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Parameter Estimation of the Two-Parameter Exponential Distribution under Three Step-Stress Accelerated Life Test

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Abstract

In life testing, the lifetimes of test units under the usual conditions are so long that life testing at usual conditions is impractical. Testing units are subjected to conditions of high stress to yield informations quickly. In this paper, the inferences of parameters on the three step-stress accelerated life testing are studied. The two-parameter exponential distribution with a failure rate function that a log-quadratic function of stress and the tempered failure rate model are considered. We obtain the maximum likelihood estimators of the model parameters and their confidence regions. A numerical example will be given to illustrate the proposed inferential procedures.

Keywords : Accelerated life test, Confidence region, Maximum likelihood estimator, Tampered failure rate model, Three step-stress, Two-parameter exponential distribution

1. Introduction

In many reliability studies, the life testing were made under environment conditions. But for extremely reliable units, it is in general impossible to make life testing at use stress because the lifetimes of test units at use stress tend to be long and then the testing time may be very long. As a common approach to avoid this problem, the accelerated life testings(ALTs) are widely used. Testing units

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are subjected to conditions of greater stress than use stress and then accelerated life testing quickly yields information on test unit.

Widely used methods of applying stress to test units are constant-stress test, step-stress test, varying-stress test. In step-stress testing, the stress on unfailed units is allowed to change at preassigned times or upon occurrences of specified number of failures until they fail. In an ordinary step-stress ALTs, the test units are simultaneously put on a stress x_1, and run until a preassigned time or upon occurrences of fixed number of failures and the failure times of those failing in this interval are observed. And then at the change point, surviving units are subjected to the different(stronger in general) stress x_2 , and so on.

There are three types of models that have been commonly used on analysis of step-stress ALTs. They are the tampered random variable(TRV) model by DeGroot and Goel(1979), the cumulative exposure(CE) model by Nelson(1980), the tampered failure rate(TFR) model by Bhattacharyya and Soejoeti(1989).

Meeker(1984) discussed the design for Type I censored constant stress ALTs. Nelson(1980, 1990) presented the cumulative exposure model and studied the design to determine the optimal stress change time for two-step stress ALTs. Bai, Kim and Lee(1989a, 1989b) extended their results to the case of Type Ⅰ censoring. Bai and Chun(1991) studied optimum plan searching change time that minimizes the sum of asymptotic variances of maximum likelihood estimators of the log mean lifetime at the usual condition for Type Ⅰ censored two-step stress ALTs. Bai and Chung(1992) studied two optimal designs for two-step stress and constant stress partially ALTs under the tampered random variable model and compared their performances. Moon(2004a, 2004b) studied the optimal designs for M-level constant-stress ALTs with k-stress variables and the optimal design for M-level constant-stress ALTs with a polynomial stress model under Weibull distribution. Wu(2002) considered parameter estimations for the two step-stress ALTs considering cumulative exposure model for a two-parameter exponential distribution with Type Ⅱ censoring.

In this paper, we consider the parameter estimation for three step-stress accelerated life tests, assuming that the lifetime of test units follows a two-parameter exponential distribution under the tampered failure rate model. In section 2, we describe the model and some necessary assumptions. Maximum likelihood estimators of the parameters are obtained in section 3. The confidence regions for the parameters are derived in section 4. In section 5, the proposed inferential procedures are illustrated with a simulated data set.

2. Model and Assumptions

For three step-stress ALTs, all test units are simultaneously run on use stress x_1 until a preassigned time τ_1, but if all units do not fail before time τ_1, the surviving units are subjected to a stronger stress x_2 and observed until time τ_2. For still functioning units at time τ_2, the stress is also changed to a more stronger stress x_3 and units are observed until all units are failed.

By changing stresses at preassigned times, the failure rate functions at stresses x_2 and x_3 are assumed to be expressed as the initial failure rate function $1/\Theta_1$ multiplied by unknown factors α_1 and α_2 . Acceleration factors α_1 and α_2 depend on stresses and possibly τ_1 and τ_2 . Also α_1 and α_2 will be greater than 1 because the effect of changing stresses to the higher level is to subject test units to greater failure conditions.

Some useful notations in constructing maximum likelihood estimators are introduced as follows.

(1) n_i is the number of failed units at the stress x_i , $i=1,2,3$.

(2) T_{ij} is the lifetime of a unit at stress x_i , $j=1,2,\cdots,n_i$.

Suppose that the failure rate function of each test unit has the log-quadratic relationship with the stress variable x_i , which is given by

$$\log\frac{1}{\Theta_i}=\beta_0+\beta_1 x_i+\beta_2 x_i^2\ ,\quad i=1,2,3 \tag{1}$$

where β_0 , β_1 and β_2 are unknown parameters.

By the TFR model under three step-stress ALTs, the distribution function of lifetime which follows a two-parameter exponential distribution is given by

$$\overline{F}(t)=\begin{cases}\exp\left(-\dfrac{t-\mu}{\Theta_1}\right) & \text{if } t\le\tau_1\ ,\\ \exp\left(-\dfrac{\tau_1-\mu}{\Theta_1}-\dfrac{t-\tau_1}{\Theta_2}\right) & \text{if } \tau_1<t\le\tau_2\ ,\\ \exp\left(-\dfrac{\tau_1-\mu}{\Theta_1}-\dfrac{\tau_2-\tau_1}{\Theta_2}-\dfrac{t-\tau_2}{\Theta_3}\right) & \text{if } \tau_2<t\ .\end{cases}$$

Hence, the corresponding probability density function of lifetime is obtained by

$$f(t)=\begin{cases}\dfrac{1}{\Theta_1}\exp\left(-\dfrac{t-\mu}{\Theta_1}\right) & \text{if } t\le\tau_1\ ,\\ \dfrac{1}{\Theta_2}\exp\left(-\dfrac{\tau_1-\mu}{\Theta_1}-\dfrac{t-\tau_1}{\Theta_2}\right) & \text{if } \tau_1<t\le\tau_2\ ,\\ \dfrac{1}{\Theta_3}\exp\left(-\dfrac{\tau_1-\mu}{\Theta_1}-\dfrac{\tau_2-\tau_1}{\Theta_2}-\dfrac{t-\tau_2}{\Theta_3}\right) & \text{if } \tau_2<t\ .\end{cases} \tag{2}$$

3. Maximum Likelihood Estimators of Parameters

Suppose that $T_{11}<T_{12}<\cdots<T_{1n_1}<T_{21}<\cdots<T_{2n_2}<T_{31}<\cdots<T_{3n_3}$ are the lifetimes of the test units observed to fail at stress x_i , $i=1,2,3$, where $n=n_1+n_2+n_3$. Thus, the likelihood function for T_{ij} , $j=1,2,\cdots,n_i$, $i=1,2,3$ is given by

$$\begin{aligned} L(\Theta_1,\Theta_2,\Theta_3)= & \frac{1}{\Theta_1^{n_1}}\frac{1}{\Theta_2^{n_2}}\frac{1}{\Theta_3^{n_3}}\exp\left[-\frac{1}{\Theta_1}\left(\sum_{j=1}^{n_1}t_{ij}+(n-n_1)\tau_1-n\mu\right)\right] \\ & \times\exp\left[-\frac{1}{\Theta_2}\left(\sum_{j=1}^{n_2}t_{2j}+(n-n_1-n_2)\tau_2-(n-n_1)\tau_1\right)\right] \\ & \times\exp\left[-\frac{1}{\Theta_3}\left(\sum_{j=1}^{n_3}t_{3j}-(n-n_1-n_2)\tau_2\right)\right], \quad t_{ij}\ge\mu \end{aligned} \tag{3}$$

where $n_i>0$, $i=1,2,3$.

Since $\mu\le t_{11}<t_{12}<\cdots<t_{1n_1}<t_{21}<\cdots<t_{2n_2}<t_{31}<\cdots<t_{3n_3}$, the MLE for μ is $\hat{\mu}=T_{11}$. Substituting $\hat{\mu}$ for μ and (1) for Θ_1 and Θ_2 in (3), the loglikelihood function is a function of unknown parameters β_0 , β_1 and β_2 given by as follows:

$$\begin{aligned} \log L(\beta_0,\beta_1,\beta_2)= & n_1(\beta_0+\beta_1x_1+\beta_2x_1^2)+n_2(\beta_0+\beta_1x_2+\beta_2x_2^2) \\ & +n_3(\beta_0+\beta_1x_3+\beta_2x_3^2)-U_1\exp(\beta_0+\beta_1x_1+\beta_2x_1^2) \\ & -U_2\exp(\beta_0+\beta_1x_2+\beta_2x_2^2)-U_3(\beta_0+\beta_1x_3+\beta_2x_3^2), \end{aligned}$$

where $$U_1=\sum_{j=1}^{n_1}t_{1j}+(n-n_1)\tau_1-n\hat{\mu}, \tag{4}$$

$$U_2=\sum_{j=1}^{n_2}t_{2j}+(n-n_1-n_2)\tau_2-(n-n_1)\tau_1, \tag{5}$$

and

$$U_3=\sum_{j=1}^{n_3}t_{3j}-(n-n_1-n_2)\tau_2. \tag{6}$$

Differentiating the loglikelihood function in (3) for β_0 , β_1 and β_2 and letting $\frac{\partial}{\partial\beta_i}\log L(\beta_0,\beta_1,\beta_2)=0$, $i=0,1,2$, we can get the MLEs for β_0 , β_1 and β_2 as follows:

$$\begin{aligned} \hat{\beta}_0= & \frac{x_2x_3}{(x_3-x_1)(x_2-x_1)}\log\frac{n_1}{U_1}-\frac{x_2x_3}{(x_3-x_1)(x_2-x_1)}\log\frac{n_2}{U_2} \\ & +\frac{x_1x_2}{(x_3-x_1)(x_3-x_2)}\log\frac{n_3}{U_3}, \end{aligned}$$

$$\hat{\beta}_1 = -\frac{x_2 + x_3}{(x_3 - x_1)(x_2 - x_1)}\log\frac{n_1}{U_1} + \frac{x_1 + x_3}{(x_3 - x_2)(x_2 - x_1)}\log\frac{n_2}{U_2} - \frac{x_1 + x_2}{(x_3 - x_1)(x_3 - x_2)}\log\frac{n_3}{U_3},$$

$$\hat{\beta}_2 = \frac{1}{(x_3 - x_1)(x_2 - x_1)}\log\frac{n_1}{U_1} - \frac{1}{(x_3 - x_2)(x_2 - x_1)}\log\frac{n_2}{U_2} + \frac{1}{(x_3 - x_1)(x_3 - x_2)}\log\frac{n_3}{U_3}.$$

4. Confidence Regions

The joint confidence regions for parameters μ, β_0, β_1 and β_2 are given in this section. Let the random variables Y be defined as

$$Y = \begin{cases} \dfrac{T-\mu}{\theta_1}, & \mu \le T < \tau_1, \\ \dfrac{\tau_1-\mu}{\theta_1} + \dfrac{T-\tau_1}{\theta_2}, & \tau_1 \le T < \tau_2, \\ \dfrac{\tau_1-\mu}{\theta_1} + \dfrac{\tau_2-\tau_1}{\theta_2} + \dfrac{T-\tau_2}{\theta_3}, & \tau_2 \le T < \infty, \end{cases} \tag{7}$$

where T has the probability density function in (2).

If $y \le \dfrac{\tau_1-\mu}{\theta_1}$, then

$$P(Y \le y) = P(T \le \mu + \theta_1 y) = \int_{\mu}^{\mu+\theta_1 y} \frac{1}{\theta_1} e^{-\frac{t-\mu}{\theta_1}} dt = 1 - e^{-y}.$$

If $\dfrac{\tau_1-\mu}{\theta_1} < y \le \dfrac{\tau_2-\tau_1}{\theta_2}$, then

$$\begin{aligned} P(Y \le y) &= P(\mu \le T \le \tau_1) + P\left(\tau_1 < T \le \tau_1 + \theta_2\left(y - \frac{\tau_1-\mu}{\theta_1}\right)\right) \\ &= \int_{\mu}^{\tau_1} \frac{1}{\theta_1} e^{-\frac{t-\mu}{\theta_1}} dt + \int_{\tau_1}^{\tau_1+\theta_2\left(y-\frac{\tau_1-\mu}{\theta_1}\right)} \frac{1}{\theta_2} e^{-\frac{\tau_1-\mu}{\theta_1} - \frac{t-\tau_1}{\theta_2}} dt \\ &= 1 - e^{-y}. \end{aligned}$$

If $y > \dfrac{\tau_2-\tau_1}{\theta_2}$, then

$$
\begin{aligned}
P(Y\le y) &= P(\mu\le T\le \tau_1)+P(\tau_1<T\le\tau_2)+P\left(T\le \tau_2+\Theta_3\left(y-\frac{\tau_2-\tau_1}{\Theta_2}-\frac{\tau_1-\mu}{\Theta_1}\right)\right)\\
&= \int_{\mu}^{\tau_1}\frac{1}{\Theta_1}e^{-\frac{t-\mu}{\Theta_1}}dt+\int_{\tau_1}^{\tau_2}\frac{1}{\Theta_2}e^{-\frac{\tau_1-\mu}{\Theta_1}-\frac{t-\tau_1}{\Theta_2}}dt\\
&\quad+\int_{\tau_2}^{\tau_2+\Theta_3\left(y-\frac{\tau_2-\tau_1}{\Theta_2}-\frac{\tau_1-\mu}{\Theta_1}\right)}\frac{1}{\Theta_3}e^{-\frac{\tau_1-\mu}{\Theta_1}-\frac{\tau_2-\tau_1}{\Theta_2}-\frac{t-\tau_2}{\Theta_3}}dt\\
&= 1-e^{-y}.
\end{aligned}
$$

We can see that the random variable Y defined in (7) has an exponential distribution with mean 1.

To derive the joint confidence regions for the parameters μ , β_0 , β_1 and β_2 , the following lemma is necessary.

Lemma 1. Let $Y_{(1)}$, $Y_{(2)}$, $\cdots$, $Y_{(r)}$ be the first r ordered observations of a size n random sample from the exponential distribution with mean 1. Let $D=\sum_{i=1}^{r}Y_{(i)}+(n-r)Y_{(r)}-nY_{(1)}$. Then $Y_{(1)}$ and D are independent, and $2nY_{(1)}$ and $2D$ are distributed as $\chi^2_{(2)}$ and $\chi^2_{(2r-2)}$, respectively.

Proof. Let $Z_1=nY_{(1)}$ and $Z_i=(n-i+1)(Y_{(i)}-Y_{(i-1)})$, $i=2.3.\cdots,r$. Since

$$W=\sum_{i=1}^{r}Y_{(i)}+(n-r)Y_{(r)}=\sum_{i=1}^{r}Z_i$$

and the Jacobian is $\frac{(n-r)!}{n!}$, the joint probability density function of $Z_1,Z_2,\cdots,Z_r$ is given by

$$f(z_1,z_2,\cdots,z_r)=\exp\left(-\sum_{i=1}^{r}z_i\right) .$$

Then $Z_1,Z_2,\cdots,Z_r$ are independent and identically distributed as an exponential distribution with mean 1, and $2W$ has a chi-square distribution with $2r$ d.f. Hence $2nY_{(1)}=2Z_1\sim\chi^2_{(2)}$ and $2D=2W-2Z_1=\sum_{i=2}^{r}Z_i\sim\chi^2_{(2r-2)}$.

The joint confidence region for μ and β_0 , the joint confidence region for μ and β_1 , the joint confidence region for μ and β_2 are given in next three theorems.

Let $F_{\alpha(v_1,v_2)}$ be the upper α percentage point of the F distribution wit v_1 and v_2 degrees of freedom(d.f.) and let $\chi^2_{\alpha(v)}$ be the upper α percentage point of the chi-square distribution with v degrees of freedom.

Theorem 1. Suppose that T_{ij} , $j=1,2,\cdots,n_i$, $i=1,2,3$ are the n ordered failure times of a size n sample from a distribution with the probability density function in (2). Then for any $0<\alpha<1, n_1>0, n_2>0$ and $n_3>0$, a $(1-\alpha)\times 100\%$ joint confidence region for μ and β_2 are given as follows.

$$\Bigg[\ \hat{\mu}-\frac{n_1\hat{\theta}_1}{n(n_1-1)}F_{\frac{\alpha}{8}(2,2n_1-2)}<\mu<\hat{\mu}-\frac{n_1\hat{\theta}_1}{n(n_1-1)}F_{1-\frac{\alpha}{8}(2,2n_1-2)},$$

$$\frac{1}{(x_3-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{2U_3}\left(\chi^2_{1-\frac{\alpha}{8}(2n-2)}-\chi^2_{\frac{\alpha}{8}(2r-2)}\right)\right)-\log\left(\frac{\chi^2_{\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)$$

$$-\frac{1}{(x_2-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{\frac{\alpha}{8}(2r-2)}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{1-\frac{\alpha}{8}(2)}\right)\right)-\log\left(\frac{\chi^2_{1-\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)$$

$$<\ \beta_2\ <$$

$$\frac{1}{(x_3-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{2U_3}\left(\chi^2_{\frac{\alpha}{8}(2n-2)}-\chi^2_{1-\frac{\alpha}{8}(2r-2)}\right)\right)-\log\left(\frac{\chi^2_{1-\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)$$

$$-\frac{1}{(x_2-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{1-\frac{\alpha}{8}(2r-2)}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{\frac{\alpha}{8}(2)}\right)\right)-\log\left(\frac{\chi^2_{\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)\Bigg],$$

where U_1, U_2 and U_3 are defined in (4), (5) and (6), and $r=n_1+n_2$, $\hat{\mu}=T_{11}$ and $\hat{\theta}_1=\frac{1}{n_1}\left[\sum_{j=1}^{n_1}T_{1j}+(n-n_1)T_{1n_1}-nT_{11}\right]$, respectively.

Proof. Let the order statistic $Y_{(j)}$, $j=1,2,\cdots,n$ be defined as follows: $Y_{(j)}=\frac{T_{1j}-\mu}{\theta_1}$, $j=1,2,\cdots,n_1$, $Y_{(n_1+j)}=\frac{\tau_1-\mu}{\theta_1}+\frac{T_{2j}-\tau_1}{\theta_2}$, $j=1,2,\cdots,n_2$ and $Y_{(n_2+j)}=\frac{\tau_1-\mu}{\theta_1}+\frac{\tau_2-\tau_1}{\theta_2}+\frac{T_{3j}-\tau_2}{\theta_3}$, $j=1,2,\cdots,n_3$, where $n=n_1+n_2+n_3$. Then $Y_{(1)}<Y_{(2)}<\cdots<Y_{(n)}$ are the n order statistics from an exponential distribution with mean 1. From Lemma 1, $2nY_{(1)}=\frac{2n(\hat{\mu}-\mu)}{\theta_1}$ has a chi-square distribution with 2 d.f. Let $D_1=\frac{U_1}{\theta_1}+\frac{U_2}{\theta_2}$ and $D_2=\frac{U_1}{\theta_1}+\frac{U_2}{\theta_2}+\frac{U_3}{\theta_3}$. Then $D_1=\sum_{i=1}^{r}Y_{(i)}+(n-r)Y_{(r)}-nY_{(1)}$, where $r=n_1+n_2$ and $D_2=\sum_{i=1}^{n}Y_{(i)}-nY_{(1)}$, and hence, $2D_1$ has a chi-square distribution with $(2r-2)$ d.f. and $2D_2$ has a chi-square distribution with $(2n-2)$ d.f. from Lemma 1, and $Y_{(1)}$ and D_1 , D_2 are independent. Furthermore, we only consider the ordered observations $T_{11}, T_{12}, \cdots, T_{1n_1}$ at stress x_1 and we can treat these observations as a Type-Ⅱ censored sample

from an exponential distribution with parameters μ and Θ_1. By Lawless(1982), $\frac{n(n_1-1)(\hat{\mu}-\mu)}{n_1\hat{\Theta}_1}$ has a F distribution with 2 and $(2n_1-2)$ d.f., where $\hat{\mu}=T_{11}$ and $\hat{\Theta}_1=\frac{1}{n_1}[\sum_{j=1}^{n_{1j}}T_{1j}+(n-n_1)T_{1n_1}-nT_{11}]$.

Now, we are going to derive a joint confidence region for μ and β_2 with confidence coefficient at least $(1-\alpha)$, where $0<\alpha<1$. By Bonferroni's inequality(Ross(1976)), we have

$$
\begin{aligned}
1-\alpha \le P\Bigg(& F_{1-\frac{\alpha}{8}(2,2n_1-2)} < \frac{n(n_1-1)(\hat{\mu}-\mu)}{n_1\hat{\Theta}_1} < F_{\frac{\alpha}{8}(2,2n_1-2)}, \\
& \chi^2_{1-\frac{\alpha}{8}(2)} < \frac{2n(\hat{\mu}-\mu)}{\Theta_1} < \chi^2_{\frac{\alpha}{8}(2)},\ \chi^2_{1-\frac{\alpha}{8}(2r-2)} < 2\left(\frac{U_1}{\Theta_1}+\frac{U_2}{\Theta_2}\right) < \chi^2_{\frac{\alpha}{8}(2r-2)}, \\
& \chi^2_{1-\frac{\alpha}{8}(2n-2)} < 2\left(\frac{U_1}{\Theta_1}+\frac{U_2}{\Theta_2}+\frac{U_3}{\Theta_3}\right) < \chi^2_{\frac{\alpha}{8}(2n-2)}\Bigg) \\
\le P\Bigg(& \hat{\mu}-\frac{n_1\hat{\Theta}_1}{n(n_1-1)}F_{\frac{\alpha}{8}(2,2n_1-2)} < \mu < \hat{\mu}-\frac{n_1\hat{\Theta}_1}{n(n_1-1)}F_{1-\frac{\alpha}{8}(2,2n_1-2)}, \\
& \log\left(\frac{\chi^2_{1-\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right) < \beta_0+\beta_1x_1+\beta_2x_1^2 < \log\left(\frac{\chi^2_{\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right), \\
& \log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{1-\frac{\alpha}{8}(2r-2)}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{\frac{\alpha}{8}(2)}\right)\right) < \beta_0+\beta_1x_2+\beta_2x_2^2 < \\
& \log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{\frac{\alpha}{8}(2r-2)}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{1-\frac{\alpha}{8}(2)}\right)\right), \\
& \log\left(\frac{1}{2U_3}\left(\chi^2_{1-\frac{\alpha}{8}(2n-2)}-\chi^2_{\frac{\alpha}{8}(2r-2)}\right)\right) < \beta_0+\beta_1x_3+\beta_2x_3^2 < \\
& \log\left(\frac{1}{2U_3}\left(\chi^2_{\frac{\alpha}{8}(2n-2)}-\chi^2_{1-\frac{\alpha}{8}(2r-2)}\right)\right)\Bigg) \\
\le P\Bigg(& \hat{\mu}-\frac{n_1\hat{\Theta}_1}{n(n_1-1)}F_{\frac{\alpha}{8}(2,2n_1-2)} < \mu < \hat{\mu}-\frac{n_1\hat{\Theta}_1}{n(n_1-1)}F_{1-\frac{\alpha}{8}(2,2n_1-2)}, \\
& \frac{1}{(x_3-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{2U_3}\left(\chi^2_{1-\frac{\alpha}{8}(2n-2)}-\chi^2_{\frac{\alpha}{8}(2r-2)}\right)\right)-\log\left(\frac{\chi^2_{\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right) \\
& -\frac{1}{(x_2-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{\frac{\alpha}{8}(2r-2)}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{1-\frac{\alpha}{8}(2)}\right)\right)-\log\left(\frac{\chi^2_{1-\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right) \\
& < \beta_2 < \\
& \frac{1}{(x_3-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{2U_3}\left(\chi^2_{\frac{\alpha}{8}(2n-2)}-\chi^2_{1-\frac{\alpha}{8}(2r-2)}\right)\right)-\log\left(\frac{\chi^2_{1-\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right) \\
& -\frac{1}{(x_2-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{1-\frac{\alpha}{8}(2r-2)}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{\frac{\alpha}{8}(2)}\right)\right)-\log\left(\frac{\chi^2_{\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)\Bigg).
\end{aligned}
$$

This completes the proof.

Theorem 2. Suppose that T_{ij}, $j=1,2,\cdots,n_i$, $i=1,2,3$ are the n ordered failure times of a size n sample from a distribution with the probability density function in (2). Then for any $0<\alpha<1, n_1>0, n_2>0$ and $n_3>0$, the $(1-\alpha)\times 100\%$ joint confidence regions for μ and β_1 are given as follows.

$$\Bigg[\hat{\mu}-\frac{n_1\hat{\theta}_1}{n(n_1-1)}F_{\frac{\alpha}{8}(2,2n_1-2)} < \mu < \hat{\mu}-\frac{n_1\hat{\theta}_1}{n(n_1-1)}F_{1-\frac{\alpha}{8}(2,2n_1-2)},$$

$$\frac{x_1+x_3}{(x_2-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{1-\frac{\alpha}{8}(2r-2}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{\frac{\alpha}{8}(2)}\right)\right)-\log\left(\frac{\chi^2_{\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)$$

$$-\frac{x_1+x_2}{(x_3-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{2U_3}\left(\chi^2_{\frac{\alpha}{8}(2n-2)}-\chi^2_{1-\frac{\alpha}{8}(2r-2)}\right)\right)-\log\left(\frac{\chi^2_{1-\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)$$

$$< \beta_1 <$$

$$-\frac{x_1+x_3}{(x_2-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{\frac{\alpha}{8}(2r-2)}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{1-\frac{\alpha}{8}(2)}\right)\right)-\log\left(\frac{\chi^2_{1-\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)$$

$$-\frac{x_1+x_2}{(x_3-x_1)(x_3-x_2)}\left(\log\left(\frac{1}{2U_3}\left(\chi^2_{1-\frac{\alpha}{8}(2n-2)}-\chi^2_{\frac{\alpha}{8}(2r-2)}\right)\right)-\log\left(\frac{\chi^2_{\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)\Bigg],$$

where U_1, U_2 and U_3 are defined in (4), (5) and (6), and $r=n_1+n_2$, $\hat{\mu}=T_{11}$ and $\hat{\theta}_1=\frac{1}{n_1}\left[\sum_{j=1}^{n_1}T_{1j}+(n-n_1)T_{1n_1}-nT_{11}\right]$, respectively.

Proof. From the proof of Theorem 1, $2nY_{(1)}=\frac{2n(\hat{\mu}-\mu)}{\theta_1}$ is distributed as $\chi^2_{(2)}$, $2D_1$ is distributed as $\chi^2_{(2r-2)}$, $2D_2$ is distributed as $\chi^2_{(2n-2)}$ and $\frac{n(n_1-1)(\hat{\mu}-\mu)}{n_1\hat{\theta}_1}$ is distributed as $F_{(2,2n_1-2)}$, where D_1 and D_2 are defined in Theorem 1. By Bonferroni's inequality(Ross(1976)), a joint confidence region for μ and β_1 with confidence coefficient at least $(1-\alpha)$, where $0<\alpha<1$, is derived.

Theorem 3. Suppose that T_{ij}, $j=1,2,\cdots,n_i$, $i=1,2,3$ are the n ordered failure times of a size n sample from a distribution with the probability density function in (2). Then for any $0<\alpha<1, n_1>0, n_2>0$ and $n_3>0$, the $(1-\alpha)\times 100\%$ joint confidence regions for μ and β_0 are given as follows.

$$\Bigg[\hat{\mu}-\frac{n_1\hat{\theta}_1}{n(n_1-1)}F_{\frac{\alpha}{8}(2,2n_1-2)} < \mu < \hat{\mu}-\frac{n_1\hat{\theta}_1}{n(n_1-1)}F_{1-\frac{\alpha}{8}(2,2n_1-2)},$$

$$\frac{x_2}{(x_3-x_1)(x_3-x_2)}\left(x_1\log\left(\frac{1}{2U_3}\left(\chi^2_{1-\frac{\alpha}{8}(2n-2)}-\chi^2_{\frac{\alpha}{8}(2r-2)}\right)\right)-x_3\log\left(\frac{\chi^2_{\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)$$

$$-\frac{x_3}{(x_2-x_1)(x_3-x_2)}\left(x_1\log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{\frac{\alpha}{8}(2r-2)}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{1-\frac{\alpha}{8}(2)}\right)\right)-x_2\log\left(\frac{\chi^2_{1-\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)$$

$$< \beta_0 <$$

$$\frac{x_2}{(x_3-x_1)(x_3-x_2)}\left(x_1\log\left(\frac{1}{2U_3}\left(\chi^2_{\frac{\alpha}{8}(2n-2)}-\chi^2_{1-\frac{\alpha}{8}(2r-2)}\right)\right)-x_3\log\left(\frac{\chi^2_{1-\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)$$

$$-\frac{x_3}{(x_2-x_1)(x_3-x_2)}\left(x_1\log\left(\frac{1}{U_2}\left(\frac{1}{2}\chi^2_{1-\frac{\alpha}{8}(2r-2)}-\frac{U_1}{2n(\hat{\mu}-\mu)}\chi^2_{\frac{\alpha}{8}(2)}\right)\right)-x_2\log\left(\frac{\chi^2_{\frac{\alpha}{8}(2)}}{2n(\hat{\mu}-\mu)}\right)\right)\Bigg],$$

where U_1, U_2 and U_3 are defined in (4), (5) and (6), and $r=n_1+n_2$, $\hat{\mu}=T_{11}$ and $\hat{\theta}_1=\frac{1}{n_1}\left[\sum_{j=1}^{n_1}T_{1j}+(n-n_1)T_{1n_1}-nT_{11}\right]$, respectively.

Proof. From the proof of Theorem 1 and by Bonferroni's inequality(Ross(1976)), the proof is completed.

5. A Numerical Example

The 40 simulated sample from model (2) is given in Table 1 to illustrate the use of joint confidence regions for μ, β_0, β_1 and β_2 . These data are simulated from the exponential distribution with mean 1, and then transformed by (7) to get the sample from (2) based on $\beta_0=-3.85135$, $\beta_1=0.8393$, $\beta_2=0.1216$ and $\mu=50$, $x_1=0.5$, $x_2=1.0$, $x_3=2.0$ and $\tau_1=53$, $\tau_2=57$.

<Table 1> Simulated failure times

stress	failure times					
x_1	50.48	50.99	51.36	51.42	52.81	
x_2	53.44	53.68	53.79	53.95	54.66	54.70
	54.99	55.53	55.77	55.84	56.14	56.44
x_3	57.10	57.39	57.40	57.58	57.63	57.93
	58.11	58.13	58.42	58.43	58.59	58.61
	58.85	59.29	59.52	59.56	59.57	60.33
	61.19	61.21	61.61	61.62		

The MLEs of μ , β_0 , β_1 and β_2 are $\hat{\mu}=50.48$, $\hat{\beta}_0=-3.51$, $\hat{\beta}_1=1.09$ and $\hat{\beta}_2=0.14$, respectively.

Now we construct 90% joint confidence regions for μ and β_0 , μ and β_1 , μ and β_2 . Note that

$\chi^2_{(0.0125,78)}=108.6327$, $\chi^2_{(0.9875,78)}=52.7203$, $\chi^2_{(0.0125,32)}=52.5393$,
$\chi^2_{(0.9875,32)}=16.7959$, $\chi^2_{(0.0125,2)}=8.7641$, $\chi^2_{(0.9875,2)}=0.0252$,
$F_{(0.0125,2,32)}=7.9629$, $F_{(0.9875,2,32)}=0.0126$.

Then by theorem 1, a 90% joint confidence region for μ and β_2 is given as follows.

$$\Big[\ 45.8682<\mu<50.4760,\ 0.67\left(-2.7157-\log\left(\frac{0.1096}{50.48-\mu}\right)\right)-2\left(\log\left(0.2286-\frac{0.0003}{50.48-\mu}\right)-\log\left(\frac{0.0003}{50.48-\mu}\right)\right) < \beta_2 < 0.67\left(-0.0103-\log\left(\frac{0.0003}{50.48-\mu}\right)\right)-2\left(\log\left(0.0731-\frac{0.0884}{50.48-\mu}\right)-\log\left(\frac{0.1096}{50.48-\mu}\right)\right)\Big].$$

And by theorem 2, a 90% joint confidence region for μ and β_1 is given as follows.

$$\Big[\ 45.8682<\mu<50.4760,\ 5\left(\log\left(0.0731-\frac{0.0884}{50.48-\mu}\right)-\log\left(\frac{0.1096}{50.48-\mu}\right)\right)-\left(-0.0103-\log\left(\frac{0.0003}{50.48-\mu}\right)\right) < \beta_1 < 5\left(\log\left(0.2286-\frac{0.0003}{50.48-\mu}\right)-\log\left(\frac{0.0003}{50.48-\mu}\right)\right)-\left(-2.7157-\log\left(\frac{0.1096}{50.48-\mu}\right)\right)\Big].$$

Finally by theorem 3, a 90% joint confidence region for μ and β_0 is given as follows.

$$\Big[\ 45.8682<\mu<50.4760,\ 0.67\left(-1.3578-2\times\log\left(\frac{0.1096}{50.48-\mu}\right)\right)-4\left(0.5\times\log\left(0.2286-\frac{0.0003}{50.48-\mu}\right)-\log\left(\frac{0.0003}{50.48-\mu}\right)\right) < \beta_0 < 0.67\left(-0.0051-2\times\log\left(\frac{0.0003}{50.48-\mu}\right)\right)-4\left(0.5\times\log\left(0.0731-\frac{0.0884}{50.48-\mu}\right)-\log\left(\frac{0.1096}{50.48-\mu}\right)\right)\Big].$$

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