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Surface Gravity Waves with Strong Frequency Modulation

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ABSTRACT: *Modulation theory describes propagation of surface waves with deep wave number and frequency modulation. Locally spectrally narrow wave packet can have accumulated large scale frequency shift of carrier wave during propagation. Some important nonlinear modulation effects, such as negative frequencies, phase kinks, crest pairing, etc., often observed experimentally at long fetch propagation of finite amplitude surface wave trains, are reproduced by the proposed theory. The presented model permits also to analyze the appropriately short surface wave packets and modulation periods. Solutions show the wave phase kinks to arise on areas' of relatively small free surface displacement in complete accordance with the experiments.*

1. Introduction

For the past two decades experiments on the nonlinear wave propagation on water surfaces have revealed a number of modulation effects that have not been explained by theorists. Lake and Yuen (1978) were the first who observed the wave crests 'lost' at a quasi-zero amplitude (node) of deeply modulated wave trains. Then a similar effect was found by Ramamonjiarisoa and Mollo-Christensen (1979), Mollo-Christe nsen and Ramamonjiarisoa (1982) and Chereskin, T., Mollo-Christensen, E. (1985) in wind sea waves and under laboratory conditions: a nonlinear surface wave merged with the foregoing one and then disappeared. As a result of such a 'crest pairing' the wave period is doubled instantaneously by Yuen, H. and Lake, B. (1975).

Melville (1983) investigated thoroughly the evolution of an initially uniform train of Stokes surface waves in a long laboratory tank. At the wave fetch beginning, he observed the development of the well-known Benjamin-Feir (1967) sideband instability resulting in weak amplitude-frequency modulation of the wave train. As nonlinear effects accumulated along the tank, an asymmetry arose in the wave envelope related to its maximum. The relative phase between modulations of wave amplitude and frequency also changed, the amplitude was modulated with phase opposite to that of frequency modulation. At a longer distance, Melville observed the most prominent features of that mode, the so-called phase 'reversals' accompanied by very large variations in the wave number, frequency and phase velocity. In particular, the frequency turned out even to be negative near these local phase kinks.

The main goal of the present paper is to derive and study a general set of the third order equations for slowly modulated wave trains propagating on the surface of deep water. As distinct from other works, the proposed model should allow for a variety of uniformly valid solutions with the wave number and frequency having relative variations of the order of unit over the 'slow' coordinate and time.

The paper is organized in the following manner. Section 2 contains the problem's initial statement including its scaling and assumptions necessary to derive the governing modulation equations for the first order potential amplitude, wave number and frequency. Restricted traveling surface wave solutions are found and qualitatively analyzed in section 3 with the use of the phase plane of potential and velocity amplitudes, depending on the fluxes of wave energy and action, as well as on the frequency detuning from the group resonance. Various wave modulation regimes are exhibited and discussed in section 4. In section 5 we present concluding remarks.

2. Main Equations

The set of equations for potential motion of an ideal

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incompressible infinite-depth fluid with the free surface is given by the Laplace equation

$$\phi_{xx}+\phi_{zz}=0 \qquad -\infty < z < \eta(z,t) \tag{2.1}$$

the boundary conditions at the free surface.

$$\phi_t + g\eta + 1/2(\phi_x^2 + \phi_z^2) = 0, \ z=\eta(x,t) \tag{2.2}$$

$$\eta_t + \phi_x \eta_x = \phi_z \qquad z=\eta(x,t) \tag{2.3}$$

and at the bottom

$$\phi = 0, \quad z = -\infty \tag{2.4}$$

Here $\phi(x,z,t)$ and $\eta(x,z,t)$ are the velocity potential and the free surface displacement.
Let us normalize the variables as follows:

$$\phi = a_0\sqrt{gk_0}\phi' = \epsilon\sqrt{gk_0^3}\ \phi',\ \eta = a_0\eta' = \epsilon k_0 \eta',$$
$$t = 1\sqrt{gk_0}t' \quad z = z'k_0 \quad x = x'k_0 \tag{2.5}$$

where $\epsilon = a_0 k_0$ is the conventional average wave steepness parameter, and the dimentionless quantities are primed. Then, the set Eq. (2.1)~Eq. (2.4) is reduced to the form.

$$\phi_{xx}+\phi_{zz}=0\ ,\ -\infty < z < \epsilon\eta(x,t) \tag{2.6}$$

$$-\eta = \phi_t + 1/2(\phi_x^2+\phi_z^2)\ ,\ z=\epsilon\eta(x,t) \tag{2.7}$$

$$\eta_t + \epsilon\phi_x\eta_x = \phi_z\ ,\ z=\epsilon\eta(x,t) \tag{2.8}$$

$$\Phi = 0, \qquad z=-\infty \tag{2.9}$$

where the primes are omitted. Further analysis is based on the assumption of small parameter $\epsilon \ll 1$, therefore the weakly nonlinear surface wave train is described by a solution to Eq. (2.6), Eq. (2.9) expanded into a Stokes series in terms of ϵ.

Assuming also the wave motion phase $\theta = \theta(x,t)$, we define the wave number and frequency in the usual way as

$$k=\theta_x, \quad w=-\theta_t \tag{2.10}$$

These main wave parameters will be considered further as slowly varying with the characteristic scale $O(\epsilon^{-1})$ longer than the primary wavelength and period (Chu and Mei 1970):

$$\phi_0 = \phi_0(\epsilon x, \epsilon t),\ k = k(\epsilon x,\epsilon t),\ w = w(\epsilon x, \epsilon t) \tag{2.11}$$

The solution to the problem, uniformly valid to $O(\epsilon^3)$, is found by a two-scale expansion with the differentiation

$$\begin{aligned}&\partial/\partial t = \omega\partial/\partial\theta + \epsilon\partial/\partial T, \partial/\partial x = k\partial/\partial\theta + \epsilon\partial/\partial X\\ &T=\epsilon t, X=\epsilon x\end{aligned} \tag{2.12}$$

We search the velocity potential in the form:

$$\phi = \phi_0 e^{kz} sin\theta + \epsilon(\gamma z + \delta z^2)e^{kz}cos\theta + \ldots \tag{2.13}$$

Substituting potential Eq. (2.13) into the Laplace Eq. (2.6) and equating the coefficients at the same orders in ϵ, we find directly the modulational corrections to the vertical velocity profile

$$\gamma = -\phi_{0x}, \delta = -1/2k_x\phi_0. \tag{2.14}$$

The free surface displacement $\eta = \eta(x,t)$ is also sought as an asymptotic series,

$$\eta = \eta_0 + \epsilon\eta_1 + \epsilon^2\eta_2 + \ldots \tag{2.15}$$

Substitution of velocity potential Eq. (2.13), Eq. (2.14) into the kinematic boundary condition Eq. (2.8) gives the following relationships between the modulation characteristics:

$$\omega^2 = \kappa + \omega^2\kappa^3\phi_0^2 + \phi_{0TT}/\phi_0, \tag{2.16}$$

$$(\omega\phi_0)_T + \omega\phi_{0T} + \phi_{0X} = 0 \tag{2.17}$$

The first of these formulas, representing the dispersion relation with the total second-order amplitude-phase dispersion included, simplifies to

$$\omega^2 = \kappa + \kappa^4\phi_0^2 + \phi_{0TT}/\phi_0. \tag{2.18}$$

Eq. (2.17) yields the known wave action conservation law.

$$(\omega\phi_0^2)_T + \frac{1}{2}(\phi_0^2)_X = 0 \tag{2.19}$$

Modulation Eq. (2.18) and Eq. (2.19) are closed the wave phase conservation that follows from Eq. (2.10) as a compatibility condition

$$k_T + \omega\chi = 0 \tag{2.20}$$

Closed set Eq. (2.18), Eq. (2.20) defines all three unknown modulation functions: the order potential amplitude, wave number and frequency.

3. Travelling Wave Solution

Let us find traveling wave solutions to the problem Eq. (2.18) and Eq. (2.20), supposing all the unknown functions to depend on the single variable $\xi = X - cT$, where c is the velocity of a reference frame where the waves are stationary. Then, after integrating Eq. (2.19) and Eq. (2.20) the problem has the form

$$\omega^2 = k + k^4\phi_0^2 + c^2\phi_{0\xi\xi}/\phi_0 \tag{3.1}$$

$$(-c\omega + 1/2)\phi_0^2 = A, \tag{3.2}$$

$$-ck + \omega = \Omega \tag{3.3}$$

where A and Ω are the integration free constants. We renormalize the constants and variables by following way:

$$A = c^6\,\tilde{A}\,,\ \phi_0 = c^3\tilde{\phi}_0,\quad \xi = c^2\tilde{\xi},\ \triangle = 1/2 - c\Omega \tag{3.4}$$

Now, omitting tildes we write down the main equation to be analyzed:

$$\phi_{0\xi\xi} = \phi_0(1/2 - A\phi_0^2)^2 - \phi_0(\triangle - A\phi_0^2) - \phi_0^3(\triangle - A\phi_0^2)^4 \tag{3.5}$$

We note at once that Eq. (3.5) has the first integral

$$\phi_{0\xi}^2 = -1/2\triangle^4\phi_0^4 + 1/2A^4/\phi_0^4 + (1/4 - \triangle + 4\triangle^3 A)\phi_0^2 - A^2 - 4\triangle A^3\phi_0^2 - 12\triangle^2 A^2 ln\phi_0 + E \tag{3.6}$$

The structure of this solution to Eq. (3.5) allows us to consider the phase plane (Fig. 1).

The sought solutions form a family with two main types of trajectories:

(i) close type I curves that describe periodic structure behavior of velocity amplitude, surface displacement and frequency functions

(ii) type II curves which are singular at the point $\phi_0 = 0$ with infinite horizontal velocities $\phi_{0\xi} \to \infty$.

The integral cycles I of Fig. 1 are defined by two real roots of the equation

$$\phi_{0\xi\xi} = 0 \tag{3.7}$$

solution exists in the region of parameters A and $\triangle$ between two curves plotted in Fig. 2.

Pointing to the existence of two extremes for the right-hand side of Eq. (3.6), which mean periodic

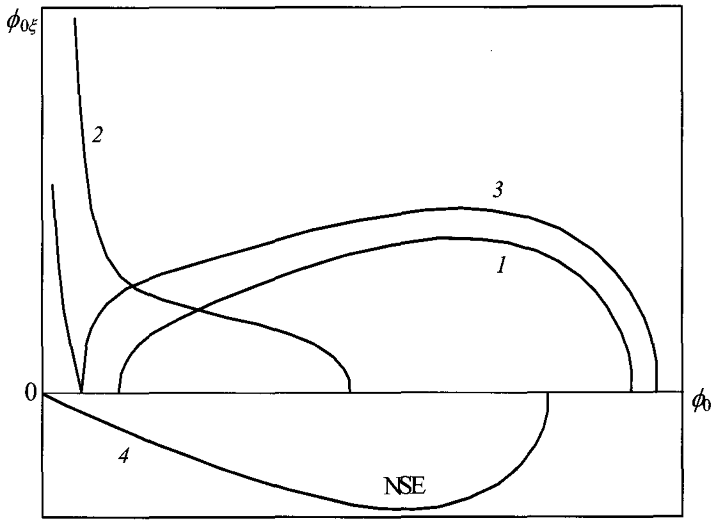

Fig. 1 Phase curves of Eq. (3.6):
(1) periodic solutions;
(2) solutions with singularity in the origin;
(3) solitary solutions;
(4) NSE soliton solution (A=0, E=0)

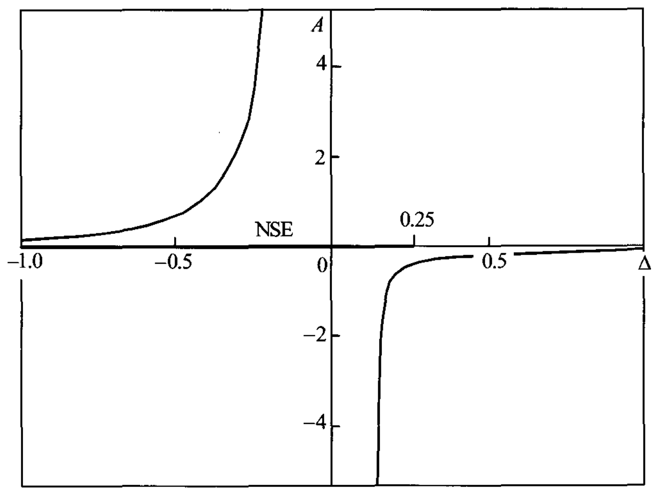

Fig. 2 Plane of parameters for Eq. (3.8). Constricted solutions exist for the region between curves. Solid line describes solutions of NSE equation

modulation of the fluid velocity amplitude.

After excluding the trivial root $\phi_0 = 0$, Eq. (3.7) takes the form of the quadratic equation in the squared variable ϕ_0^2.

$$1/4 - A + \frac{A^2}{\phi_0^4} - \phi_0^2 (A - \frac{A}{\phi_0^2}) = 0 \quad (3.8)$$

Solving Eq. (3.8) one can find that periodic constricted.

4. Analysis of Solutions and Discussion

All properties of the restricted traveling wave solutions are analyzed by integrating Eq. (3.6) in quadrature for various combinations of the three controlling parameters $(\triangle, A, E)$. The most interesting stationary solutions are illustrated by the plots of modulation parameters: the reduced wave frequency

$$\omega(\xi) = \frac{1}{c} (\frac{1}{2} - \frac{A}{\phi_0^2(\xi)}) \quad (4.1)$$

and phase

$$\Theta(\xi) = \frac{1}{c\epsilon} \int_0^{\xi} [\frac{1}{2} - \frac{A}{\phi_0^2(\xi)}] d\xi \quad (4.2)$$

as well as by the profiles of two lowest harmonics of the free displacement:

$$\eta^{(1)}(\xi) = w\phi_0 cos\theta - c\phi_{0\xi} sin\theta - 3/8k^2 w^3 \phi_0^3 cos\theta \quad (4.3)$$
$$\eta^{(2)}(\xi) = 1/2kw^2\phi_0^2 cos2\theta \quad (4.4)$$

Let us first consider using concrete examples the main properties of permanent form periodic solutions for the positive wave action flux A > 0, when the group velocity exceeds the velocity of observer, $c_g = (2w)^{-1} > c$ In this case, it follows from Eq. (4.1) that the instantaneous frequency is higher for larger wave amplitudes. Thus as nonlinear effects grow quite slowly (quasi-stationary) at a constant and positive wave action flux, one should observe a specific average frequency upshift in traveling surface waves.

We will start our consideration from the wave self-modula ted solutions to the classical NSE equation. Fig. 3 shows two lowest harmonics of the surface profile together with the reduced frequency and phase for the following parameters: $A = 0.30, \triangle = 0.22, E = 0$.

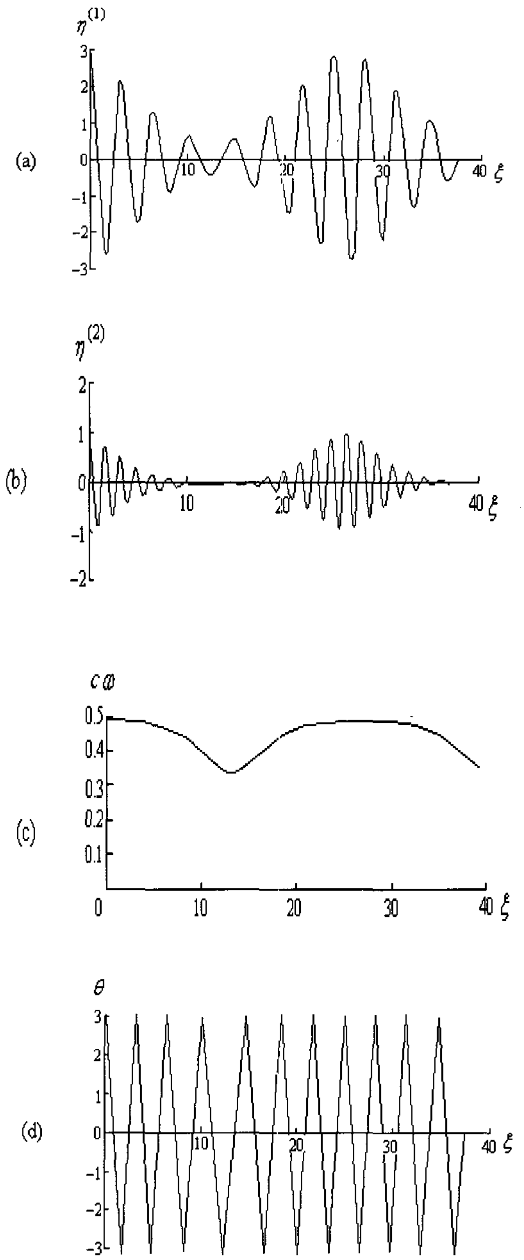

Fig. 3 Surface waves modulation for A=0.30, $\Delta = 0.22$, $E = 0$, (a) surface profile for first harmonics, (b) surface profile for second harmonics, (c) frequency, (d) phase of the first harmonics

One sees from Fig. 3 that the wave system as a whole is modulated smoothly in a regular manner close to the NSE-type 'cnoidal' envelope because the amplitude of wave noticeably exceeds the critical value.

According to the previous analysis, we expect a deep wave modulation to 'smooth' quasi-periodically the water free surface if the integral cycle approaches the critical line

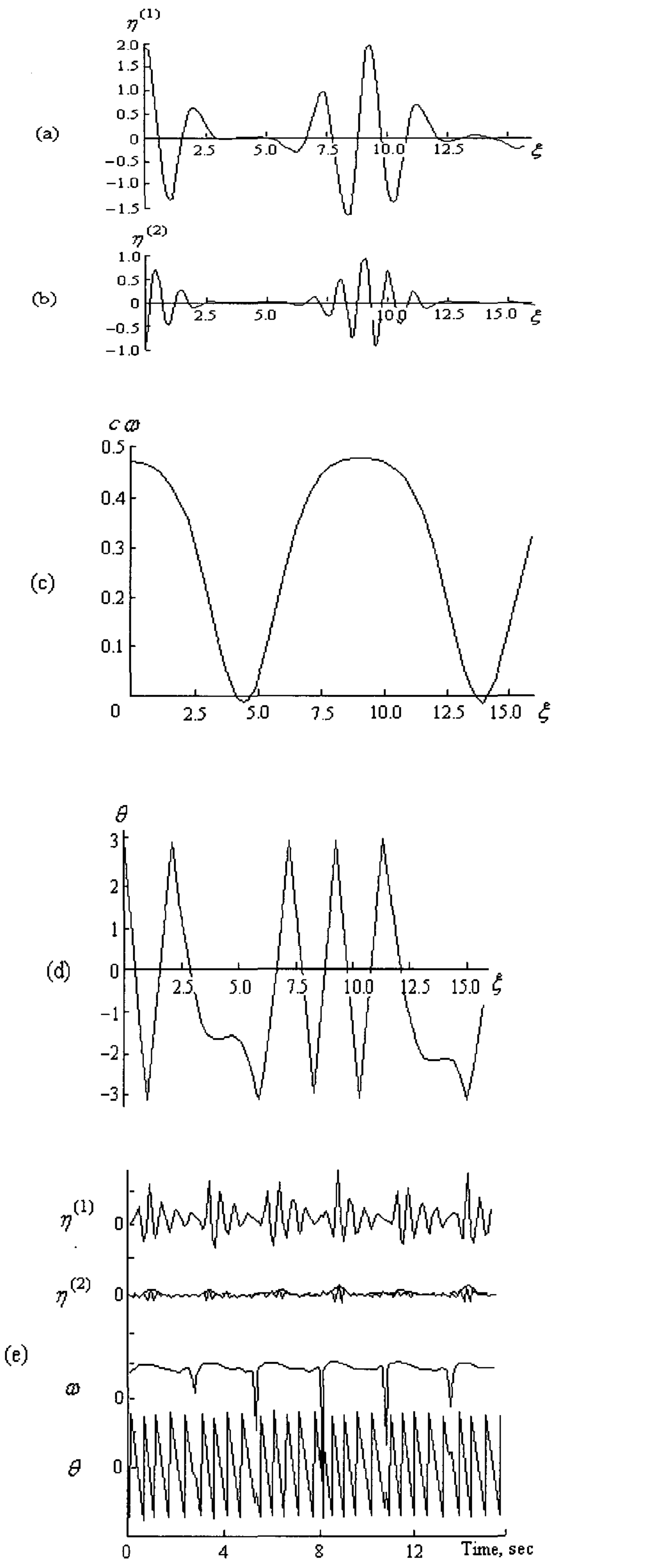

Fig. 4 Surface waves modulation for $A=0.35$, $\Delta=-0.5$, $E=0.2$, $\epsilon=0.15$ (a) surface profile for first harmonics, (b) surface profile for second harmonics, (c) frequency, (d) phase of the first harmonics, (e) experiment results of Melville(1983)

$\phi_{0c}=(2A)^{1/2}$ on the phase plane. Such a wave mode is shown in Fig. 4 for parameters $A=0.35, \Delta=-0.5$, $E=0.2$, $\epsilon=0.15$ A solution of this type has all the main indications of the phase reversals observed in the experiments by Melville (1983)(Fig. 4e): the wave frequency (Fig. 5c) locally drops from the almost maximum $cw \approx 0.48$ to zero, while the phase (Fig. 5d) has sharp kinks localized exactly at the minimum of surface elevation. The phase delays of order $\pi/2$ take approximately one wave period, during which the wave crests are merging together.

5. Concluding Remarks

We have analyzed various types of permanent envelopes traveling on the surface of deep water, which can be treated as solutions to the equations of the third order approximation in wave steepness, assuming the wave number and frequency variations are not small. Some important nonlinear modulation effects, such as negative frequencies, phase kinks, crest pairing, etc., often observed experimentally at long-fetch propagation of finite-amplitude surface wave trains, are reproduced by the proposed theory. The observation of various deeply modulated wave motions predicted in this paper is of general interest. The simply scaled model free parameters can facilitate the choice of dynamic and kinematics wave characteristics appropriate for a relevant laboratory experiment as well as the search for optimum conditions for the field wave measurements.

Acknowledgement

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