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Bayesian Hypothesis Testing for the Ratio of Exponential Means

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Abstract

This paper considers testing for the ratio of two exponential means. We propose a solution based on a Bayesian decision rule to this problem in which no subjective input is considered. The criterion for testing is the Bayesian reference criterion (Bernardo, 1999). We derive the Bayesian reference criterion for testing the ratio of two exponential means. Simulation study and a real data example are provided.

Keywords : Bayesian reference criterion, Intrinsic discrepancy loss, Ratio of exponential means, Reference prior

1. Introduction

Consider two independent random samples $(X_{11},\cdots,X_{1n})$ and $(X_{21},\cdots,X_{2n})$, where the X_{1j} are independent and identically exponentially distributed $\varepsilon(\lambda)$ with mean λ and the X_{2j} are independent and identically exponentially distributed $\varepsilon(\eta\lambda)$ with mean $\eta\lambda$. The parameter of interest is η, the ratio of two exponential means. Our goal of this paper is to test the null hypothesis $H_1:\eta=\eta_0$ versus the alternative $H_2:\eta\neq\eta_0$.

In Bayesian model selection or hypothesis testing problems, the Bayes factor

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under proper priors has been shown very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' priors or reference priors (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. To avoid this difficulty several alternative approaches have been proposed: the Schwarz (1978) asymptotic approximation, namely the Bayesian information criterion; the fractional Bayes factor and the intrinsic Bayes factor proposed by O'Hagan (1995) and Berger and Pericchi (1996). For criticisms and comparisons of these methods relevant references include Berger and Pericchi (1997), De Santis and Spezzaferri (1999), and O'Hagan (1995,1997).

On the other hand, Bernardo(1999) and Bernardo and Rueda (2002) introduced a new Bayesian hypothesis testing as decision problem. They justified the choice of a particular continuous invariant difference loss function, the intrinsic discrepancy loss. This is combined with reference analysis to propose an attractive Bayesian solution to the problem of hypothesis testing, defined as the problem of deciding whether or not available data are statistically compatible with the hypothesis that the parameters of the model belong to some subset of the parameter space. That is, to decide whether or not some data $\boldsymbol{x}$ are compatible with the null hypothesis $H_0 : \Theta = \Theta_0$, they computes the reference posterior expectation of the intrinsic discrepancy loss which could be loss if the null hypothesis were used. This provides an attractive non-negative test function which is invariant under reparametrization. The corresponding Bayes decision rule, the Bayesian reference creterion (BRC), indicates that the null should only be rejected if the posterior expected loss of information from using the null is too large.

The comparison of two exponential distributions is often important in statistical analyses of lifetime data. In particular, the problem of estimating the ratio of exponential means is one-to-one problem of estimating the reliability of a exponential stress-strength system. The estimation of ratio of two exponential means was given by Lawless (1982) and Cox and Reid (1987). Mukerjee and Dey (1993) derived the matching prior. Using the orthogonal parametrization, Datta and Ghosh (1995) derived the reference prior. Recently, Kim and Chung (2004) proposed the Bayesian reference criterion for testing the equality of two exponential parameters.

In this paper, we construct an objective Bayesian testing procedure based on the BRC. The outline of the remaining sections are as follows. In Section 2, a brief summary of the BRC is given and we derive expression for the BRC to solve our problem. In Section 3, we provide an example and simulation study for illustration.

2. Bayesian Hypothesis Testing

2.1 Bayesian Reference Criterion

Let $\{p(\boldsymbol{x} \mid \Theta, \omega, \Theta \in \Theta, \omega \in \Omega)\}$, be a statistical model which is assumed to have been generated some data $\boldsymbol{x} \in X$, and consider a precise value $\Theta = \Theta_0$ among those which remain possible after $\boldsymbol{x}$ has been observed. Here Θ is considered to be the vector of interest and ω be the vector of nuisance parameters.

To decide whether or not the precise value Θ_0 may be used as a proxy for the unknown value of Θ,

(i) compute the intrinsic discrepancy $\delta(\Theta_0, \Theta, \omega)$;

$$\delta(\Theta_0, \Theta, \omega) = \min_{\omega_0 \in \Omega} \delta\{p(\boldsymbol{x} \mid \Theta, \omega), p(\boldsymbol{x} \mid \Theta_0, \omega_0)\},$$

where $\delta\{p_1(\boldsymbol{x}), p_2(\boldsymbol{x})\} = \min\left\{ \int p_1(\boldsymbol{x}) \log \frac{p_1(\boldsymbol{x})}{p_2(\boldsymbol{x})} dx, \int p_2(\boldsymbol{x}) \log \frac{p_2(\boldsymbol{x})}{p_1(\boldsymbol{x})} dx \right\}.$

Intrinsic discrepancy $\delta(\Theta_0, \Theta, \omega)$ between $p(\boldsymbol{x} \mid \Theta, \omega)$ and $p(\boldsymbol{x} \mid \Theta_0, \omega_0)$ is the intrinsic discrepancy between the assumed probability density $p(\boldsymbol{x} \mid \Theta, \omega)$ and its closest approximation with Θ_0.

(ii) derive the corresponding reference posterior expectation $d(\Theta_0, \boldsymbol{x})$,

$$d(\Theta_0, \boldsymbol{x}) = \int_\Theta \int_\Omega \delta(\Theta_0, \Theta, \omega) \pi_\delta(\Theta, \omega \mid \boldsymbol{x}) \, d\Theta \, d\omega,$$

where $\pi_\delta(\Theta, \omega \mid \boldsymbol{x})$ is the posterior distribution which corresponds to the δ-reference prior $\pi_\delta(\Theta, \omega)$ and state this number as a measure of evidence against the null hypothesis $H_0 : \Theta = \Theta_0$.

(iii) If a formal decision is required, reject the null if, and only if, $d(\Theta_0, \boldsymbol{x}) > d^*$, for some context dependent d^*. The values $d^* \approx 1.0$ (no evidence against the null), $d^* \approx 2.5$ (mild evidence against the null) and $d^* \approx 5.0$ (significant evidence against the null) may conveniently be used for scientific communication.

2.2 Bayesian Reference Criterion for Testing the Ratio of Two Exponential Means

We consider two independent random samples $(X_{11},\cdots,X_{1n})$ and $(X_{21},\cdots,X_{2n})$, where the X_{1j} are independent and identically exponentially distributed $\varepsilon(\lambda)$ with mean λ and the X_{2j} are independent and identically exponentially distributed $\varepsilon(\eta\lambda)$ with mean $\eta\lambda$. Then the joint probability distribution of $X_{11},\cdots,X_{1n}$ and $X_{21},\cdots,X_{2n}$ is

$$p(\boldsymbol{x}\mid\lambda,\eta)=\lambda^{-2n}\eta^{-n}\exp\left\{-\frac{n\overline{x}_1}{\lambda}-\frac{n\overline{x}_2}{\eta\lambda}\right\}, \tag{1}$$

where $\boldsymbol{x}=(x_{11},\cdots,x_{1n},x_{21},\cdots,x_{2n})$, $\overline{x}_1=\sum_{j=1}^{n}x_{1j}/n$, $\overline{x}_2=\sum_{j=1}^{n}x_{2j}/n$, $\lambda>0$ and $\eta>0$. The parameter η is the ratio of two exponential means. The reference prior (Datta and Ghosh, 1995) for η is given by

$$\pi(\lambda,\eta)\propto\eta^{-1}\lambda^{-1}. \tag{2}$$

We want to test whether or not the value $\eta=\eta_0$ is compatible with those observations. To derive the BRC, firstly, we compute the directed logarithmic divergence of $p(\boldsymbol{x}\mid\eta_0,\lambda_0)$ from $p(\boldsymbol{x}\mid\eta,\lambda)$, $k(\eta_0,\lambda_0\mid\eta,\lambda)$, is given by

$$\begin{aligned}k(\eta_0,\lambda_0\mid\eta,\lambda) &= \int_0^{\infty}\cdots\int_0^{\infty}p(\boldsymbol{x}\mid\eta,\lambda)\log\frac{p(\boldsymbol{x}\mid\eta,\lambda)}{p(\boldsymbol{x}\mid\eta_0,\lambda_0)}\,dx_{11}\cdots dx_{2n}\\ &= n\left[\log\frac{\eta_0\lambda_0^2}{\eta\lambda^2}+\frac{\lambda}{\lambda_0}+\frac{\eta\lambda}{\eta_0\lambda_0}-2n\right].\end{aligned}$$

This is minimized when $\lambda_0=\dfrac{\lambda(\eta_0+\eta)}{2\eta_0}$, to yield

$$\inf_{\lambda\in R^+}k(\eta_0,\lambda_0\mid\eta,\lambda) = n\left[\log\frac{(\eta_0+\eta)^2}{4\eta_0\eta}\right].$$

Also the directed logarithmic divergence of $p(\boldsymbol{x}\mid\eta,\lambda)$ from $p(\boldsymbol{x}\mid\eta_0,\lambda_0)$, $k(\eta_0,\lambda_0\mid\eta,\lambda)$, is given by

$$k(\eta,\lambda \mid \eta_0,\lambda_0) = \int_0^\infty \cdots \int_0^\infty p(\boldsymbol{x} \mid \eta_0,\lambda_0)\log\frac{p(\boldsymbol{x} \mid \eta_0,\lambda_0)}{p(\boldsymbol{x} \mid \eta,\lambda)}\,dx_{11}\cdots dx_{2n}$$
$$= n\left[\log\frac{\eta\lambda^2}{\eta_0\lambda_0^2} + \frac{\lambda_0}{\lambda} + \frac{\eta_0\lambda_0}{\eta\lambda} - 2n\right].$$

This is minimized when $\lambda_0 = \dfrac{2\eta\lambda}{\eta_0 + \eta}$, to yield

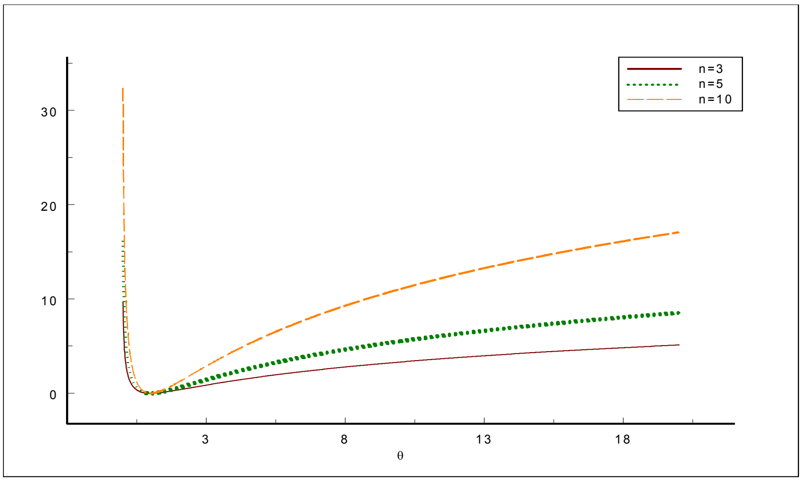
$$\inf_{\lambda\in R^+} k(\eta,\lambda \mid \eta_0,\lambda_0) = n\left[\log\frac{(\eta_0+\eta)^2}{4\eta_0\eta}\right].$$

Therefore the intrinsic discrepancy loss $\delta(\eta_0,\eta,\lambda)$ is given by

$$\begin{aligned}\delta(\eta_0,\eta,\lambda) &= \min_{\lambda_0\in\Lambda}\delta\{p(\boldsymbol{x} \mid \eta,\lambda), p(\boldsymbol{x} \mid \eta_0,\lambda_0)\}\\ &= n\left[\log\frac{(\eta_0+\eta)^2}{4\eta_0\eta}\right] \qquad (3)\\ &= n\left[\log\frac{(1+\Theta)^2}{4\Theta}\right],\end{aligned}$$

where $\Theta = \eta_0/\eta$. Thus the $\delta(\eta_0,\eta,\lambda)$ only depends on the ratio $\Theta = \eta_0/\eta$. Figure 1 represents the intrinsic loss function (3), as a function Θ, for several values of n. As one might expect, $\delta(\eta_0,\eta,\lambda)$ increases with $\Theta<1$ and $\Theta>1$.

Next we compute reference posterior of η when δ is the quantity of interest. From (2) the reference prior for η is $\pi(\eta,\lambda) \propto \eta^{-1}\lambda^{-1}$.



<Figure 1> Intrinsic Discrepancy Loss as a function of $\Theta = \eta_0/\eta$

Moreover, the quantity δ is a piecewise invertible function of η. So the joint reference prior when δ is the parameter of interest is

$$\pi_\delta(\eta,\lambda) \propto \eta^{-1}\lambda^{-1}. \tag{4}$$

From the likelihood function (1) and the reference prior (4), the joint posterior for η and λ is given by

$$\pi_\delta(\eta,\lambda \mid \boldsymbol{x}) \propto \lambda^{-2n-1}\eta^{-n-1}\exp\left\{-\frac{n\,\overline{x}_1}{\lambda}-\frac{n\,\overline{x}_2}{\eta\lambda}\right\},$$

where $\overline{x}_1 = \sum_{j=1}^{n} x_{1j}/n$ and $\overline{x}_2 = \sum_{j=1}^{n} x_{2j}/n$. This posterior is proper if $n \geq 1$. Therefore the reference posterior expectation, test statistic, is

$$\begin{aligned} d(\eta_0, \boldsymbol{x}) &= \int_0^\infty \int_0^\infty \delta(\eta_0,\eta,\lambda)\pi_\delta(\eta,\lambda \mid \boldsymbol{x})\,d\eta\, d\lambda \\ &= n\,C\int_0^\infty \left[\log\frac{(1+\Theta)^2}{4\Theta}\right]\Theta^{n-1}(\eta_0\,\overline{x}_1+\Theta\,\overline{x}_2)^{-2n}d\Theta, \end{aligned}$$

where $C = 2^{2n-1}[\eta_0\,\overline{x}_1\,\overline{x}_2]^n\Gamma(n+1/2)/[\Gamma(n)\Gamma(1/2)]$. The exact value of test statistic can easily be found by one dimensional numerical integration.

3. Numerical Studies

We consider the hypothesis testing problem for the ratio of exponential means under the reference prior for several configurations, (η,λ) and n using BRC. This is done numerically. For our simulation, we take $\lambda = 1$.

To see the performance of tests by BRC for the hypothesis $H_0{:}\eta=\eta_0$ against $H_1{:}\eta\neq\eta_0$ for fixed (η,λ) and n, we take 100,000 independent random samples of $\boldsymbol{X}$ from the model (1). In particular, six samples of sizes, 2, 5, 10, 20, 30 and 50 are taken. Table 1, 2 and 3 give correspondence between the threshold value d^* for test statistic $d(\eta_0, \boldsymbol{x})$, and type I error probabilities, $P[d > d^* \mid H_0]$. Here threshold value takes over the range of values from 1 to 9. For the cases presented in Table 1, 2, and 3, we see that type I error probabilities are smaller as sample size increases. Note that there is no anymore a one-to-one correspondence between d^*-values and significance levels; indeed, our procedure recommends rejecting the null whenever $d>5$, which implies type I error probabilities of 0,0029 0.0020, 0.0023, 0.0025, 0.0021 and 0.0026, when the sample size is 2, 5, 10, 20, 30 and 50 with Table 1, respectively.

<Table 1> Correspondence between the Threshold Value d^* of the Test Statistic, $d(\eta=0.1, \boldsymbol{x})$ and Type I Error Probabilities, $P[d>d^* \mid H_0]$

d^*	$n=2$	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$
1	0.32784	0.31846	0.31714	0.31838	0.31773	0.31734
2	0.08163	0.08068	0.08192	0.08216	0.08335	0.08164
2.42	0.04888	0.04748	0.04830	0.04959	0.04992	0.04982
3	0.02477	0.02379	0.02334	0.02465	0.02506	0.02490
4	0.00849	0.00723	0.00757	0.00768	0.00852	0.00829
5	0.00279	0.00256	0.00244	0.00251	0.00256	0.00257
6	0.00109	0.00084	0.00080	0.00098	0.00098	0.00093
7	0.00038	0.00030	0.00020	0.00029	0.00026	0.00027
9	0.00000	0.00001	0.00003	0.00004	0.00005	0.00005

<Table 2> Correspondence between the Threshold Value d^* of the Test Statistic, $d(\eta=1.0, \boldsymbol{x})$ and Type I Error Probabilities, $P[d>d^* \mid H_0]$

d^*	$n=2$	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$
1	0.32815	0.32167	0.31529	0.31824	0.32071	0.31640
2	0.08144	0.08056	0.08161	0.08128	0.08120	0.08185
2.42	0.04893	0.04798	0.04789	0.04877	0.04880	0.05056
3	0.02501	0.02400	0.02456	0.02469	0.02467	0.02494
4	0.00824	0.00754	0.00692	0.00825	0.00819	0.00825
5	0.00311	0.00289	0.00236	0.00248	0.00261	0.00239
6	0.00117	0.00082	0.00073	0.00077	0.00094	0.00104
7	0.00026	0.00036	0.00032	0.00034	0.00028	0.00025
9	0.00000	0.00005	0.00004	0.00003	0.00006	0.00004

<Table 3> Correspondence between the Threshold Value d^* of the Test Statistic, $d(\eta=10.0, \boldsymbol{x})$ and Type I Error Probabilities, $P[d>d^* \mid H_0]$

d^*	$n=2$	$n=5$	$n=10$	$n=20$	$n=30$	$n=50$
1	0.32925	0.31983	0.32031	0.31831	0.31747	0.31799
2	0.08119	0.08157	0.08094	0.08328	0.08086	0.08326
2.42	0.04833	0.04736	0.04905	0.04906	0.04944	0.04998
3	0.02490	0.02419	0.02395	0.02409	0.02500	0.02426
4	0.00769	0.00731	0.00776	0.00752	0.00785	0.00782
5	0.00305	0.00238	0.00232	0.00274	0.00261	0.00258
6	0.00120	0.00088	0.00095	0.00083	0.00097	0.00089
7	0.00040	0.00027	0.00036	0.00043	0.00033	0.00034
9	0.00000	0.00004	0.00000	0.00007	0.00001	0.00002

Example 1. The following data, given by Lawless (1982), are failure times (in minutes) for two types of electrical insulation in which the insulation was subjected to an increasing voltage stress. The original dataset is assumed to have two parameter exponential distributions. We subtracted from the data to the MLE for the location parameter. Thus we may assume that the transformed data follow one parameter exponential distributions heuristically. For the following data, Kim and Kim (2000) derived the intrinsic priors for testing two exponential means with the fractional Bayes factor.

Type A ($\boldsymbol{x_1}$) : 9.5, 58.4, 12.1, 126.3, 139.6, 63.0, 83.2, 85.8, 30.9, 16.3, 34.6

Type B ($\boldsymbol{x_2}$) : 200.8, 60.9, 67.5, 131.7, 3.2, 103.4, 22.0, 128.6, 16.6, 23.8, 30.2

The sample means are $\overline{x}_1 = 55.0$ ($n_1 = 11$) and $\overline{x}_2 = 65.7$ ($n_2 = 11$).

We want to test $H_1 : \eta = 1$ versus $H_2 : \eta \neq 1$. The the reference posterior expectation, test statistic, is $d = 0.594\ (<1)$. Thus the BRC criterion suggests that the null hypothesis, $H_1 : \eta = 1$, is accepted. Also the Bayes factor with fractional prior for testing $H_1 : \mu_1 = \mu_2$ versus $H_1 : \mu_1 \neq \mu_2$ is $B^I_{21} = 0.280$ (Kim and Kim, 2000). Therefore the Bayes factor and the BRC criterion give the same conclusion and we may conclude that there is little difference between the two types of electrical insulation.

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