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Bayesian Test for the Equality of Gamma Means[1)]

Sang Gil Kang[2)]

Abstract

When X and Y have independent gamma distributions, we develop a Bayesian procedure for testing the equality of two gamma means. The reference prior is derived. Using the derived reference prior, we propose a Bayesian test procedure for the equality of two gamma means using fractional Bayes factor and intrinsic Bayes factor. Simulation study and a real data example are provided.

Keywords : Fractional Bayes factor, Gamma mean, Intrinsic Bayes factor, Reference prior

1. Introduction

A general review of the gamma distribution including several references to applications in diverse fields is given by Johnson, Kotz and Balakrishnan (1994). In particular, the gamma distribution has been suggested as the failure time model, and also received considerable attention in the area of ecology and weather analysis.

The present paper focuses on Bayesian test for the equality of two gamma means. In Bayesian testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegalhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

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1) This research was supported by Sangji University Research Fund, 2005
2) Assistant Professor, Department of Applied Statistics, Sangji University, Wonju, 220-702, Korea
E-mail : sangkg@mail.sangji.ac.kr

Spiegalhalter and Smith (1982) used the device of imaginary training samples in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so, there is no guarantee that the Bayes factor of Spiegalhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang, Kim and Lee, 2005, 2006).

For two sample gamma models, Shiue and Bain (1983) derived an approximate F test for testing equality of means when the shape parameters are equal. Shiue, Bain and Engelhardt (1988) extended the method to the case where the shape parameters are unequal. Booth, Hobert and Ohman (1999) gave a detailed review of existing methods for inference concerning the ratio of two means when the shape parameters are equal and are in proportion. Simulation studies in Booth, Hobert and Ohman (1999) suggest that intervals obtained by extending the method in Jensen (1986) and those obtained by bootstrap calibration have similar performance in terms of length and coverage. Even though the bootstrap calibration method is computationally intensive, they still recommended it over the extended Jensen's method because it is much simpler to implement and is more versatile. Wong, Wu and Sun (2004) proposed a method based on the modified signed log-likelihood ratio statistic for small sample inference concerning the ratio of two means when the shape parameters are equal and are unequal. They argued that the proposed method gave extremely accurate coverage in simulation studies, and was more direct and less computational intensive than the calibrated bootstrap method (Booth, Hobert and Ohman, 1999).

Almost all the work mentioned above is the analysis based on the classical point of view, there is a little work on this problem from the viewpoint of the objective Bayesian framework. So we feel a strong necessity to develop objective Bayesian test procedure for the equality of two gamma means. For dealing this problem, we use the fractional Bayes factor (O'Hagan, 1995) and the intrinsic Bayes factor (Berger and Pericchi, 1996).

The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian model selection based on the Bayes factor. In Section 3, for some case, we derive the reference prior. Using the reference prior, we provide the Bayesian testing procedure based on the fractional Bayes factor and intrinsic Bayes factors for testing for the equality of two gamma mean parameters. In Section 4, simulation study and a real example are given.

2. Intrinsic and Fractional Bayes Factors

Hypotheses H_1, $H_2, \cdots$, H_q are under consideration, with the data $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)$ under model $H_i, i = 1, 2, \cdots, q$. The parameter vectors $\boldsymbol{\theta}_i$ are unknown. Let $\pi_i(\boldsymbol{\theta}_i)$ be the prior distribution of model H_i, and let p_i be the prior probabilities of model H_i, $i = 1, 2, \cdots, q$. Then the posterior probability that the model H_i is true is

$$P(H_i \mid \boldsymbol{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{1}$$

where B_{ji} is the Bayes factor of model H_j to model H_i defined by

$$B_{ji} = \frac{\int f_j(\boldsymbol{x} \mid \boldsymbol{\theta}_j) \pi_j(\boldsymbol{\theta}_j) d\boldsymbol{\theta}_j}{\int f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i) \pi_i(\boldsymbol{\theta}_i) d\boldsymbol{\theta}_i} = \frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})}. \tag{2}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\boldsymbol{\theta}_i)$ and $\pi_j(\boldsymbol{\theta})$. Usually, one can use the noninformative prior such as uniform prior, Jeffreys prior or reference prior in Bayesian analysis. Denote it as π_i^N. The use of noninformative priors $\pi_i^N(\cdot)$ in (2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\boldsymbol{x}(l)$ denote the part of the data to be so used and let $\boldsymbol{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\boldsymbol{x}(l)) < \infty, i = 1, \cdots, q. \tag{3}$$

In view (3), the posteriors $\pi_i^N(\theta_i \mid \boldsymbol{x}(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $\boldsymbol{x}(-l)$, using $\pi_i^N(\theta_i \mid \boldsymbol{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int_{\Theta_j} f(\boldsymbol{x}(-l) \mid \theta_j, \boldsymbol{x}(l)) \pi_j^N(\theta_j \mid \boldsymbol{x}(l)) d\theta_j}{\int_{\Theta_i} f(\boldsymbol{x}(-l) \mid \theta_i, \boldsymbol{x}(l)) \pi_i^N(\theta_i \mid \boldsymbol{x}(l)) d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\boldsymbol{x}(l)) \tag{4}$$

where

$$B_{ji} = B_{ji}^N(\boldsymbol{x}) = \frac{m_j^N(\boldsymbol{x})}{m_i^N(\boldsymbol{x})} \quad \text{and} \quad B_{ij}^N(\boldsymbol{x}(l)) = \frac{m_i^N(\boldsymbol{x}(l))}{m_j^N(\boldsymbol{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\boldsymbol{x}$ and training samples $\boldsymbol{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\boldsymbol{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the Arithmetic Intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \cdot \frac{1}{L}\sum_{l=1}^{L} B_{ij}^N(\boldsymbol{x}(l)). \tag{5}$$

where L is the number of all possible minimal training samples. Also the Median Intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \cdot ME[B_{ij}^N(\boldsymbol{x}(l))], \tag{6}$$

where ME indicates the median, here to be taken over all the training sample Bayes factors. So we can also calculate the posterior probability of H_i using (1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (5) and (6).

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b, of each likelihood function, $L(\boldsymbol{\theta_i}) = f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of model H_j versus model H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int L^b(\boldsymbol{x} \mid \boldsymbol{\theta}_i)\pi_i^N(\boldsymbol{\theta}_i)d\boldsymbol{\theta}_i}{\int L^b(\boldsymbol{x} \mid \boldsymbol{\theta}_j)\pi_j^N(\boldsymbol{\theta}_j)d\boldsymbol{\theta}_j} = B_{ji}^N \cdot \frac{m_i^b(\boldsymbol{x})}{m_j^b(\boldsymbol{x})},$$

and $f_i(\boldsymbol{x} \mid \boldsymbol{\theta}_i)$ is the likelihood function and b specifies a fraction of the likelihood which is to be used as a prior density. He proposed three ways for the choice of the fraction b. One common choice of b is$b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. (see O'Hagan, 1995, 1997, and the discussion by Berger and Mortera of O'Hagan, 1995).

3. Bayesian Test Procedures

Let X be a gamma distribution with density function

$$f(x \mid \mu,\nu) = \left(\frac{\nu}{\mu}\right)^{\nu} \frac{x^{\nu-1}}{\Gamma(\nu)} \exp\{-\frac{\nu}{\mu}\mathrm{x}\},\ \ x>0,$$

where $\mu > 0$ is the mean parameter and $\nu > 0$ is the shape parameter. Suppose that $X_1,\cdots,X_{n_1}$ denote independent random samples from gamma distribution with the shape parameter ν_x and the mean μ_x, and $Y_1,\cdots,Y_{n_2}$ denote independent random samples from gamma distribution with the shape parameter ν_y and the mean μ_y. Then the joint probability density function is

$$f(\boldsymbol{x},\boldsymbol{y} \mid \mu_x,\mu_y,\nu) = \left(\frac{\nu_x}{\mu_x}\right)^{n_1\nu_x} \left[\prod_{i=1}^{n_1} \frac{x_i^{\nu_x-1}}{\Gamma(\nu_x)}\right] \exp\left\{-\sum_{\mathrm{i}=1}^{\mathrm{n}_1} \frac{\nu_{\mathrm{x}}}{\mu_{\mathrm{x}}}\mathrm{x}_{\mathrm{i}}\right\} \times \left(\frac{\nu_y}{\mu_y}\right)^{n_2\nu_y} \left[\prod_{i=1}^{n_2} \frac{y_i^{\nu_y-1}}{\Gamma(\nu_y)}\right] \exp\left\{-\sum_{\mathrm{i}=1}^{\mathrm{n}_2} \frac{\nu_{\mathrm{y}}}{\mu_{\mathrm{y}}}\mathrm{y}_{\mathrm{i}}\right\}, \tag{7}$$

where $\mu_x > 0, \mu_y > 0$, $\nu_x > 0$ and $\nu_y > 0$. We want to test the hypotheses $H_1 : \mu_x = \mu_y$ vs. $H_2 : \mu_x \neq \mu_y$. Our interest is to develop a Bayesian test procedure based on the fractional Bayes factor and the intrinsic Bayes factor under the noninformative prior.

3.1 Bayesian Test Procedure based on the Fractional Bayes Factor

We now derive the reference priors for different groups of ordering of (μ,ν_x,ν_y) under the hypothesis H_1. Reference priors introduced by Bernardo (1979), and extended further by Berger and Bernardo (1989,1992) have become very popular over the years for the development of noninformative priors.

Under H_1, the joint density is given by

$$f(\boldsymbol{x},\boldsymbol{y} \mid \mu,\nu_x,\nu_y) = \frac{\mu^{-(n_1\nu_x+n_2\nu_y)}\nu_x^{n_1\nu_x}\nu_y^{n_2\nu_y}}{\Gamma(\nu_x)^{n_1}\Gamma(\nu_y)^{n_2}} [\prod_{i=1}^{n_1} x_i]^{\nu_x-1} [\prod_{i=1}^{n_2} y_i]^{\nu_y-1} \times \exp\{-\frac{1}{\mu}[\sum_{\mathrm{i}=1}^{\mathrm{n}_1}\nu_{\mathrm{x}}\mathrm{x}_{\mathrm{i}} + \sum_{\mathrm{i}=1}^{\mathrm{n}_2}\nu_{\mathrm{y}}\mathrm{y}_{\mathrm{i}}]\}. \tag{8}$$

Based on (8), the Fisher information matrix is given by

$$\boldsymbol{I} = \begin{pmatrix} \frac{n_1\nu_x + n_2\nu_y}{\mu^2} & 0 & 0 \\ 0 & n_1[\psi'(\nu_x) - \nu_x^{-1}] & 0 \\ 0 & 0 & n_2[\psi'(\nu_y) - \nu_y^{-1}] \end{pmatrix},$$

where $\psi'(\cdot)$ is the trigamma function. From the above Fisher information matrix $\boldsymbol{I}$, μ, ν_x and ν_y are mutually orthogonal in the sense of Cox and Reid(1987). Then due to the orthogonality of the parameters, following Datta and Ghosh (1995), choosing rectangular compacts for each μ, μ_x and ν_y, the reference priors are given by as follows.

For the gamma populations (8), the reference prior distributions for group of ordering of $\{(\nu_x, \nu_y), \mu\}$ is

$$\pi_{R1}(\mu, \nu_x, \nu_y) \propto \mu^{-1}[\psi'(\nu_x) - \nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y) - \nu_y^{-1}]^{\frac{1}{2}}.$$

For group of ordering of $\{\nu_x, \nu_y, \mu\}$ and $\{\mu, \nu_x, \nu_y\}$, the reference prior is

$$\pi_{R2}(\mu, \nu_x, \nu_y) \propto \mu^{-1}[\psi'(\nu_x) - \nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y) - \nu_y^{-1}]^{\frac{1}{2}}.$$

Note that the two group reference prior and the one-at-a-time reference prior are the same. Thus the reference prior for the hypothesis H_1 is

$$\pi_1(\mu, \nu_x, \nu_y) = \mu^{-1}[\psi'(\nu_x) - \nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y) - \nu_y^{-1}]^{\frac{1}{2}},$$

where $\psi'(\cdot)$ is the trigamma function. The likelihood function under H_1 is

$$L(\mu, \nu_x, \nu_y \mid \boldsymbol{x}, \boldsymbol{y}) = \frac{\mu^{-(n_1\nu_x + n_2\nu_y)}\nu_x^{n_1\nu_x}\nu_y^{n_2\nu_y}}{\Gamma(\nu_x)^{n_1}\Gamma(\nu_y)^{n_2}}[\prod_{i=1}^{n_1} x_i]^{\nu_x - 1}[\prod_{i=1}^{n_2} y_i]^{\nu_y - 1}$$

$$\times \exp\{-\frac{1}{\mu}[\sum_{i=1}^{n_1}\nu_x x_i + \sum_{i=1}^{n_2}\nu_y y_i]\}.$$

Then the element of fractional Bayes factor under H_1 is given by

$$m_1^b(\boldsymbol{x},\boldsymbol{y}) = \int_0^\infty\int_0^\infty\int_0^\infty L^b(\mu,\nu_x,\nu_y \mid \boldsymbol{x},\boldsymbol{y})\pi_1(\mu,\nu_x,\nu_y)d\mu d\nu_x d\nu_y$$

$$= \int_0^\infty\int_0^\infty \frac{\Gamma(b(n_1\nu_x+n_2\nu_y))}{\Gamma(\nu_x)^{n_1 b}\Gamma(\nu_y)^{n_2 b}} \frac{[\prod_{i=1}^{n_1} x_i]^{b(\nu_x-1)}[\prod_{i=1}^{n_2} y_i]^{b(\nu_y-1)}}{[\sum_{i=1}^{n_1}\nu_x x_{i+}\sum_{i=1}^{n_2}\nu_y y_i]^{b(n_1\nu_x+n_2\nu_y)}}$$

$$\times b^{-b(n_1\nu_x+n_2\nu_y)}[\psi'(\nu_x)-\nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y)-\nu_y^{-1}]^{\frac{1}{2}}d\nu_x d\nu_y.$$

For the hypothesis H_2, the reference prior for μ_x, μ_y and ν is

$$\pi_2(\mu_x,\mu_y,\nu_x,\nu_y) = \mu_x^{-1}\mu_y^{-1}[\psi'(\nu_x)-\nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y)-\nu_y^{-1}]^{\frac{1}{2}}.$$

This reference prior is derived by Liseo (1993). The likelihood function under H_2 is

$$L(\mu_x,\mu_y,\nu_x,\nu_y \mid \boldsymbol{x},\boldsymbol{y}) = \frac{\mu_x^{-n_1\nu_x}\mu_y^{-n_2\nu_y}\nu_x^{n_1\nu_x}\nu_y^{n_2\nu_y}}{\Gamma(\nu_x)^{n_1}\Gamma(\nu_y)^{n_2}}[\prod_{i=1}^{n_1} x_i]^{\nu_x-1}[\prod_{i=1}^{n_2} y_i]^{\nu_y-1}$$

$$\times\exp\{-\frac{\nu_x}{\mu_x}\sum_{i=1}^{n_1} x_i\}\exp\{-\frac{\nu_y}{\mu_y}\sum_{i=1}^{n_2} y_i\}.$$

Thus the element of fractional Bayes factor under H_2 gives as follows.

$$m_2^b(\boldsymbol{x},\boldsymbol{y}) = \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty L^b(\mu_x,\mu_y,\nu_x,\nu_y \mid \boldsymbol{x},\boldsymbol{y})$$

$$\times\pi_2(\mu_x,\mu_y,\nu_x,\nu_y)d\mu_x d\mu_y d\nu_x d\nu_y$$

$$= \int_0^\infty\int_0^\infty \frac{\Gamma(n_1 b\nu_x)\Gamma(n_2 b\nu_y)}{\Gamma(\nu_x)^{n_1 b}\Gamma(\nu_y)^{n_2 b}} \frac{[\prod_{i=1}^{n_1} x_i]^{b(\nu_x-1)}[\prod_{i=1}^{n_2} y_i]^{b(\nu_y-1)}}{[\sum_{i=1}^{n_1} x_i]^{n_1 b\nu_x}[\sum_{i=1}^{n_2} y_i]^{n_2 b\nu_y}}$$

$$\times b^{-n_1 b\nu_x - n_2 b\nu_y}[\psi'(\nu_x)-\nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y)-\nu_y^{-1}]^{\frac{1}{2}}d\nu_x d\nu_y.$$

Therefore the B_{21}^N is given by

$$B_{21}^{N} = \frac{S_2(\boldsymbol{x},\boldsymbol{y})}{S_1(\boldsymbol{x},\boldsymbol{y})},$$

where

$$S_1(\boldsymbol{x},\boldsymbol{y}) = \int_0^\infty \int_0^\infty \frac{\Gamma((n_1\nu_x + n_2\nu_y))}{\Gamma(\nu_x)^{n_1}\Gamma(\nu_y)^{n_2}} \frac{[\prod_{i=1}^{n_1} x_i]^{(\nu_x-1)}[\prod_{i=1}^{n_2} y_i]^{(\nu_y-1)}}{[\sum_{i=1}^{n_1}\nu_x x_{i+}\sum_{i=1}^{n_2}\nu_y y_i]^{(n_1\nu_x+n_2\nu_y)}}$$

$$\times [\psi'(\nu_x) - \nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y) - \nu_y^{-1}]^{\frac{1}{2}} d\nu_x d\nu_y$$

and

$$S_2(\boldsymbol{x},\boldsymbol{y}) = \int_0^\infty \int_0^\infty \frac{\Gamma(n_1\nu_x)\Gamma(n_2\nu_y)}{\Gamma(\nu_x)^{n_1}\Gamma(\nu_y)^{n_2}} \frac{[\prod_{i=1}^{n_1} x_i]^{(\nu_x-1)}[\prod_{i=1}^{n_2} y_i]^{(\nu_y-1)}}{[\sum_{i=1}^{n_1} x_i]^{n_1\nu_x}[\sum_{i=1}^{n_2} y_i]^{n_2\nu_y}}$$

$$\times [\psi'(\nu_x) - \nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y) - \nu_y^{-1}]^{\frac{1}{2}} d\nu_x d\nu_y.$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\boldsymbol{x},\boldsymbol{y})}{m_2^b(\boldsymbol{x},\boldsymbol{y})} = \frac{S_1(\boldsymbol{x},\boldsymbol{y};b)}{S_2(\boldsymbol{x},\boldsymbol{y};b)},$$

where

$$S_1(\boldsymbol{x},\boldsymbol{y};b) = \int_0^\infty \int_0^\infty \frac{\Gamma(b(n_1\nu_x + n_2\nu_y))}{\Gamma(\nu_x)^{n_1 b}\Gamma(\nu_y)^{n_2 b}} \frac{[\prod_{i=1}^{n_1} x_i]^{b(\nu_x-1)}[\prod_{i=1}^{n_2} y_i]^{b(\nu_y-1)}}{[\sum_{i=1}^{n_1}\nu_x x_{i+}\sum_{i=1}^{n_2}\nu_y y_i]^{b(n_1\nu_x+n_2\nu_y)}}$$

$$\times b^{-b(n_1\nu_x+n_2\nu_y)}[\psi'(\nu_x) - \nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y) - \nu_y^{-1}]^{\frac{1}{2}} d\nu_x d\nu_y$$

and

$$S_2(\boldsymbol{x},\boldsymbol{y};b) = \int_0^\infty \int_0^\infty \frac{\Gamma(n_1 b\nu_x)\Gamma(n_2 b\nu_y)}{\Gamma(\nu_x)^{n_1 b}\Gamma(\nu_y)^{n_2 b}} \frac{[\prod_{i=1}^{n_1} x_i]^{b(\nu_x-1)}[\prod_{i=1}^{n_2} y_i]^{b(\nu_y-1)}}{[\sum_{i=1}^{n_1} x_i]^{n_1 b\nu_x}[\sum_{i=1}^{n_2} y_i]^{n_2 b\nu_y}}$$

$$\times b^{-n_1 b\nu_x - n_2 b\nu_y}[\psi'(\nu_x) - \nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_y) - \nu_y^{-1}]^{\frac{1}{2}} d\nu_x d\nu_y.$$

Thus the fractional Bayes factor of H_2 versus H_1 is given by

$$B_{21}^F = \frac{S_2(\boldsymbol{x},\boldsymbol{y};b)}{S_1(\boldsymbol{x},\boldsymbol{y};b)} \cdot \frac{S_1(\boldsymbol{x},\boldsymbol{y})}{S_2(\boldsymbol{x},\boldsymbol{y})}. \tag{9}$$

Note that the calculation of the fractional Bayes factor of H_2 versus H_1 requires two dimensional integration.

3.2 Bayesian Test Procedure based on the Intrinsic Bayes Factor

The element B_{21}^N of the intrinsic Bayes factor is computed in the derivation of fractional Bayes factor. So using minimal training sample, we only calculate the marginal densities under H_1 and H_2, respectively. The marginal density of $(X_i, X_j, X_k\, Y_l, Y_m, Y_n)$ is finite for all $1 \leqq i < j < k \leqq n_1, 1 \leqq l < m < n \leqq n_2$ under each hypothesis (see Liseo, 1993). Thus we conclude that any training sample of size six is a minimal training sample.

The marginal density $m_1^N(x_i,x_j,x_k,y_l,y_m,y_n)$ under H_1 is given by

$$\begin{aligned}
&m_1^N(x_i,x_j,x_k,y_l,y_m,y_n)\\
&= \int_0^\infty\int_0^\infty\int_0^\infty f(x_i,x_j,x_k,y_l,y_m,y_n \mid \mu,\nu_x,\nu_y)\pi_1(\mu,\nu)d\mu d\nu_x d\nu_y\\
&= \int_0^\infty\int_0^\infty \frac{\Gamma(3\nu_x+3\nu_y)}{\Gamma(\nu_x)^3\Gamma(\nu_y)^3}\frac{[x_ix_jx_k]^{\nu_x-1}[y_ly_my_n]^{\nu_y-1}}{[\nu_x(x_i+x_j+x_k)+\nu_y(y_l+y_m+y_n)]^{3(\nu_x+\nu_y)}}\\
&\quad \times[\psi'(\nu_x)-\nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_x)-\nu_x^{-1}]^{\frac{1}{2}}d\nu_x d\nu_y\\
&\equiv T_1(x_i,x_j,x_k,y_l,y_m,y_n),
\end{aligned}$$

where $1 \leqq i < j < k \leqq n_1, 1 \leqq l < m < n \leqq n_2$. And the marginal density $m_2^N(x_i,x_j,x_k,y_l,y_m,y_n)$ under H_2 is given by

$$\begin{aligned}
m_2^N(x_i,x_j,x_k,y_l,y_m,y_n) &= \int_0^\infty\int_0^\infty\int_0^\infty\int_0^\infty f(x_i,x_j,x_k,y_l,y_m,y_n \mid \mu_x,\mu_y,\nu_x,\nu_y)\\
&\quad \times\pi_2(\mu_x,\mu_y,\nu_x,\nu_y)d\mu_x d\mu_y d\nu_x d\nu_y\\
&= \int_0^\infty\int_0^\infty \frac{\Gamma(3\nu_x)\Gamma(3\mu_y)}{\Gamma(\nu_x)^3\Gamma(\nu_y)^3}\frac{[x_ix_jx_k]^{\nu_x-1}[y_ly_my_n]^{\nu_y-1}}{[x_i+x_j+x_k]^{3\nu_x}[y_l+y_m+y_n]^{3\nu_y}}\\
&\quad \times[\psi'(\nu_x)-\nu_x^{-1}]^{\frac{1}{2}}[\psi'(\nu_x)-\nu_x^{-1}]^{\frac{1}{2}}d\nu_x d\nu_y\\
&\equiv T_2(x_i,x_j,x_k,y_l,y_m,y_n).
\end{aligned}$$

Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{S_2(\boldsymbol{x},\boldsymbol{y})}{S_1(\boldsymbol{x},\boldsymbol{y})} \cdot \left[\frac{1}{L} \sum_{i,j,kl,m,n} \sum \frac{T_1(x_i,x_j,x_k,y_l,y_m,y_n)}{T_2(x_i,x_j,x_k,y_l,y_m,y_n)} \right]. \tag{10}$$

where $L = n_1(n_1-1)(n_1-2)n_2(n_2-1)(n_2-2)/36$. And the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(\boldsymbol{x},\boldsymbol{y})}{S_1(\boldsymbol{x},\boldsymbol{y})} \cdot ME\left[\frac{T_1(x_i,x_j,x_k,y_l,y_m,y_n)}{T_2(x_i,x_j,x_k,y_l,y_m,y_n)} \right]. \tag{11}$$

Note that the calculations of the AIBF and MIBF of H_2 versus H_1 require two dimensional integration. In Section 4, we investigate our testing procedures.

4. Numerical Studies

In order to assess the Bayesian test procedures, we evaluate the posterior probability for several configurations (μ_x,μ_y), (ν_x,ν_y) and (n_1,n_2). In particular, for fixed (μ_x,μ_y) and (ν_x,ν_y), we take 200 independent random samples of X and Y from the model (7). In our simulation, we examine the cases when $(\mu_x,\mu_y)=(1,1),(1,3),(1,5),(\nu_x,\nu_y)=(0.5,1),(0.5,3),(1,3)$ and $(n_1,n_2)=(5,5)$, $(5,10)$, $(10,10)$.

The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. Table 1 shows the results of the averages and the standard deviations in parentheses of posterior probabilities. From the Table 1, the fractional Bayes factor and the intrinsic Bayes factors give fairly reasonable answers. Also the fractional Bayes factor and intrinsic Bayes factors give similar results for all sample sizes.

Example 1. The data in Table 2 from Cameron and Pauling (1978) are survival times obtained from the data of first hospital attendance, of six women with terminal ovarian cancer who were treated with supplemental vitamin C. Along with each of these six survival times is the mean survival time of 10 individually matched controls. Booth, Hobert and Ohman (1999) analyzed this data by assuming the survival time of a vitamin C patient follow the gamma distribution with mean μ_y and shape ν, and the mean survival times of 10 independent control patients follows the gamma distribution with mean μ_x and shape 10ν.

Booth, Hobert and Ohman (1999) reported the 95% confidence interval for μ_y/μ_x is (0.102, 1.050). And the confidence intervals based on the signed log-likelihood ratio method and the modified signed log-likelihood ratio method by Wong, Wu and Sun (2004) are (0.141, 1.001) and (0.105, 1.032), respectively.

The value of fractional Bayes factor of H_2 versus H_1 is $B_{12}^F = 0.803$. We assume that the prior probabilities are equal. Then the posterior probability for H_2 is 0.555. Also the values of AIBF and MIBF of H_2 versus H_1 are $B_{12}^{AI} = 1.035$ and $B_{12}^{MI} = 0.838$, respectively. We assume that the prior probabilities are equal. Then the posterior probabilities for H_2 are 0.493 and 0.544, respectively. Thus there is slightly evidence for H_2 in terms of the posterior probability.

<Table 1> The averages and the standard deviations in parentheses of posterior probabilities

(ν_x,ν_y)	(μ_x,μ_y)	(n_1,n_2)	$P^F(H_1 \mid \boldsymbol{x},\boldsymbol{y})$	$P^{AI}(H_1 \mid \boldsymbol{x},\boldsymbol{y})$	$P^{MI}(H_1 \mid \boldsymbol{x},\boldsymbol{y})$
0.5, 1.0	1,1	5, 5	0.552 (0.082)	0.571 (0.092)	0.571 (0.084)
		5,10	0.609 (0.133)	0.587 (0.124)	0.585 (0.115)
		10,10	0.616 (0.136)	0.645 (0.145)	0.644 (0.140)
	1,3	5, 5	0.482 (0.114)	0.492 (0.125)	0.495 (0.113)
		5,10	0.490 (0.174)	0.471 (0.153)	0.478 (0.137)
		10,10	0.372 (0.201)	0.387 (0.214)	0.397 (0.208)
	1,5	5, 5	0.408 (0.143)	0.412 (0.152)	0.421 (0.140)
		5,10	0.398 (0.194)	0.387 (0.168)	0.400 (0.153)
		10,10	0.239 (0.172)	0.245 (0.182)	0.261 (0.179)
0.5, 3.0	1,1	5, 5	0.556 (0.077)	0.580 (0.080)	0.578 (0.074)
		5,10	0.615 (0.130)	0.588 (0.111)	0.584 (0.106)
		10,10	0.629 (0.121)	0.667 (0.117)	0.665 (0.113)
	1,3	5, 5	0.470 (0.107)	0.491 (0.108)	0.482 (0.103)
		5,10	0.490 (0.173)	0.474 (0.140)	0.472 (0.134)
		10,10	0.367 (0.208)	0.396 (0.217)	0.405 (0.214)
	1,5	5, 5	0.386 (0.138)	0.406 (0.139)	0.397 (0.133)
		5,10	0.364 (0.177)	0.377 (0.141)	0.374 (0.134)
		10,10	0.196 (0.152)	0.218 (0.163)	0.235 (0.170)
1.0, 3.0	1,1	5, 5	0.559 (0.077)	0.587 (0.069)	0.583 (0.066)
		5,10	0.621 (0.109)	0.607 (0.088)	0.601 (0.085)
		10,10	0.629 (0.122)	0.671 (0.110)	0.663 (0.109)
	1,3	5, 5	0.420 (0.134)	0.451 (0.126)	0.444 (0.121)
		5,10	0.387 (0.179)	0.410 (0.144)	0.412 (0.136)
		10,10	0.230 (0.186)	0.265 (0.196)	0.269 (0.190)
	1,5	5, 5	0.314 (0.125)	0.338 (0.123)	0.338 (0.118)
		5,10	0.277 (0.162)	0.307 (0.131)	0.314 (0.123)
		10,10	0.085 (0.089)	0.097 (0.102)	0.104 (0.104)

<Table 2> Survival Times (Days) of Ovarian Cancer Patients

Patient pair	1	2	3	4	5	6
Mean survival time of matched controls (x)	307	690	285	244	371	368
Supplemental vitamin C (y)	1234	89	201	356	2970	456

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