Lambda^{D}_{A,\vec{\xi}_B}(\alpha)$ which we need to compute VMREF $\vec{\xi}^{D}_{A,B}(\alpha)$.

Table 6.5 The values of $\Lambda_{\vec{\rho}} = \Lambda^{D}_{A,\vec{\xi}_B}(\alpha)$.

	$B=\{1,2\}$			$B=\{1,3\}$		
A	Λ^H_ρ	Λ^I_ρ	Λ^C_ρ	Λ^H_ρ	Λ^I_ρ	Λ^C_ρ
{1,2}	0.0544	0.0406	0.0361	0.0744	0.0606	0.0561
{2,3}	0.0516	0.0399	0.0314	0.0716	0.0599	0.0514
{1,3}	0.0484	0.0337	0.0267	0.0684	0.0537	0.0467
	$B=\{2,3\}$			$B=\{1,2,3\}$		
A	Λ^H_ρ	Λ^I_ρ	Λ^C_ρ	Λ^H_ρ	Λ^I_ρ	Λ^C_ρ
{1,2}	0.0644	0.0506	0.0461	0.1344	0.1206	0.1161
{2,3}	0.0616	0.0499	0.0414	0.1316	0.1199	0.1114
{1,3}	0.0584	0.0437	0.0367	0.1284	0.1137	0.1067
{1,2,3}	NA	NA	NA	0.1003	0.0772	0.0729

Based on the results presented in Table 6.5, one can conclude that:

1. The improved system that can be obtained by improving components 1 and 2, $A = \{1, 2\}$, according to HDM, has the same mean time to failure of that system which can be obtained by doing one of the following:

   (i) Reducing the failure rates of components 1 and 2, $B = \{1, 2\}$, by the vector $\vec{\xi} = (\rho_1, \rho_2)$, which satisfies $0.07\rho_1 + 0.06\rho_2 = 0.0544$. That is, by reducing the failure rate of component 2 by an arbitrary factor $\rho_2 \in (0, 1)$, reducing the failure rate of component 1 by the factor $\rho_1 = \frac{1}{0.07}(0.0544 - 0.06\rho_2)$.

   (ii) Reducing the failure rates of components 1 and 3, $B = \{1, 3\}$, by the vector $\vec{\xi} = (\rho_1, \rho_3)$, which satisfies $0.07\rho_1 + 0.08\rho_3 = 0.0744$. Namely, by reducing the failure rate of component 3 by an arbitrary factor $\rho_3 \in (0, 1)$, and reducing the failure rate of component 1 by the factor $\rho_1 = \frac{1}{0.07}(0.0744 - 0.08\rho_3)$.

   (iii) Reducing the failure rates of components 2 and 3, $B = \{2, 3\}$, by the vector $\vec{\xi} = (\rho_1, \rho_3)$, which satisfies $0.06\rho_2 + 0.08\rho_3 = 0.0644$.

   (iv) Reducing the failure rates of components 1, 2 and 3, $B = \{1, 2, 3\}$, by the vector $\vec{\xi} = (\rho_1, \rho_2, \rho_3)$, which satisfies $0.07\rho_1 + 0.06\rho_2 + 0.08\rho_3 = 0.1344$. That is, by reducing the failure rates of components 2, 3 by arbitrary factors $\rho_2 \in (0, 1)$, $\rho_3 \in (0, 1)$ and reducing the failure rate of component 1 by the factor $\rho_1 = \frac{1}{0.07}(0.1344 - 0.06\rho_2 - 0.08\rho_3)$.

2. The improved system that can be obtained by improving components 1, 2 and 3, $A = \{1, 2, 3\}$, according to HDM, has the same mean time to failure of that

system which can be obtained by Reducing the failure rates of components 1,2 and 3, $B = \{1, 2, 3\}$, by the vector $\vec{\rho} = (\rho_1, \rho_2, \rho_3)$ which satisfies $0.07\rho_1 + 0.06\rho_2 + 0.08\rho_3 = 0.10027$.

3. Similarly, one can read the rest of the results obtained assuming HDM, CDM and ISDM.

4. The notation NA in Table 6.5, means that the value of $\Lambda_{A,B}$ is not available and therefore there is possible equivalence between the system improved by RM and that system improved by using the redundancy method.

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