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Sequential Design of Inspection Times in Optimally Spaced Inspection

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Abstract

The spacing of inspection times in intermittent inspection is of great interest, and several ways for the determination of inspection times have been proposed. In most inspection schemes including equally spaced inspection and optimally spaced inspection, the best inspection times in each inspection scheme depend on the unknown parameter, and we need an initial guess of the unknown parameter for practical use. Thus it is evident that the efficiency of the resulting inspection scheme highly depends on the choice of the initial value. However, since we can obtain some information about the unknown parameter at each inspection, we may use the accumulated information and adjust the next inspection time. In this paper, we study this sequential determination of the inspection times in optimally spaced inspection.

Keywords : Censored data; Fisher information; Grouped data; Life testing; Order statistics.

1. Introduction

Many lifetime experiments employ the intermittent inspection scheme rather than the continuous one for its convenience and saving costs. The efficiency of the intermittent inspection scheme highly depends on the chosen inspection times, and several intermittent inspection schemes have been discussed by many authors. Kulldorff (1961) and Ehrenfeld (1962) proposed the concept of the optimal inspection, and Nelson (1977) studied the optimum demonstration tests with grouped data when the lifetime distribution is the exponential one. Other kinds of inspection schemes including equally spaced inspection and equal probability inspection have been also discussed by many authors. However, since the best inspection times in each inspection scheme depend on the unknown parameter, we need an initial guess of the unknown parameter for practical use. If the initial

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guess is close to the true parameter value, then we have the nearly optimal inspection scheme.

In this paper, we propose to determine the next inspection time just after each inspection instead of determining all inspection times before the first inspection, and call this inspection scheme *sequential inspection scheme.* Then we can estimate the unknown parameter and determine the next inspection time just after each inspection. We first study how the initial guess affects the efficiency of the optimally-spaced inspection. Then we investigate whether the sequential inspection increases the efficiency for an exponential distribution.

2. Inspection schemes

Suppose that we need to determine K inspection times $t_1, t_2, \cdots, t_K$ to estimate the scale parameter in an exponential distribution, $f(t;\theta) = \exp(-t/\theta)/\theta,\ t \geq 0$. There are several inspection schemes including equally spaced, equal probability, equally spaced in logtime and optimal spacing inspection.

2.1 Equally-spaced inspection scheme

In equally-spaced inspection scheme, the K inspection times are equally spaced as $t_i = i \cdot d,\ i = 1, \cdots, K$ where d needs to be determined. The best value of d can be determined by minimizing the asymptotic variance of the maximum likelihood estimator (MLE) based on the equally spaced inspection times in Nelson (1977),

$$\sigma^2_{\hat{\theta}}(\theta) = \frac{\theta^2 \sinh^2[d/(2\theta)]}{n[1-\exp(-Kd/\theta)](d/2\theta)^2}$$

where n is the total number of items.

However, the resulting best value depends on the unknown parameter, and we need an initial guess of the unknown parameter for practical use. The effect of a wrong guess has been discussed in Kulldorff (1961).

2.2 Equal-probability inspection scheme

In equal-probability inspection scheme, the inspection times are chosen such that the mean number of failures is the same within each interval between the K inspection such that $t_i = -\theta \log(1 - i/K),\ i = 1, 2, \ldots, K-1$ for the exponential distribution. We see that the equal-probability inspection times are easy to compute but also depend on the unknown parameter. Shapiro and Gualti (1996) discussed that just two inspections result in a moderate loss of information. Meeker (1986) discussed that when t_1 and t_K are determined, the equal-probability inspection scheme and the equal-spaced inspection scheme in logtime with the same t_1 and t_K have almost identical properties. Moreover, Kim and Yum (2000) concluded that though the equal-probability inspection is neither more efficient nor simpler to compute than the equal-spaced inspection scheme. Thus the equally-spaced inspection scheme in logtime is regarded as an alternative for the equal-probability inspection scheme since they share similar properties.

2.3 Optimal spacing inspection scheme

Kulldorff (1961) proposed the optimal inspection scheme, and Ehrenfeld (1962) and Nelson (1977) discussed that the optimal inspection times are determined by minimizing the asymptotic variance of the maximum likelihood estimator, which is equivalent to maximizing the asymptotic Fisher information. The asymptotic Fisher information in $t_1, t_2, \cdots, t_K$ about θ can be written in general as

$$I_{t_1 t_2 \cdots t_K}(\theta) = n \sum_{i=1}^{K+1} \frac{\left(\frac{\partial F(t_i;\theta)}{\partial \theta} - \frac{\partial F(t_{i-1};\theta)}{\partial \theta} \right)^2}{F(t_i;\theta) - F(t_{i-1};\theta)}$$

where $t_0 = -\infty$, $t_{K+1} = \infty$.

For the exponential distribution, the Fisher information can be obtained as

$$I_{t_1 t_2 \cdots t_K}(\theta) = \frac{n}{\theta^2} \sum_{i=1}^{K+1} \frac{(t_{i-1}\exp(-t_{i-1}/\theta) - t_i \exp(-t_i/\theta))^2}{\exp(-t_{i-1}/\theta) - \exp(-t_i/\theta)} . \tag{1}$$

Approximate solutions $t_1, t_2, \cdots, t_K$ to maximizing (1) has been studied by Ehrenfeld (1962), Saleh et al (1966). Park and Kim (2006) recently suggested a simple way to find the optimal $t_1, t_2, \cdots, t_K$. They defined q_i^* to be the maximizing solution to E_i ,

$$E_i = \frac{q_i (\log q_i)^2}{(1-q_i)} + q_i E_{i-1}, \ i = 1, 2, \ldots, \infty , \text{ where } E_0 = 0,$$

then the percentile $p_{i:K}^*$'s , which are optimal spacings of size K, can be obtained as $p_{i:K}^* = 1 - q_{K-i+1}^* q_{K-i+2}^* \cdots q_K^*$ and the corresponding inspection times are $t_i = F^{-1}(p_{i:K}^*;\theta) = -\theta \log(1 - p_{i:K}^*)$.

3. Sequential optimal spacing inspection scheme

3.1 Sequential inspection scheme

In intermittent inspection schemes, the whole inspection times are predetermined with an initial guess of the unknown parameter. Thus we are exposed to a risk resulting from a wrong guess. With or without prior knowledge of the parameter, we can get some additional information about the parameter at each inspection after the experiment starts. Thus we like to use the information to adjust the predetermined inspection times, and consider the sequential inspection scheme as follows:

With the initial value of the parameter $\tilde{\theta}_0$, only the first inspection time is determined under the optimal spacing inspection scheme. Then we have (t_1, x_1) after the first inspection, which give a new estimator $\tilde{\theta}_1$ where x_i is the number of failures in $(t_{i-1}, t_i]$. The second inspection time is determined under the optimal spacing inspection scheme with $\tilde{\theta}_1$. Since the distribution of the failures after the first inspection is still exponential with no memory property of the exponential distribution, determining the second inspection time in K inspections in sequential optimal spacing inspection scheme is equivalent to determining the first inspection time in $K-1$ inspections with $\tilde{\theta}_1$. Thus the ith optimal inspection time can be determined to be $t_i = t_{i-1} - \theta \log q^*_{K-i+1}$. After ith inspection, the experimenter can update the parameter with $(t_1, x_1, \cdots, t_i, x_i)$.

3.2 Simulation results

Components in the experiment we consider are the total inspection times K, true parameter θ and the initial parameter $\tilde{\theta}_0$ as follows:

$$K = 5, 7, 9, 11, \quad \theta = 2, 3, 4, 5, \quad \tilde{\theta}_0 = 2, 3, 4, 5.$$

While the whole inspection times in the predetermined optimal spacing inspection scheme are determined with the initial guess $\tilde{\theta}_0$, the inspection times in the sequential optimal inspection scheme are sequentially determined with the updated parameter estimator. The parameter estimator is chosen here to be the maximum likelihood estimator, which is the solution to the following likelihood equation,

$$\sum_{i=1}^{k} \frac{x_i(t_i - t_{i-1})}{1 - \exp(-(t_i - t_{i-1})/\theta)} = \sum_{i=1}^{k} x_i t_i + x_{k+1} t_k.$$

The maximum likelihood estimator does not have a closed form solution, but the new updated parameter estimator after the first inspection can be obtained in a closed form as $\tilde{\theta}_1 = -t_1/\log(1 - x_1/\sum_{i=1}^{k+1} x_i)$. We note that the conditional maximum likelihood estimator after each inspection can be obtained in a closed form as $\hat{\theta}_i = -(t_i - t_{i-1})/\log(1 - x_i/\sum_{j=i}^{k+1} x_j)$ in a similar way, but the relation with these conditional maximum likelihood estimator and the maximum likelihood estimator has not been yet studied.

The bias and variance of each maximum likelihood estimator have been calculated with 10,000 Monte Carlo simulated samples for the predetermined and sequential optimal inspection schemes. Table 1 shows the biases of the maximum likelihood estimators based on two inspection schemes. We see that the bias for the predetermined inspection scheme become large when the initial guess is too small from the true value. We can also see from the table that the sequential optimal inspection scheme produces less bias. Table 2 shows the asymptotic variances of the maximum likelihood estimators. We can see that the asymptotic variances of the sequential optimal inspection scheme are a little smaller than those of the predetermined optimal inspection scheme for all combinations. Thus we can conclude that the sequential inspection scheme reduces much of the bias rather than the variance.

<Table 1> Average biases based on 10,000 simulations

	$\tilde{\theta}_0$	Sequential Optimal Inspection				Predetermined Optimal Inspection			
	θ	2.0	3.0	4.0	5.0	2.0	3.0	4.0	5.0
$K=5$	2.0	0.0007	0.0068	0.0029	0.0172	0.0327	0.0648	0.1021	0.1069
	3.0	0.0045	0.0108	0.0020	0.0000	0.2541	0.0552	0.0754	0.1165
	4.0	0.0035	0.0023	0.0069	0.0032	1.0203	0.2029	0.0541	0.0838
	5.0	0.0202	0.0429	0.0137	0.0011	2.5671	0.6152	0.1605	0.0934
$K=7$	2.0	0.0011	0.0132	0.0047	0.0054	0.0344	0.0532	0.0649	0.0944
	3.0	0.0038	0.0169	0.0114	0.0076	0.2874	0.0454	0.0636	0.0694
	4.0	0.0017	0.0051	0.0147	0.0044	1.1523	0.2520	0.0585	0.0626
	5.0	0.0043	0.0046	0.0013	0.0024	2.8765	0.7252	0.2079	0.0855
$K=9$	2.0	0.0061	0.0027	0.0049	0.0084	0.0558	0.0276	0.0548	0.0780
	3.0	0.0221	0.0046	0.0020	0.0171	0.4110	0.0764	0.0447	0.0367
	4.0	0.0112	0.0165	0.0216	0.0076	2.4702	0.3005	0.0729	0.0674
	5.0	0.0148	0.0009	0.0001	0.0118	3.5105	0.9820	0.3017	0.1441

4. Concluding Remarks

We consider the sequential optimal inspection scheme where the inspection times are sequentially determined, and see that the sequential optimal inspection scheme reduces much of the bias of the maximum likelihood estimator. In exponential life testing, it is easy to determine the next inspection time the sequentially due to no memory property.

<Table 2> Average variances based on 10,000 simulations.

	$\tilde{\theta}_0$ / θ	Sequential Optimal Inspection				Predetermined Optimal Inspection			
		2.0	3.0	4.0	5.0	2.0	3.0	4.0	5.0
$K=5$	2.0	0.0307	0.0348	0.0382	0.0416	0.0309	0.0351	0.0394	0.0432
	3.0	0.0624	0.0690	0.0753	0.0808	0.0633	0.0694	0.0761	0.0822
	4.0	0.1040	0.1140	0.1225	0.1312	0.1087	0.1150	0.1235	0.1322
	5.0	0.1562	0.1689	0.1806	0.1918	0.1686	0.1733	0.1816	0.1934
$K=7$	2.0	0.0279	0.0313	0.0343	0.0369	0.0281	0.0315	0.0349	0.0379
	3.0	0.0575	0.0629	0.0678	0.0725	0.0585	0.0632	0.0682	0.0735
	4.0	0.0971	0.1046	0.1122	0.1186	0.1013	0.1059	0.1124	0.1191
	5.0	0.1472	0.1568	0.1664	0.1747	0.1581	0.1607	0.1676	0.1754
$K=9$	2.0	0.0264	0.0292	0.0317	0.0341	0.0265	0.0293	0.0321	0.0348
	3.0	0.0549	0.0591	0.0633	0.0675	0.0559	0.0596	0.0637	0.0680
	4.0	0.0934	0.0993	0.1053	0.1111	0.0976	0.1011	0.1062	0.1115
	5.0	0.1425	0.1497	0.1573	0.1643	0.1531	0.1540	0.1592	0.1653

In estimating the mean of the exponential distribution based on grouped data, the maximum likelihood estimator does not have a closed form solution. Thus the approximation to the maximum likelihood estimator has been studied by Tallis (1967), Kendall and Anderson (1971), and Nelson (1982). In the sequential optimal inspection. We note that the conditional maximum likelihood estimator after each inspection has a closed form solution as $\hat{\theta}_i = -(t_i - t_{i-1})/\log(1 - x_i/\sum_{j=i}^{k+1} x_j)$. If we can find a suitable weight function on these conditional maximum likelihood estimators, we can have an approximate linear maximum likelihood estimator and sequentially update the parameter estimator with additional inspection. This plausible future work will complete the practical aspects of the sequential inspection scheme.

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