2차원 동적 진동문제의 공간-시간 유한요소법 적용

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An Application of Space and Time Finite Element Method for Two-Dimensional Transient Vibration

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Abstract : This paper deals with the space-time finite element analysis of two-dimensional vibration problem with a single variable. The method of space-time finite elements enables the simpler solution than the usual finite element analysis with discretization in space only. We present a discretization technique in which finite element approximations are used in time and space simultaneously for a relatively large time period. The weighted residual process is used to formulate a finite element method for a space-time domain. A stability problem is described and some investigations for chosen type of rectangular space-time finite elements are carried out. Instability is caused by a too large time step of successive time steps in the traditional time-dependent problems. It has been shown that the numerical stability of time-stepping on the larger time steps is quite good. The unstructured space-time finite element not only overcomes the shortcomings of the stability in the traditional numerical methods, but it is also endowed with the features of an effective computational technique. Some numerical examples have been presented to illustrate the efficiency of the described method.

초 록 : 본 논문은 2차원 동적 진동문제를 공간-시간 유한요소법으로 해석하고 있다. 공간-시간 유한요소법은 공간만 분할하는 재래식 유한요소해석에 비해 보다 해를 빠르고 쉽게 얻을 수 있다. 상대적으로 큰 시간간격에 대해서 공간과 시간을 동시에 분할하는 공간-시간 유한요소 근사법을 제시한다. 가중잔차법으로 공간-시간 영역에 대해 유한요소법을 정식화 하였으며 선형 사변형 공간-시간 유한요소를 선택하여 해의 안정성에 관하여 언급하였다. 일반적 동적문제에서서는 상대적인 큰 시간간격으로 인하여 해의 불안정을 야기 시키고 있으나 본 연구에서는 수치의 안정성을 보여주고 있다. 비구조 공간-시간 유한요소법은 재래식 수치해석에서 흔히 발생하는 해의 불안정성에 대한 결점을 보완함은 물론 효과적인 계산방법을 지니고 있다. 이 방법의 효율성을 위해 수치예제들을 제시하였다.
Key Words : space-time finite element, weighted residual, vibration, stability, unstructured

1. Introduction

Following the more and more wide-spread idea that finite element methods can be successfully applied to a great variety of problems, we try to use space-time finite elements to solve a problem in two-dimensional vibrating membrane[1]. Most commonly used numerical methods for vibrating equations are based upon a space discretization which is independent of time, where we discretize in space using the finite element method and then uses some time-stepping method for the resulting system of ordinary differential equations in time to obtain algebraic equations, which are then solved for the nodal values[2,3]. This will be a stiff system which will pose extra requirements on the stability of the methods to be used for the time discretization. In this paper we consider simultaneous discretization of space and time variables to be uniform by formulating space-

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time finite element methods[4-10]. A space-time finite element is a finite element in which an additional time dimension is considered. As an example, one-dimensional beam element has a two-dimensional shape in time and space. The idea is to use the method of weighted residuals to treat space and time in a uniform manner and thus integrate both the spatial and temporal variations of the unknown quantities simultaneously. A complete space-time finite element discretization eliminates the need for an additional ordinary differential equation solver to discretize time. The approximations are continuous with respect to the space variables for each fixed time at each time step. The discretization is repeated or adjusted for subsequent time slabs using continuous finite element approximations. In problems that involve two or more space dimensions, this allows the use of small elements in regions where the gradients of the solution are large and large elements where the solution has small gradients. The rectangular division of the space-time is assumed and the method will be a conditionally stable one with regard to the relatively large time step.

2. The Model Equation

The second-order partial differential equation for wave propagation can be written as

$$\frac{\partial^2 u}{\partial t^2}(x,y,t) - c\nabla^2 u(x,y,t) = q(x,y,t) \text{ in } \Omega\times I \tag{1}$$

with boundary and initial conditions as follows:

$u = \bar{u}$ on $\Gamma_n \times I$ all $t \geq 0$

$u(x, y, 0) = u_o(x, y)$ in $\Omega(0)$, x, $y \subseteq \Omega$

$u_t(x, y, 0) = v_o(x, y)$ in $\Omega(0)$, x, $y \subseteq \Omega$

where $u(x, y, t)$ is the displacement(cm) of the point(x, y) on any cross-section for $x, y \subseteq \Omega$ at time $t \subseteq I = [0, T]$ and where T is a given time, subscript variables indicate partial differentiation with respect to time. The quantity c = $\sqrt{\tau/\rho}$ is called the velocity of elastic wave propagation(cm/sec). τ is the tension(kg/cm), $\rho = \rho(x, y) > 0$ is the density of material (g/cm^2), q is the external force(kg), and the $\bar{u}$ is specified boundary values of u, and $u_o(x, y)$ and $v_o(x, y)$ are the prescribed initial values of u and its time derivative respectively.

3. Space and Time Discretization

We shall now consider a domain for space and time(Fig. 1). Let us consider a time interval $I = \{t: 0 < t < T\}$ be partitioned into N pieces by the partition $0 = t_o < t_1 < t_2 < \cdots < t_N = T$ and let open spatial interval $\Omega(t)$ be described by $\Omega = \{x: 0 < x < A,\ y: 0 < y < B\}$ and be partitioned into M pieces by the partition $0 = x_o^n < x_1^n < x_2^n < \cdots < x_i^n < x_{i+1}^n < x_M^n = A$ for $0 \leq i \leq M\text{-}1$ and be partitioned into L pieces by the partition and $0 = y_o^n < y_1^n < y_2^n < \cdots < y_j^n < y_{j+1}^n < y_L^n = B$ for $0 \leq j \leq L\text{-}1$ and $n \geq 0$. Then the space-time domain is the product space $\Omega \times I$. For the n-th space-time domain, let the spatial domain be subdivided into n_{el} elements, Ω_n^i, I = 1, 2, 3, $\cdots$, n_{el}. Then, for the n-th space-time element the domain is

$$G_n^i = \Omega_n^i \times I_n \quad i = 1, 2, 3, \cdots, n_{el} \tag{2}$$

with boundary $\Gamma(t)$. Within each space-time element the trial solution and weighting functions are approximated by interpolation polynomials. These functions are assumed to be continuous within each space-time domain and across the interfaces of the space-time domains, namely at times t_1, t_2, $\cdots$, $t_{N\text{-}1}$. We denote by G_n^i an arbitrary bilinear element with vertices $u_{i,n}$, $u_{i+1,n}$, $u_{i+1,n+1}$ and $u_{i,n+1}$, where i is a space index and n a time index. For G_n^i we take the displacement field $u^e(x, t) \subseteq G_n^i$ to be approximated by

$$u^e(x, t) = \sum_{k=1}^{m} N_k(G_n^i)\, u_k \tag{3}$$

where m is the number of node for each element, $N_k(G_n^i)$ are interpolating polynomials of degree r defined over the region G_n^i and u_k are the nodal values of the field $u(x, t)$. The weighting functions $\Psi_k(G_n^i)$ is also defined over the space G_n^i.

4. Weighted Residual Process

On an element basis, an algebraic relation among

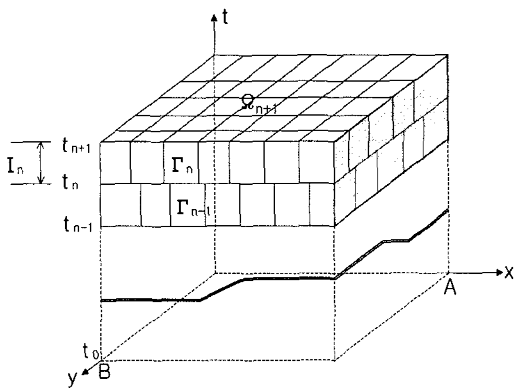

Fig. 1. The discretization of the domain in space and time.

the u_k can be obtained by the method of weighted residuals. The Galerkin method for Eq.(1) can now be placed in a weighted residual formulation for seeking a function u^e:

$$\int_{G^e}[c(\frac{\partial}{\partial x}(\frac{\partial u^e}{\partial x})+\frac{\partial}{\partial y}(\frac{\partial u^e}{\partial y}))-\frac{\partial}{\partial t}(\frac{\partial u^e}{\partial t})+q] \phi_i^e\ (x,y,t)\ dG^e=0 \quad i=1,2,3,\cdots,m \tag{4}$$

where G^e is the domain for element e. Ψ_i^e are the weight functions and m is the number of nodes in element G^e. If the function u^e were exact solution, then the term in the parenthesis would be identically zero for all choices of the weight function $\phi_i^e(x, y, t)$. This integral condition requires the error in satisfying the governing differential equation to be orthogonal to the weight functions Ψ_i^e. If the weight functions are complete so that there exists no admissible function that is orthorgonal to all Ψ_i^e except the null function, then u^e would be the exact solution. The first and the second term on the left-hand side of Eq.(4) contains second derivatives of the dependent variables. Integration by parts for the first term on the left-hand side gives

$$\int_{G^e} c\frac{\partial}{\partial x}(\frac{\partial u^e}{\partial x})\ \phi_i^e\ (x,y,t)\ dG^e = -\int_{G^e} c\frac{\partial u^e}{\partial x}\frac{\partial \psi_i^e}{\partial x}\ dG^e + \int_{\Gamma^e} n_x c\frac{\partial u^e}{\partial x}\ \psi_i^e\ d\Gamma^e \tag{5}$$

and hence Eq.(4) becomes in the rearranged form

$$\int_{G^e} c\frac{\partial u^e}{\partial x}\frac{\partial \phi_i^e}{\partial x}dG^e - \int_{\Gamma^e} n_x c\frac{\partial u^e}{\partial x}\psi_i^e\ d\Gamma^e + \int_{G^e} c\frac{\partial u^e}{\partial y}\frac{\partial \phi_i^e}{\partial y}dG^e - \int_{\Gamma^e} n_y c\frac{\partial u^e}{\partial y}\psi_i^e\ d\Gamma^e + \int_{G^e}\frac{\partial u^e}{\partial t}\frac{\partial \phi_i^e}{\partial t}dG^e - \int_{\Gamma^e} n_t\frac{\partial u^e}{\partial t}\psi_i^e\ d\Gamma^e - \int_{G^e} q\psi_i^e\ dG^e = 0 \quad i=1,2,3,\cdots,m \tag{6}$$

where Γ^e is the element boundary of G^e in x-y-t plane, n_x, n_y and n_t are the outward unit normal vectors of Γ^e. Note that the second integral in Eq.(6) carries information about kinematic boundary conditions. Evidently, this second integral does not contribute anything to the equation at the internal grid points. When a grid point lies on a boundary and if $\partial u/\partial n$ is specified on that boundary, then the integral can be evaluated. The last integral carries information about the forcing function.

5. Finite Element Formulation

We now describe the isoparametric formulation of the stiffness matrix for the space-time elements. The solution domain is discretized by using three-dimensional bilinear quadrilateral elements for a general region specified by a global coordinates system(x, y, t).

$$x=\sum_{i=1}^{8}N_i x_i,\ y=\sum_{i=1}^{8}N_i y_i,\ t=\sum_{i=1}^{8}N_i t_i \tag{7}$$

where x_i, y_i, t_i are nodal coordinates in the global coordinate system. This transformation from the x-y-t region to the 2×2×2 cubic in natural coordinate system(ξ, η, ζ) can be expressed in matrix form as follows:

$$\begin{Bmatrix} x \\ y \\ t \end{Bmatrix} = [N]\{c\} \tag{8}$$

where $\{c\} = \{x_{i,j,n}\,y_{i,j,n}\,t_{i,j,n}\cdots\ x_{i,j+1,n+1}\,y_{i,j+1,n+1}\,t_{i,j+1,n+1}\}^T$ and the interpolation function $N_i(\xi, \eta, \zeta)$ for node i has the following form

$$N_i(\xi,\eta,\zeta) = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta) \quad i = 1 \text{ to } 8 \tag{9}$$

The unknown displacement is interpolated by the same functions, making this an isoparametric formulation.

$$u^e(x,y,t) = [N(\xi,\eta,\zeta)]\{u\} \tag{10}$$

If the weighting functions $\phi^e_i(x, y, t)$ are taken as the interpolation function N_i, then Eq.(6) becomes

$$\sum_{J=1}^{8}\int_{G^e} cN_{j,x}N_{i,x}u_j^e\,dG^e + \sum_{J=1}^{8}\int_{G^e} cN_{j,y}N_{i,y}u_j^e\,dG^e + \sum_{J=1}^{8}\int_{G^e} N_{j,t}N_{i,t}u_j^e\,dG^e - \sum_{J=1}^{8}\int_{\Gamma^e} cn_x N_{j,x}N_i u_j^e\,d\Gamma^e - \sum_{J=1}^{8}\int_{\Gamma^e} cn_y N_{j,y}N_i u_j^e\,d\Gamma^e - \sum_{J=1}^{8}\int_{\Gamma^e} n_t N_{j,t}N_i u_j^e\,d\Gamma^e + \int_{G^e} qN_i\,dG^e = 0 \quad i = 1,\ldots,8 \tag{11}$$

Equation(11) is rewritten in matrix form as

$$[K]\{u\} + [M]\{u\} + [BX]\{u\} + [BT]\{u\} + \{p\} = 0 \tag{12}$$

or

$$[E]\{u\} = [p] \tag{13}$$

where

$$[E] = [K] + [M] + [BX] + [BT]$$
$$\{u\} = \{u_{i,n}\ u_{i+1,n}\ u_{i+1,n+1}\ u_{i+1,n}\}^T$$

The individual matrices are given as

$$[K] = \int_{G^e} cN_{j,x}N_{i,x}\,dG^e + \int_{G^e} cN_{j,y}N_{i,y}\,dG^e$$

$$[M] = \int_{G^e} N_{j,t}N_{i,t}\,dG^e$$

$$[BX] = -\sum_{J=1}^{8}\int_{\Gamma^e} cn_x N_{j,x}N_i\,d\Gamma^e - \sum_{J=1}^{8}\int_{\Gamma^e} cn_y N_{j,y} N_i\,d\Gamma^e$$

$$[BT] = -\sum_{J=1}^{8}\int_{\Gamma^e} n_t N_{j,t}N_i\,d\Gamma^e$$

$$\{p\} = -\int_{G^e} qN_i\,dG^e$$

The integration will be integrated numerically using Gauss-Legendre technique of selecting integration points and weighting factors in the ξ - τ space. Using this approach, the matrix $[E^e]$ only is considered. The other matrices can be performed in a same manner.

$$[E^e] = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} W_i W_j W_k [N^T(\xi_i,\eta_j,\zeta_k)]^T\{A_1\}^T \{A_1\}[N^T(\xi_i,\eta_j,\zeta_k)]\ |\ J\ (\xi_i,\eta_j,\zeta_k)\ | \tag{14}$$

where

$$[A_1] = [J]^{-1}$$

$$[J] = \begin{pmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial t}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial t}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial t}{\partial \zeta} \end{pmatrix}$$

where W_i, W_j and W_k are Gauss weights, ξ_i, η_j and ζ_k are the coordinates of the Gauss points, and p is the number of Gauss points in each integration direction.

6. Assembling the Finite Element Equations

The assemblage process to obtain the system matrix E is symbolically written

$$E = \sum_{i}^{TE} E_i^e$$

where the TE is the total number of total elements and summation goes over all elements in the assemblage. In an analogous manner the load vectors Q are assembled from the element load vectors. For each element we have equations of the form of Eq.13. At $t = 0$, we know all values of u. We need to assemble these individual element equations so that the values of u can be calculated at the end of the first time step. This continues so that during the n-th time step we know the values of u at $u_{i,n}$ and want to calculate $u_{i,n+1}$. Define a global nodal displacement vector as

$$\{P\} = \{p_1\, p_2\, p_3 \cdots p_N\}^T$$
$$\{U\} = \{u_1\, u_2\, u_3 \cdots u_N\}^T$$

The assembled equations at the n-th time level will have the matrix form as

$$[S_2]\,\{U\}^{n+1} = \{P\}^{n+1} - [S_1]\,\{U\}^{n} \tag{15}$$

The assembled matrices $[S]$ are related to time level n and $n+1$ and $\{Q\}$ is assemblage vector of nodal force. These equations can be eventually simplified into a set of linear algebraic equations for the unknown displacements in the time slab. The equation set which is not yet modified to included boundary conditions is rewritten in the final form as

$$[S_2]\,\{U\}^{n+1} + [S_1]\,\{U\}^{n} = \{P\}^{n+1} \tag{16}$$

Such a formulation enables step-by-step solution. Given a known initial displacement u_o at time $n = t_o$ and a time step $\triangle t$, Eq.(16) can be solved for the unknown nodal values at time step $n+1 = t_o + \triangle t$ by summing around the nodes at this latter step and then solving the system of simultaneous linear algebraic equations that results. At each new time step, an identical procedure is used until a required time is reached.

7. Numerical Results and Discussion

To illustrate the formulation of the method and its efficiency the free vibrations of a rectangular membrane of homogeneous material, of dimensions a = 1.2m by b = 0.6m is chosen and fixed on all its edges. Assume the tension in the membrane is τ = 182.5N/m and the density is ρ = 405.5kg/m^2. The initial deflection of the membrane is assumed to be

$$u_o(x,y) = 0.1(1.2x - x^2)(0.6y - y^2)$$

and the initial velocity is $v_o(x, y) = 0$. Since we will verify numerical solutions of this computational model against analytical solutions, only case with the homogeneous boundary condition was considered in the x - y coordinate system. The analytical solution of this problem is

$$u(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{m\pi y}{b}\, p_{mn} \tag{22}$$

The coefficients p_{mn} is given by the equations

$$p_{mn} = \frac{4}{ab}\left[\frac{(-1)^n(0.6n^2b^2\pi^2 + 2b^3 - b^3n^2\pi^2) - 2b^3}{n^3\pi^3}\right]\left[\frac{(-1)^m(1.2m^2a^2\pi^2 + 2a^3 - a^3m^2\pi^2) - 2a^3}{m^3\pi^3}\right]$$

Fig. 2 shows the displacement curves of the centerline for various fixed values of the time t with relatively large time increment $\triangle t$ = 0.05 second and compares the exact solution for t = 0.30 second only. The present method from the space-time finite element is in good agreement with the analytical solution for a short time. As for the critical time step, a large time step is taken for numerical stability considerations. The critical time step for stability in the Newmark time integration scheme can be found as $\triangle t_{cr} = [\frac{1}{2}\omega_{\max}^2(\alpha - \gamma)]^{-1/2}$, where $\omega_{\max}$ is the maximum eigenvalue of the corresponding discrete eigenvalue problem. For a 10×10 mesh of linear rectangular elements, $\omega_{\max}$ = 105.53, and $\triangle t_{cr}$ = 0.0328 for the linear acceleration scheme(α = 0.5, γ = 1/3). Even with $\triangle t$ = 0.07 second which is about double of critical time step these results are still within 2.42% of the exact solution at t = 0.210 second. To study the convergence of the present method, the results in terms of particle velocity for time step of 0.03 second at time t = 0.30, 0.36, 0.42 second were obtained. These results are plotted in Fig. 3. There is no plottable difference between the analytical exact solution and the results of the present method. It is noted that in all cases the solution is very accurately predicted, even with the large time step. However, for

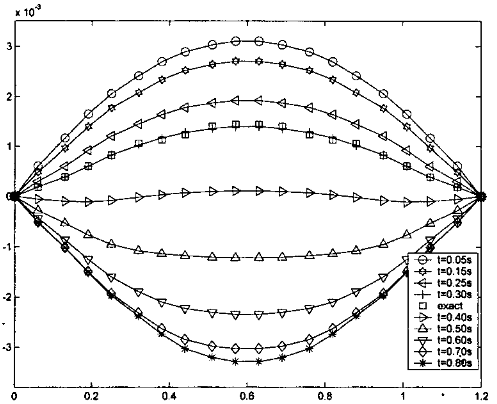

Fig. 2. Comparison of the centerline deflection and variation of the deflection as a function of position and time.

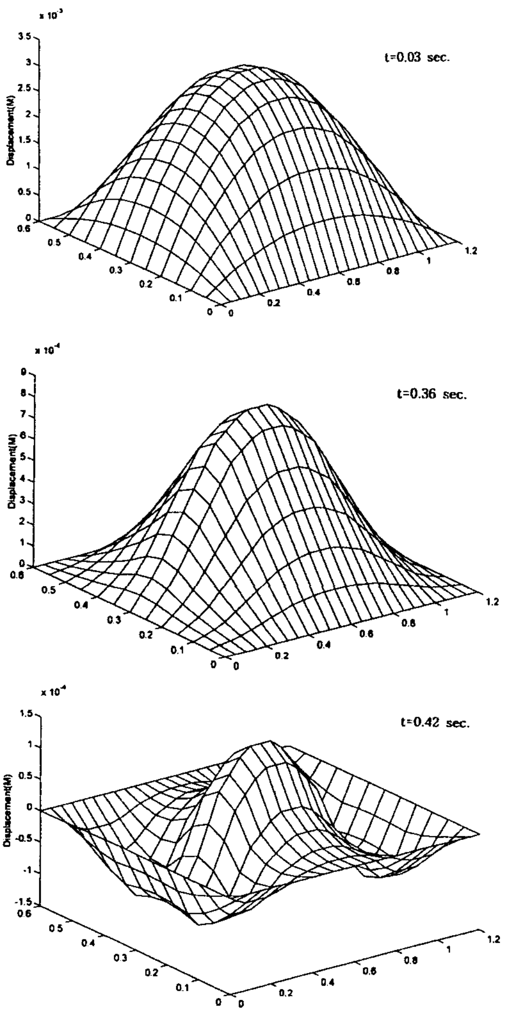

Fig. 3. Developing deflections for vibration in rectangular membrane with various time *t*.

accurate prediction of the response, a finer finite element discretization and smaller space interval along with smaller space interval must be employed. No significant instability problems and much more rapid convergence to the analytical solution were experienced in this approach.

8. Conclusions

A Galerkin procedure for space-time finite element formulation of transient vibration problem in two dimensions has been presented. In this method a complete space-time finite element discretization is generated and it eliminates the need for any additional ordinary differential equation solver to resolve the temporal behavior of the problem. It has been shown that the numerical stability of time-stepping on the larger time steps is quite good. The main disadvantage, from the practical point of view, of any space-time finite element formulation is the additional dimension of time. Although more computer storage capacity was required to handle the present algorithm, this disadvantage was easily offset by the substantial savings in the CPU time due to rapid convergence of the solution. In this present approach, large time step increments were allowed for achieving stable solutions regardless of the maximum permissible time step in traditional numerical methods. Furthermore, numerical results have shown that the space-time Galerkin formulation is effective for localizing oscillations due to sharp gradients along with the fined mesh. It is apparent that the space-time finite method can be extended to analyze other eigenvalue and time-dependent problems and provide a powerful means of finding approximate solutions.

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