Journal of the Korean
Data & Information Science Society
2006, Vol. 17, No. 4, pp. 1397 ~ 1403

Simulation Study on the Scale Change Test for Autoregressive Models with Heavy-Tailed Innovations[1)]

Siyun Park[2)] · Sangyeol Lee[3)]

Abstract

This paper considers the testing problem for scale changes in autoregressive processes with heavy-tailed innovations. For a test, we propose the CUSUM test statistic based on the trimmed residuals. We perform a simulation study for the mixture normal and Cauchy innovations.

Keywords : CUSUM test, Heavy-tailed innovations, Infinite variance process, Normal mixture distribution, Scale change

1. Introduction

In this paper, we consider the problem of testing scale changes in heavy-tailed autoregressive processes. It is well known that the data observed in finance and computer networking follow heavy-tailed distributions. A typical example of this phenomenon can be found in high frequency financial and network traffic data. The techniques for the analysis of the heavy tailed data have been developed for decades and well summarized in Samorodnitsky and Taqqu (1994). Adler, Feldman and Taqqu (1998) proposed statistical techniques and applications based on their theoretical background. In particular, Calder and Davis (1998) considered the parameter estimation in stable ARMA time series.

---

1) This work was supported by grant No. R01-2006-000-10545-0 from the Basic Research Program of the Korea Science and Engineering Foundation.

2) Post doctor, Department of Statistics, Seoul National University, San 56-1, Sillim-dong, Gwanak-gu, Seoul 151-742, Korea.
Email : siyun.park@gmail.com

3) Professor, Department of Statistics, Seoul National University, San 56-1, Sillim-dong, Gwanak-gu, Seoul 151-742, Korea.
Email : sylee@stats.snu.ac.kr

Since Hsu, Miller and Wichern (1974) modeled stock returns by using normal probability model with changing variances, many articles are devoted to detecting variance changes in time series. For instance, Tsay (1988) studied ARMA models making allowance for outliers and variance changes. Inclán and Tiao (1994) proposed the following test statistic, IT_n below for variance changes in iid sample as a centered version of the CUSUM test statistic proposed by Brown, Durbin and Evans (1975):

$$IT_n = \max_{1 \le k \le n} \sqrt{\frac{n}{2}} \left| \frac{\sum_{j=1}^{k} \epsilon_j^2}{\sum_{j=1}^{n} \epsilon_j^2} - \frac{k}{n} \right|, \tag{1.1}$$

where $\epsilon_j \sim iid\, N(0, \sigma^2)$. They showed that the test statistic IT_n has the same limiting distribution of $\sup_{0 \le t \le 1} |B^0(t)|$, where B^0 denotes a Brownian bridge. They also presented the algorithm for the detection of the multiple variance changes and the location in which the variance change occurs. Lee and Park (2001) extended Inclán and Tiao's method to infinite order moving average processes, and proposed a CUSUM test based on the squares of the trimmed observations, which is applicable to the processes contaminated by outliers. According to their study, the CUSUM test does not perform well in highly correlated processes. For this reason, Park, Lee and Jeon (2000) adopted the approach of using the AR(∞) model to discard the effects caused by correlations. However, their paper does not cover the infinite variance case, so here we study the CUSUM test for autoregressive processes with infinite variance.

In this paper, we study the performance of the CUSUM test within the frame work of autoregressive models with normal mixture and Cauchy innovations, and demonstrate that the CUSUM test statistic based on the trimmed residuals work adequately for those models. This paper is organized as follows. In Section 2, we propose the test statistic and illustrate how the test works for mixture normal process and infinite variance process. In Section 3, we present the simulation results on the performance of our test.

2. CUSUM Test for Scale Changes

In this section, we introduce the CUSUM test for scale changes in autoregressive models with heavy-tailed innovations. Let us consider the stationary AR(q) model satisfying the difference equation:

$$X_t - \sum_{j=1}^{q} \beta_j X_{t-j} = \epsilon_t, \tag{2.1}$$

where ϵ_t are iid random variables with a common distribution function F. The heavy-tailed distribution under consideration in this paper is as follows:

CASE 1. F is a mixture normal distribution with the density of the form

$$f = \sum_{i=1}^{K} \lambda_i f_i, \text{ where } f_i \text{ denotes the pdf of } N(\mu_i, \sigma_i^2), \text{ and } \sum_{i=1}^{K} \lambda_i = 1. \tag{2.2}$$

CASE 2. F is an α-stable distribution satisfying

$$x^{\alpha} P(|\epsilon_t| > x) \to C \quad \text{as } x \to \infty, \tag{2.3}$$

where $0 < \alpha < 2$ and C is a finite positive constant,

In order to construct the CUSUM test, we obtain the LSE

$$\hat{\underline{\beta}}^{LS} = \left(\sum_{t=q+1}^{n} X_{t-1} X'_t \right)^{-1} \sum_{t=q+1}^{n} X_{t-1} X_t$$

for $\underline{\beta} = (\beta_1, \cdots, \beta_q)'$, and calculate the residuals $\hat{\epsilon}_t = X_t - \hat{\underline{\beta}}^{LS} X_{t-1}$. Lee, Park and Jeon (2000) considered the CUSUM test in autoregressive models with finite variance. However, their method cannot be directly applicable to the infinite variance case. Further, even if the variance exists, the heavy-tailed distribution highly damages the performance of the CUSUM test (cf. Lee and Park (2001)). Therefore, we follow the approach to use trimmed residuals.

For $u \in (0,1)$, let ξ_u be a number such that $F(\xi_u) = u$. Provided $\epsilon_t, \cdots, \epsilon_n$ are given, if we set

$$\begin{aligned} \xi_{nu} &= \epsilon_{(n,[nu])}, && \text{if } nu \text{ is an integer,} \\ &= \epsilon_{(n,[nu]+1)}, && \text{if } nu \text{ is not an integer,} \end{aligned}$$

where $\epsilon_{(nu)}$ denote the ordered r.v.s of $\epsilon_1, \cdots, \epsilon_n$, and $[x]$ is the largest integer not exceeding x, ξ_{nu} is the empirical quantile estimator for ξ_u. However, since the true errors are unobservable, we replace them by the residuals and analogously define the residual empirical quantiles $\hat{\xi}_{nu}$. Based on the trimmed residuals $\hat{u}_t^2 = \hat{\epsilon}_t^2 I(\hat{\xi}_{nu} \le \hat{\epsilon}_t \le \hat{\xi}_{n\nu})$, we construct the CUSUM test in analogy of IT_n:

$$T_n = \max_{q+1 \le k \le n} \frac{\sqrt{n-q}\,\hat{\sigma}^2}{\hat{\tau}} \left| \frac{\sum_{t=1}^{k} \hat{u}_t^2}{\sum_{t=1}^{k} \hat{u}_t^2} - \frac{k-q}{n-q} \right| , \tag{2.4}$$

where $\hat{\sigma}^2 = \frac{1}{n-q} \sum_{t=q+1}^{n} \hat{u}_t^2$ and $\hat{\tau}^2 = \frac{1}{n-q} \sum_{t=q+1}^{n} \hat{u}_t^4 - (\hat{\sigma}^2)^2$.

A large value of T_n indicates a scales change in time series. Given significance level α, the critical value c_α is obtained by the equation $P(BB > c_\alpha) = \alpha$, where $BB = \sup_{0 \le t \le 1} |B^0(t)|$. For instance, $c_{0.05}$=1.358. In this paper, we do not provide the proof for the weak convergence of T_n to BB since we focus on the performance itself of the CUSUM test. We leave this as the task of future study.

3. Simulation Result and Discussion

In this section, we evaluate the performance of T_n proposed in (2.4) through a simulation study. For this task, we examine the performance of T_n with the trimming portions u=0.05 and ν=0.95 for the AR(1) process with heavy tailed distributions. We set up the null and alternative hypotheses:

$$H_0 : X_t = \phi X_{t-1} + \epsilon_t, \quad t = 1, \cdots, n, \tag{3.1}$$

and

$$H_1 : \begin{cases} X_t = \phi X_{t-1} + \epsilon_t, & t = 1, \cdots, [\theta n] \\ X_t = \phi X_{t-1} + \delta \epsilon_t, & t = [\theta n] + 1, \cdots, n, \ 0 < \theta < 1. \end{cases} \tag{3.2}$$

Here, we focus on the two cases mentioned in (2.2) and (2.3) as follows:

$$X_t = \phi X_{t-1} + \epsilon_t, \ \epsilon_t \sim \lambda N(0,1) + (1-\lambda) N(0, \sigma_v^2), \tag{3.3}$$

and

$$X_t = \phi X_{t-1} + \epsilon_t, \ \epsilon_t \sim Cauchy(0,1) \ . \tag{3.4}$$

Empirical sizes and powers are calculated by the proportions of the number of rejections of the null hypothesis (3.1) out of 1000 repetitions. The empirical powers are obtained for θ = 0.25, 0.5, 0.75. In each simulation, 100 initial observations are discarded to remove initialization effects.

Table 1 presents the empirical sizes and powers for the model in (3.3) with ϕ = 0.1, 0.5, 0.9, λ=0.9, and σ_v^2 = 25, 100 for n = 200, 300 and 500. We employ δ= 0.5, 1.8 and 2.0 to investigate the empirical powers. As seen in the table, the sizes are stable even when the observations are highly correlated. When the change occurs in the middle of the series, the test has the best powers as expected. It is also shown that the test tends to perform better when σ_v^2 = 100 than when σ_v^2 = 25. Further, it can be seen that the test can detect the changes well when n is larger than 300.

<Table 1> The empirical sizes and powers of T_n with u=0.05, ν=0.95 for (3.3)

σ_v^2 = 0.5						σ_v^2 = 100					
			ϕ						ϕ		
	n	θ	0.1	0.5	0.9		n	θ	0.1	0.5	0.9
size	200		0.044	0.051	0.049	size	200		0.031	0.045	0.036
	300		0.044	0.049	0.052		300		0.030	0.043	0.045
	500		0.051	0.038	0.045		500		0.042	0.047	0.048
power δ=0.5	200	0.25	0.553	0.569	0.547	power δ=0.5	200	0.25	0.710	0.673	0.686
		0.50	0.902	0.904	0.904			0.50	0.848	0.861	0.848
		0.75	0.674	0.679	0.680			0.75	0.616	0.615	0.622
	300	0.25	0.713	0.728	0.689		300	0.25	0.859	0.871	0.863
		0.50	0.980	0.980	0.976			0.50	0.954	0.960	0.961
		0.75	0.897	0.925	0.896			0.75	0.844	0.851	0.843
	500	0.25	0.879	0.861	0.888		500	0.25	0.985	0.978	0.968
		0.50	0.999	0.999	1.000			0.50	0.999	0.994	0.999
		0.75	0.991	0.999	0.993			0.75	0.979	0.969	0.974
power δ=1.8	200	0.25	0.478	0.454	0.435	power δ=1.8	200	0.25	0.429	0.446	0.411
		0.50	0.768	0.754	0.758			0.50	0.718	0.746	0.754
		0.75	0.439	0.447	0.437			0.75	0.544	0.543	0.585
	300	0.25	0.712	0.690	0.718		300	0.25	0.733	0.716	0.748
		0.50	0.922	0.901	0.918			0.50	0.901	0.934	0.922
		0.75	0.552	0.583	0.569			0.75	0.748	0.756	0.769
	500	0.25	0.945	0.959	0.951		500	0.25	0.950	0.944	0.953
		0.50	0.990	0.989	0.987			0.50	0.994	0.993	0.991
		0.75	0.741	0.749	0.764			0.75	0.942	0.951	0.937
power δ=2.0	200	0.25	0.596	0.616	0.624	power δ=2.0	200	0.25	0.570	0.552	0.546
		0.05	0.884	0.876	0.891			0.05	0.837	0.817	0.851
		0.75	0.574	0.531	0.536			0.75	0.681	0.690	0.655
	300	0.25	0.882	0.879	0.861		300	0.25	0.817	0.833	0.808
		0.50	0.986	0.981	0.984			0.50	0.958	0.960	0.943
		0.75	0.708	0.727	0.685			0.75	0.861	0.866	0.875
	500	0.25	0.989	0.993	0.994		500	0.25	0.984	0.977	0.971
		0.50	1.000	1.000	1.000			0.50	0.999	0.994	0.998
		0.75	0.889	0.876	0.867			0.75	0.984	0.984	0.985

Table 2 presents the empirical sized and powers for the AR(1) model with Cauchy innovations with ϕ=0.1, 0.5, and 0.8 in (3.4) for n = 300, 500, 700, and 1000. Similarly to the previous case, we obtained reasonably good results. However, we should point out that the performance in this case is not so satisfactory as the previous case. To compensate for the power loss, larger sample sizes are required. All these results enable us to conclude that the CUSUM test performs adequately for heavy-tailed autoregressive models.

<Table 2 > The empirical sizes and powers of T_n with u=0.05, ν=0.95 for (3.3)

			ϕ						ϕ		
	n	θ	0.1	0.5	0.8		n	θ	0.1	0.5	0.8
size	300		0.045	0.053	0.053	size	700		0.039	0.043	0.053
	500		0.048	0.051	0.049		1000		0.039	0.043	0.048
power δ=0.5	300	0.25	0.282	0.338	0.326	power δ=3.0	300	0.25	0.415	0.426	0.549
	300	0.50	0.462	0.467	0.439		300	0.50	0.793	0.792	0.895
	300	0.75	0.191	0.201	0.231		300	0.75	0.637	0.644	0.759
	500	0.25	0.478	0.462	0.488		500	0.25	0.722	0.737	0.824
	500	0.50	0.664	0.658	0.689		500	0.50	0.956	0.961	0.987
	500	0.75	0.361	0.364	0.384		500	0.75	0.846	0.855	0.931
	700	0.25	0.607	0.611	0.629		700	0.25	0.872	0.863	0.953
	700	0.50	0.826	0.814	0.814		700	0.50	0.996	0.994	0.999
	700	0.75	0.521	0.509	0.538		700	0.75	0.944	0.952	0.977
	1000	0.25	0.790	0.806	0.772		1000	0.25	0.973	0.976	0.988
	1000	0.50	0.947	0.945	0.933		1000	0.50	0.999	1.000	1.000
	1000	0.75	0.720	0.719	0.701		1000	0.75	0.993	0.992	1.000
power δ=2.0	300	0.25	0.178	0.181	0.224	power δ=4.0	300	0.25	0.598	0.637	0.637
	300	0.50	0.433	0.452	0.477		300	0.50	0.940	0.938	0.938
	300	0.75	0.309	0.311	0.330		300	0.75	0.802	0.830	0.830
	500	0.25	0.357	0.345	0.372		500	0.25	0.903	0.890	0.890
	500	0.50	0.651	0.664	0.665		500	0.50	0.963	0.995	0.995
	500	0.75	0.474	0.460	0.512		500	0.75	0.973	0.964	0.964
	700	0.25	0.480	0.486	0.472		700	0.25	1.000	0.978	0.978
	700	0.50	0.806	0.815	0.788		700	0.50	1.000	1.000	1.000
	700	0.75	0.588	0.604	0.601		700	0.75	0.995	0.999	0.999
	1000	0.25	0.703	0.730	0.711		1000	0.25	0.998	0.999	0.999
	1000	0.50	0.934	0.943	0.923		1000	0.50	1.000	1.000	1.000
	1000	0.75	0.777	0.779	0.780		1000	0.75	1.000	1.000	1.000

References

1. Adler, R. J., Feldman, R. E. and Taqqu, M. S. (1998). *A Practical Guide to Heavy Tails: Statistical Techniques and Applications.* Birkhäuser, Boston.
2. Brockwell, P. J. and Davis, R.A. (1991). *Time Series: Theory and Methods, 2nd edition.* Springer, New York.
3. Brown, R. L., Durbin, J., and Evans, J. M. (1975). *Journal of the Royal Statistical Society, Ser B, 37, 149-163.*
4. Calder, M. and Davis, R.A. (1998). *Inference for Linear Processes with Stable Noise, A practical Guide to Heavy Tails: Statistical Techniques and Applications (Adler, R., Feldman, R., and Taqqu, M., editors)* Birkhäuser*, Boston, 159--176.*
5. Hsu, D. A., Miller, R. B., and Wichern, D. W. (1974). On the Stable Paretian Behavior of Stock Market Prices. (1974). *Journal of the American Statistical Association,* 76, 108-113.
6. Inclán, C. and Tiao, G. (1994). Use of Cumulative Sums of Squares for Retrospective Detection of Changes of Variance, *Journal of the American Statistical Association,* 89, 913-923.
7. Lee, S. and Park, S.. (2001). The Cusum of Squares Test for Scale Changes in Infinite Order Moving Average Processes, *Scandinavian Journal of Statistics, 29*, 625-644.
8. Park, S., Lee, S. and Jeon, J. (2000). The Cusum of Squares Test for Variance Changes in Infinite Order Autoregressive Models, *Journal of the Korean Statistical Society, 29*, 351-360.
9. Samorodnitsky, G. and Taqqu, M. S. (1994). *Stable Non-Gaussian Random Processes : Stochastic Models with Infinite Variance.* Chapman and Hill, New York.
10. Tsay, R. Outliers, Level Shifts, and Variance Changes in Time Series. (1988). *Journal of Forecasting,* 7, 1-20.

[received date : Sep. 2006, accepted date : Nov. 2006]