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Bayesian Model Selection in the Gamma Populations

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Abstract

When X and Y have independent gamma distributions, we consider the testing problem for two gamma means. We propose a solution based on a Bayesian model selection procedure to this problem in which no subjective input is considered. The reference prior is derived. Using the derived reference prior, we compute the fractional Bayes factor and the intrinsic Bayes factors. The posterior probability of each model is used as a model selection tool. Simulation study and a real data example are provided.

Keywords : Fractional Bayes factor, Gamma mean, Intrinsic Bayes factor, Reference prior

1. Introduction

Consider two independent random samples $(X_1, \cdots, X_{n_1})$ and $(Y_1, \cdots, Y_{n_2})$, where the X_i are independent and identically gamma distributed with the shape parameter ν and the mean μ_x, and the Y_i are independent and identically gamma distributed with the shape parameter ν and the mean μ_y. Our goal of the analysis is to test the null hypotheses $H_1 : \mu_x = \mu_y$ versus the alternative $H_2 : \mu_x \neq \mu_y$.

In a Bayesian setup, the testing problem is formulated as a model selection problem in which we have the compare the model $M_1 : \{ f_1(\boldsymbol{x} \mid \Theta_1), \pi_1^N(\Theta_1) \}$

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to M_2:$\{f_2(\boldsymbol{x} \mid \Theta_2), \pi_2^N(\Theta_2)\}$. For a given prior P on $\{M_1, M_2\}$, the posterior probability of M_1 is

$$P(M_1 \mid \boldsymbol{x}) = \left(1 + \frac{P(M_2)}{P(M_1)} B_{21}\right)^{-1},$$

where the factor

$$B_{21} = \frac{\int f_2(\boldsymbol{x} \mid \Theta_2)\pi_2(\Theta_2)d\Theta_2}{\int f_1(\boldsymbol{x} \mid \Theta_1)\pi_1(\Theta_1)d\Theta_1} \tag{1}$$

is known as the Bayes factor of M_2 and M_1. This factor encapsulates all the information that the data have about the posterior probability of M_1. It can also be defined as the ratio of posterior to prior odds.

Unfortunately, with improper priors, the Bayes factor (1) is defined up to a multiplicative constant c_2/c_1. To avoid this difficulty several alternative approaches have been proposed. Spiegalhalter and Smith (1982) used the device of imaginary training samples in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so, there is no guarantee that the Bayes factor of Spiegalhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang, Kim and Lee, 2005, 2006).

A general review of the gamma distribution including several references to applications in diverse fields is given by Johnson, Kotz and Balakrishnan (1994). In particular, the gamma distribution has been suggested as the failure time model, and also received considerable attention in the area of ecology and weather analysis. For two sample gamma models, Shiue and Bain (1983) derived an approximate F test for testing the equality of two means when the shape parameters are equal. Shiue, Bain and Engelhardt (1988) extended the method to the case where the shape parameters are unequal. Booth, Hobert and Ohman (1999) proposed a bootstrap calibration method for inference concerning the ratio of two means when the shape parameters are equal and are in proportion. Simulation studies in Booth, Hobert and Ohman (1999) suggest that intervals obtained by extending the method in Jensen (1986) and those obtained by bootstrap calibration have similar performance in terms of length and coverage. Wong, Wu and Sun (2004) proposed a method based on the modified signed log-likelihood ratio statistic for small sample inference concerning the ratio of two means when the shape parameters are equal and are unequal. They argued that the proposed method give extremely accurate coverage in simulation studies, and is more direct and less computational intensive than the calibrated bootstrap method (Booth, Hobert and Ohman, 1999).

Almost all the work mentioned above is the analysis based on the classical point of view, there is a little work on this problem from the viewpoint of the objective Bayesian framework. So we feel a strong necessity to develop objective Bayesian testing procedure for the equality of two gamma means. For dealing this problem, we use the fractional Bayes factor (O'Hagan, 1995) and the intrinsic Bayes factor (Berger and Pericchi, 1996).

The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian model selection based on the Bayes factor. In Section 3, for some case, we derive the reference prior. Using the derived reference prior, we provide Bayesian model selection procedure based on the fractional Bayes factor and intrinsic Bayes factor for the testing for the equality of two gamma mean parameters. In Section 4, simulation study and a real example are given.

2. Intrinsic and Fractional Bayes Factors

Models M_1, $M_2, \cdots$, M_q are under consideration, with the data $\boldsymbol{x}=(x_1,x_2,\cdots,x_n)$ having probability density function $f_i(\boldsymbol{x} \mid \Theta_i)$ under model M_i, $i=1,2,\cdots,q$. The parameter vectors Θ_i are unknown. Let $\pi_i(\Theta_i)$ be the prior distribution of model M_i, and let p_i be the prior probabilities of model M_i, $i=1,2,\cdots,q$. Then the posterior probability that the model M_i is true is

$$P(M_i \mid \boldsymbol{x})=\left(\sum_{j=1}^{q}\frac{p_j}{p_i}\cdot B_{ji}\right)^{-1}, \tag{2}$$

where B_{ji} is the Bayes factor of model M_j to model M_i defined by

$$B_{ji}=\frac{\int f_j(\boldsymbol{x} \mid \Theta_j)\pi_j(\Theta_j)d\Theta_j}{\int f_i(\boldsymbol{x} \mid \Theta_i)\pi_i(\Theta_i)d\Theta_i}=\frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})}. \tag{3}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution

$\pi_i(\Theta_i)$ and $\pi_j(\Theta_j)$. Often in Bayesian analysis, one can use noninformative priors π_i^N. Common choices are the uniform prior, the Jeffreys prior and the reference prior. The noninformative prior π_i^N is typically improper. Hence the use of noninformative prior $\pi_i^N(\cdot)$ in (3) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\boldsymbol{x}(l)$ denote the part of the data to be so used and let $\boldsymbol{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\boldsymbol{x}(l)) < \infty, \ i = 1, \cdots, q. \tag{4}$$

In view (4), the posteriors $\pi_i^N(\Theta_i \mid \boldsymbol{x}(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $\boldsymbol{x}(-l)$, using $\pi_i^N(\Theta_i \mid \boldsymbol{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int_{\Theta_j} f(\boldsymbol{x}(-l) \mid \Theta_j, \boldsymbol{x}(l)) \pi_j^N(\Theta_j \mid \boldsymbol{x}(l)) d\Theta_j}{\int_{\Theta_i} f(\boldsymbol{x}(-l) \mid \Theta_i, \boldsymbol{x}(l)) \pi_i^N(\Theta_i \mid \boldsymbol{x}(l)) d\Theta_i} = B_{ji}^N \cdot B_{ij}^N(\boldsymbol{x}(l)) \tag{5}$$

where

$$B_{ji} = B_{ji}^N(\boldsymbol{x}) = \frac{m_j^N(\boldsymbol{x})}{m_i^N(\boldsymbol{x})} \quad \text{and} \quad B_{ij}^N(\boldsymbol{x}(l)) = \frac{m_i^N(\boldsymbol{x}(l))}{m_j^N(\boldsymbol{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\boldsymbol{x}$ and training samples $\boldsymbol{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\boldsymbol{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of M_j to M_i is

$$B_{ji}^{AI} = B_{ji}^N \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ij}^N(\boldsymbol{x}(l)). \tag{6}$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of M_j to

M_i is

$$B_{ji}^{MI} = B_{ji}^{N} \cdot ME[B_{ij}^{N}(\boldsymbol{x}(l))], \tag{7}$$

where ME indicates the median, here to be taken over all the training sample Bayes factors. So we can also calculate the posterior probability of M_i using (2), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (6) and (7).

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b, of each likelihood function, $L(\theta_i) = f_i(\boldsymbol{x} \mid \theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of model M_j versus model M_i is

$$B_{ji}^{F} = B_{ji}^{N} \cdot \frac{\int L^b(\boldsymbol{x} \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\boldsymbol{x} \mid \theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^{N} \cdot \frac{m_i^b(\boldsymbol{x})}{m_j^b(\boldsymbol{x})},$$

and $f_i(\boldsymbol{x} \mid \theta_i)$ is the likelihood function and b specifies a fraction of the likelihood which is to be used as a prior density. He proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. (see O'Hagan, 1995, 1997, and the discussion by Berger and Mortera of O'Hagan, 1995).

3. Bayesian Model Selection Procedures

Let $X_1, \cdots, X_{n_1}$ be independent random sample from gamma distribution $G(\mu_x, \nu)$ with the shape parameter ν and the mean μ_x and $Y_1, \cdots, Y_{n_2}$ be independent random sample from gamma distribution $G(\mu_y, \nu)$ with the shape parameter ν and the mean μ_y. We are interest to testing the hypotheses $H_1: \mu_x = \mu_y$ vs. $H_2: \mu_x \neq \mu_y$.

The two default models being compared are

$$M_1: f(\boldsymbol{x}, \boldsymbol{y} \mid \theta_1) = G(\boldsymbol{x} \mid \mu, \nu)G(\boldsymbol{y} \mid \mu, \nu), \ \pi_1^N(\theta_1)$$

and

$$M_2: f(\boldsymbol{x}, \boldsymbol{y} \mid \theta_2) = G(\boldsymbol{x} \mid \mu_x, \nu)G(\boldsymbol{y} \mid \mu_y, \nu), \ \pi_2^N(\theta_2),$$

where $\boldsymbol{x}=(x_1,\cdots,x_{n_1})$, $\boldsymbol{y}=(y_1,\cdots,y_{n_2})$, $\Theta_1=(\mu,\nu)$ and $\Theta_2=(\mu_x,\mu_y,\nu)$.

3.1 Bayesian Model Selection based on the Fractional Bayes Factor

Under the Model M_1, the reference prior for $\mu(\equiv\mu_x=\mu_y)$ and ν is

$$\pi_1^N(\mu,\nu)=\mu^{-1}[\psi'(\nu)-\nu^{-1}]^{\frac{1}{2}},$$

where $\psi'(\cdot)$ is the trigamma function. This reference prior is derived by Liseo (1993). And the likelihood function under M_1 is

$$L(\mu,\nu\mid\boldsymbol{x},\boldsymbol{y})=\frac{\mu^{-(n_1+n_2)\nu}\nu^{(n_1+n_2)\nu}}{\Gamma(\nu)^{n_1+n_2}}[\prod_{i=1}^{n_1}x_i\prod_{i=1}^{n_2}y_i]^{\nu-1}\exp\{-\frac{\nu}{\mu}[\sum_{i=1}^{n_1}x_i+\sum_{i=1}^{n_2}y_i]\}.$$

Then the element of fractional Bayes factor under M_1 is given by

$$\begin{aligned} m_1^b(\boldsymbol{x},\boldsymbol{y}) &= \int_0^\infty\int_0^\infty L^b(\mu,\nu\mid\boldsymbol{x},\boldsymbol{y})\pi_1^N(\mu,\nu)\,d\mu\,d\nu \\ &= \int_0^\infty\frac{\Gamma(nb\nu)}{\Gamma(\nu)^{nb}}\frac{[\prod_{i=1}^{n_1}x_i\prod_{i=1}^{n_2}y_i]^{b(\nu-1)}}{[\sum_{i=1}^{n_1}x_i+\sum_{i=1}^{n_2}y_i]^{nb\nu}}b^{-nb\nu}[\psi'(\nu)-\nu^{-1}]^{\frac{1}{2}}d\nu, \end{aligned}$$

where $n=n_1+n_2$.

For the M_2, the reference prior for μ_x,μ_y and ν is

$$\pi_2^N(\mu_x,\mu_y,\nu)=\mu_x^{-1}\mu_y^{-1}[\psi'(\nu)-\nu^{-1}]^{\frac{1}{2}}.$$

The derivation of the reference prior for μ_x,μ_y and ν is given in Appendix. The likelihood function under M_2 is

$$\begin{aligned} L(\mu_x,\mu_y,\nu\mid\boldsymbol{x},\boldsymbol{y}) &= \frac{\mu_x^{-n_1\nu}\mu_y^{-n_2\nu}\nu^{(n_1+n_2)\nu}}{\Gamma(\nu)^{n_1+n_2}}[\prod_{i=1}^{n_1}x_i\prod_{i=1}^{n_2}y_i]^{\nu-1} \\ &\quad\times\exp\{-\frac{\nu}{\mu_x}\sum_{i=1}^{n_1}x_i\}\exp\{-\frac{\nu}{\mu_y}\sum_{i=1}^{n_2}y_i\}. \end{aligned}$$

Thus the element of fractional Bayes factor under M_2 gives as follows.

$$m_2^b(\boldsymbol{x},\boldsymbol{y}) = \int_0^\infty\int_0^\infty\int_0^\infty L^b(\mu_x,\mu_y,\nu \mid \boldsymbol{x},\boldsymbol{y})\pi_2^N(\mu_x,\mu_y,\nu)\,d\mu_x d\mu_y d\nu$$

$$= \int_0^\infty \frac{\Gamma(n_1 b\nu)\Gamma(n_2 b\mu)}{b^{nb\nu}\Gamma(\nu)^{nb}} \frac{[\prod_{i=1}^{n_1} x_i \prod_{i=1}^{n_2} y_i]^{b(\nu-1)}}{[\sum_{i=1}^{n_1} x_i]^{n_1 b\nu}[\sum_{i=1}^{n_2} y_i]^{n_2 b\nu}} [\psi'(\nu)-\nu^{-1}]^{\frac{1}{2}} d\nu,$$

where $n=n_1+n_2$. Therefore the B_{21}^N is given by

$$B_{21}^N = \frac{S_2(\boldsymbol{x},\boldsymbol{y})}{S_1(\boldsymbol{x},\boldsymbol{y})}, \tag{8}$$

where

$$S_1(\boldsymbol{x},\boldsymbol{y}) = \int_0^\infty \frac{\Gamma(n\nu)}{\Gamma(\nu)^n} \frac{[\prod_{i=1}^{n_1} x_i \prod_{i=1}^{n_2} y_i]^{\nu-1}}{[\sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} y_i]^{n\nu}} [\psi'(\nu)-\nu^{-1}]^{\frac{1}{2}} d\nu,$$

and

$$S_2(\boldsymbol{x},\boldsymbol{y}) = \int_0^\infty \frac{\Gamma(n_1\nu)\Gamma(n_2\mu)}{\Gamma(\nu)^n} \frac{[\prod_{i=1}^{n_1} x_i \prod_{i=1}^{n_2} y_i]^{(\nu-1)}}{[\sum_{i=1}^{n_1} x_i]^{n_1\nu}[\sum_{i=1}^{n_2} y_i]^{n_2\nu}} [\psi'(\nu)-\nu^{-1}]^{\frac{1}{2}} d\nu.$$

And also

$$\frac{m_1^b(\boldsymbol{x},\boldsymbol{y})}{m_2^b(\boldsymbol{x},\boldsymbol{y})} = \frac{S_1(\boldsymbol{x},\boldsymbol{y};b)}{S_2(\boldsymbol{x},\boldsymbol{y};b)},$$

where

$$S_1(\boldsymbol{x},\boldsymbol{y};b) = \int_0^\infty \frac{\Gamma(nb\nu)}{\Gamma(\nu)^{nb}} \frac{[\prod_{i=1}^{n_1} x_i \prod_{i=1}^{n_2} y_i]^{\nu-1}}{[\sum_{i=1}^{n_1} x_i + \sum_{i=1}^{n_2} y_i]^{nb\nu}} b^{-nb\nu}[\psi'(\nu)-\nu^{-1}]^{\frac{1}{2}} d\nu,$$

and

$$S_2(\boldsymbol{x},\boldsymbol{y};b) = \int_0^\infty \frac{\Gamma(n_1 b\nu)\Gamma(n_2 b\mu)}{\Gamma(\nu)^{nb}} \frac{[\prod_{i=1}^{n_1} x_i \prod_{i=1}^{n_2} y_i]^{b(\nu-1)}}{[\sum_{i=1}^{n_1} x_i]^{n_1 b\nu}[\sum_{i=1}^{n_2} y_i]^{n_2 b\nu}} b^{-nb\nu}[\psi'(\nu)-\nu^{-1}]^{\frac{1}{2}} d\nu.$$

Thus the fractional Bayes factor of M_2 versus M_1 is given by

$$B_{21}^F = \frac{S_2(\boldsymbol{x},\boldsymbol{y};b)}{S_1(\boldsymbol{x},\boldsymbol{y};b)} \cdot \frac{S_1(\boldsymbol{x},\boldsymbol{y})}{S_2(\boldsymbol{x},\boldsymbol{y})}. \tag{9}$$

Note that the calculation of the fractional Bayes factor of M_2 versus M_1 requires an one dimensional integration.

3.2 Bayesian Model Selection based on the Intrinsic Bayes Factor

The element B_{21}^{N}, (8), of the intrinsic Bayes factor is computed in the fractional Bayes factor. So using minimal training sample, we only calculate the marginal densities under M_1 and M_2, respectively. The marginal density of (X_i, X_j, Y_k, Y_l) is finite for all $1 \le i < j \le n_1, 1 \le k < l \le n_2$ under each hypothesis (see Liseo, 1993). Thus we conclude that any training sample of size four is a minimal training sample.

The marginal densities $m_1^N(x_i, x_j, y_k, y_l)$ under M_1 is given by

$$\begin{aligned} m_1^N(x_i, x_j, y_k, y_l) &= \int_0^\infty \int_0^\infty f(x_i, x_j, y_k, y_l \mid \mu, \nu) \pi_1^N(\mu, \nu) d\mu d\nu \\ &= \int_0^\infty \frac{\Gamma(4\nu)}{\Gamma(\nu)^4} \frac{[x_i x_j y_k y_l]^{\nu-1}}{[x_i + x_j + y_k + y_l]^{4\nu}} [\psi'(\nu) - \nu^{-1}]^{\frac{1}{2}} d\nu \\ &\equiv T_1(x_i, x_j, y_k, y_l), \end{aligned}$$

where $1 \le i < j \le n_1, 1 \le k < l \le n_2$. And the marginal density $m_2^N(x_i, x_j, y_k, y_l)$ under M_2 is given by

$$\begin{aligned} & m_2^N(x_i, x_j, y_k, y_l) \\ &= \int_0^\infty \int_0^\infty \int_0^\infty f(x_i, x_j, y_k, y_l \mid \mu_x, \mu_y, \nu) \pi_2^N(\mu_x, \mu_y, \nu) d\mu_x d\mu_y d\nu \\ &= \int_0^\infty \frac{\Gamma(2\nu)\Gamma(2\mu)}{\Gamma(\nu)^4} \frac{[x_i x_j y_k y_l]^{(\nu-1)}}{[x_i + x_j]^{2\nu}[y_k + y_l]^{2\nu}} [\psi'(\nu) - \nu^{-1}]^{\frac{1}{2}} d\nu \\ &\equiv T_2(x_i, x_j, y_k, y_l). \end{aligned}$$

Therefore the AIBF of M_2 versus M_1 is given by

$$B_{21}^{AI} = \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})} \cdot \left[\frac{1}{L} \sum_{i,j} \sum_{k,l} \frac{T_1(x_i, x_j, y_k, y_l)}{T_2(x_i, x_j, y_k, y_l)} \right]. \tag{10}$$

where $L = n_1(n_1 - 1)n_2(n_2 - 1)/4$. And the MIBF of M_2 versus M_1 is given by

$$B_{21}^{MI} = \frac{S_2(\mathbf{x}, \mathbf{y})}{S_1(\mathbf{x}, \mathbf{y})} \cdot ME \left[\frac{T_1(x_i, x_j, y_k, y_l)}{T_2(x_i, x_j, y_k, y_l)} \right]. \tag{11}$$

Note that the calculations of the AIBF and MIBF of M_2 versus M_1 require an one dimensional integration. In Section 4, we investigate our model selection procedures.

4. Numerical Studies

Example 1. To investigate the Bayesian model selection procedures, we examine the cases when $(\mu_x, \mu_y) = (1,1), (1,3), (1,5)$, $\nu = 0.5, 1, 3$ and $(n_1, n_2) =$ (10,10), (10,20), (20,10), (20,20). The posterior probabilities of M_1 being true are computed assuming equal prior probabilities. The Table 1 shows the results of the averages and the standard deviations in parentheses of posterior probabilities for each case based on 200 replications.

From the Table 1, the fractional Bayes factor and the intrinsic Bayes factor give fairly reasonable answers. Also for moderate sample sizes, the fractional Bayes factor and intrinsic Bayes factor give very similar results.

Example 2. Crow (1977) considered hail data from a cloud-seeding experiment conducted by the National Center for Atmospheric Research. The natural precipitation in a specified area was measured from a number of randomly selected storms, and the precipitation from the remaining storms in the area, which were seeded, was also measured. In one experiment, the hail mass was measured by hail/rail separators from 16 nonseeded days and 17 seeded days. The ratio of the mean hail mass under seeding and nonseeding was of interest, and an interval estimate of this ratio was desired in order to assess the possible effect of cloud seeding. Crow (1977) considered a gamma model for the data. For this experiment, we have $n_1 = 16, \bar{x} = 13.366, \sum \log x_i = 13.803$ and $n_2 = 17$, $\bar{y} = 13.249, \sum \log y_i = 20.637$. Shiue and Bain (1983) reported the 90% confidence interval for μ_y/μ_x is (0.376, 2.573) or equivalently the 90% confidence interval for μ_x/μ_y is (0.39, 2.66). The 90% confidence interval based on the modified signed log-likelihood ratio method (Wong, Wu and Sun, 2004) is (0.4, 2.64).

The value of fractional Bayes factor of M_2 versus M_1 is $B^F_{21} = 0.206$. We assume that the prior probabilities are equal. Then the posterior probability for M_1 is 0.829. Thus there are strong evidence for M_1 in terms of the posterior probability.

<Table 1> The averages and the standard deviations in parentheses of posterior probabilities

ν	(μ_x, μ_y)	(n_1, n_2)	$P^F(M_1 \mid \boldsymbol{x}, \boldsymbol{y})$	$P^{AI}(M_1 \mid \boldsymbol{x}, \boldsymbol{y})$	$P^{MI}(M_1 \mid \boldsymbol{x}, \boldsymbol{y})$
0.5	1, 1	10,10	0.666 (0.151)	0.671 (0.161)	0.695 (0.154)
		10,20	0.737 (0.126)	0.729 (0.134)	0.750 (0.130)
		20,10	0.743 (0.125)	0.734 (0.130)	0.757 (0.124)
		20,20	0.759 (0.119)	0.765 (0.122)	0.785 (0.119)
	1, 3	10,10	0.469 (0.240)	0.468 (0.249)	0.499 (0.245)
		10,20	0.430 (0.247)	0.415 (0.245)	0.445 (0.246)
		20,10	0.428 (0.280)	0.413 (0.282)	0.439 (0.285)
		20,20	0.373 (0.269)	0.373 (0.273)	0.397 (0.278)
	1, 5	10,10	0.288 (0.228)	0.280 (0.231)	0.312 (0.237)
		10,20	0.260 (0.214)	0.250 (0.212)	0.280 (0.220)
		20,10	0.182 (0.221)	0.171 (0.217)	0.188 (0.227)
		20,20	0.116 (0.180)	0.114 (0.180)	0.126 (0.192)
1.0	1, 1	10,10	0.665 (0.153)	0.674 (0.161)	0.685 (0.158)
		10,20	0.717 (0.133)	0.713 (0.137)	0.721 (0.135)
		20,10	0.712 (0.151)	0.709 (0.155)	0.717 (0.153)
		20,20	0.750 (0.107)	0.762 (0.106)	0.768 (0.105)
	1, 3	10,10	0.301 (0.247)	0.298 (0.253)	0.315 (0.253)
		10,20	0.256 (0.229)	0.249 (0.228)	0.262 (0.229)
		20,10	0.230 (0.245)	0.221 (0.243)	0.231 (0.247)
		20,20	0.131 (0.189)	0.131 (0.193)	0.138 (0.196)
	1, 5	10,10	0.108 (0.154)	0.104 (0.155)	0.117 (0.160)
		10,20	0.058 (0.100)	0.055 (0.099)	0.063 (0.106)
		20,10	0.036 (0.089)	0.033 (0.086)	0.037 (0.089)
		20,20	0.012 (0.044)	0.012 (0.043)	0.013 (0.046)
3.0	1, 1	10,10	0.684 (0.117)	0.701 (0.122)	0.691 (0.122)
		10,20	0.736 (0.121)	0.741 (0.123)	0.732 (0.124)
		20,10	0.733 (0.129)	0.738 (0.131)	0.729 (0.132)
		20,20	0.748 (0.129)	0.765 (0.129)	0.756 (0.130)
	1, 3	10,10	0.061 (0.098)	0.059 (0.100)	0.062 (0.100)
		10,20	0.018 (0.041)	0.017 (0.040)	0.018 (0.040)
		20,10	0.015 (0.038)	0.014 (0.037)	0.014 (0.036)
		20,20	0.002 (0.008)	0.002 (0.008)	0.002 (0.008)
	1, 5	10,10	0.003 (0.007)	0.002 (0.006)	0.003 (0.007)
		10,20	0.000 (0.001)	0.000 (0.001)	0.000 (0.001)
		20,10	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
		20,20	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)

Appendix. Derivation of the Reference prior

Reference priors introduced by Bernardo (1979), and extended further by Berger and Bernardo (1992) have become very popular over the years for the development of noninformative priors. In this Appendix, we derive the reference priors for different groups of ordering of (μ_x, μ_y, ν).

Under M_2, the joint density is given by

$$
\begin{aligned}
f(\boldsymbol{x}, \boldsymbol{y} \mid \mu_x, \mu_y, \nu) = & \frac{\mu_x^{-n_1\nu} \mu_y^{-n_2\nu} \nu^{(n_1+n_2)\nu}}{\Gamma(\nu)^{n_1+n_2}} \left[\prod_{i=1}^{n_1} x_i \prod_{i=1}^{n_2} y_i \right]^{\nu-1} \\
& \times \exp\left\{ -\frac{\nu}{\mu_x} \sum_{i=1}^{n_1} x_i \right\} \exp\left\{ -\frac{\nu}{\mu_y} \sum_{i=1}^{n_2} y_i \right\}.
\end{aligned}
\tag{12}
$$

Based on (12), the Fisher information matrix is given by

$$
\boldsymbol{I} = \begin{pmatrix} \frac{n_1\nu}{\mu_x^2} & 0 & 0 \\ 0 & \frac{n_2\nu}{\mu_y^2} & 0 \\ 0 & 0 & (n_1+n_2)[\psi'(\nu) - \nu^{-1}] \end{pmatrix},
$$

where $\psi'(\cdot)$ is the trigamma function. From the above Fisher information matrix $\boldsymbol{I}$, μ_x, μ_y and ν are mutually orthogonal in the sense of Cox and Reid(1987). Then due to the orthogonality of the parameters, following Datta and Ghosh (1995), choosing rectangular compacts for each μ_x, μ_y and ν, the reference priors are given by as follows.

For the gamma populations (12), the reference prior distributions for group of ordering of $\{(\mu_x, \mu_y), \nu\}$ is

$$
\pi_{R1}(\mu_x, \mu_y, \nu) \propto \mu_x^{-1} \mu_y^{-1} [\psi'(\nu) - \nu^{-1}]^{\frac{1}{2}}.
$$

For group of ordering of $\{\mu_x, \mu_y, \nu\}$ and $\{\mu_y, \mu_x, \nu\}$, the reference prior is

$$
\pi_{R2}(\mu_x, \mu_y, \nu) \propto \mu_x^{-1} \mu_y^{-1} [\psi'(\nu) - \nu^{-1}]^{\frac{1}{2}}.
$$

So we know that the two group reference prior and the one-at-a-time reference prior are the same.

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