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Estimation for Two-Parameter Rayleigh Distribution Based on Multiply Type-II Censored Sample

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Abstract

For multiply Type-II censored samples from two-parameter Rayleigh distribution, the maximum likelihood method does not admit explicit solutions. In this case, we propose some explicit estimators of the location and scale parameters in the Rayleigh distribution by the approximate maximum likelihood methods. We compare the proposed estimators in the sense of the mean squared error for various censored samples.

Keywords : Approximate maximum likelihood estimator, Multiply Type-II censored sample, Rayleigh distribution

1. Introduction

The random variable X has the Rayleigh distribution if it has a probability density function (pdf) of the form

$$f(x;\theta,\sigma)=\frac{x-\theta}{\sigma^2}\exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\},\quad x\geq\theta,\ \sigma>0, \tag{1.1}$$

where θ and σ are the location and the scale parameters, respectively.

It has been noted that in most cases, the maximum likelihood method does not provide explicit estimators based on complete and censored samples. Especially, when the sample is multiply censored, the maximum likelihood method does not admit explicit solutions. Hence it is desirable to develop approximations to this

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maximum likelihood method which would provide us with estimators that are explicit functions of order statistics.

For multiply Type-II censoring, Balasubramanian and Balakrishnan (1992) and Upadhyay et al. (1996) considered some estimations for the exponential distribution under multiply Type-II censoring. Kong and Fei (1996) discussed the limit theorems for the maximum likelihood estimator under general multiply Type-II censoring. Kang (2005) derived the approximate maximum likelihood estimators (AMLEs) of the scale parameter and the location parameter in the extreme value distribution based on multiply Type-II censored samples. Kang and Lee (2005) derived the AMLEs of the scale and location parameters in the two-parameter exponential distribution based on multiply Type-II censored samples. They also obtained the moments of the proposed estimators. Recently, Han and Kang (2006) derived some AMLEs of the scale parameter when the location parameter is known and also derived an AMLE of the location parameter when the scale parameter is known in the Rayleigh distribution under multiply Type-II censoring.

In this paper, we derive the AMLEs of the scale parameter σ and the location parameter θ in the two-parameter Rayleigh distribution under multiply Type-II censoring by the approximate maximum likelihood estimation method when two parameters are unknown. We also compare the proposed estimators in the sense of the MSE for various censored samples.

2. Approximate Maximum Likelihood Estimators

We assume that n items are put on a life test, but only a_1th,…,a_sth failures are observed, the rest are unobserved or missing, where $a_1, \dots, a_s$ are considered to be fixed. If this censoring arises, the scheme is known as multiply Type-II censoring scheme.

Let

$$X_{a_1:n} \leq X_{a_2:n} \leq \cdots \leq X_{a_s:n} \tag{2.1}$$

be the multiply Type-II censored sample, where $1 \leq a_1 < a_2 < \cdots < a_s \leq n$ and $X_{1:n}, \dots, X_{n:n}$ are order statistics of $X_1, \dots, X_n$.

Let $a_0 = 0$, $a_{s+1} = n+1$, $F(x_{a_0:n}) = 0$, $F(x_{a_{s+1}:n}) = 1$, then the likelihood function based on the multiply Type-II censored sample (2.1) is given by

$$\begin{aligned} L = \frac{1}{\sigma^s} & \frac{n!}{\prod_{j=1}^{s+1}(a_j - a_{j-1} - 1)!} [F(Z_{a_1:n})]^{a_1 - 1} [1 - F(Z_{a_s:n})]^{n - a_s} \\ & \times \prod_{j=1}^{s} f(Z_{a_j:n}) \left[\prod_{j=2}^{s} [F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})]^{a_j - a_{j-1} - 1} \right], \end{aligned} \tag{2.2}$$

where $Z_{i:n}=(X_{i:n}-\theta)/\sigma$, $f(z)=ze^{-z^2/2}$ and $F(z)=1-e^{-z^2/2}$ are the pdf and the cdf of the standard Rayleigh distribution, respectively.

From the equation (2.2), we obtain the likelihood equations as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma}=&-\frac{1}{\sigma}\left[2s+(a_1-1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}Z_{a_1:n}-(n-a_s)Z_{a_s:n}^2-\sum_{j=1}^{s}Z_{a_j:n}^2\right.\\&\left.+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\right]\\=&\,0\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta}=&-\frac{1}{\sigma}\left[(a_1-1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}-(n-a_s)Z_{a_s:n}+\sum_{j=1}^{s}\frac{1}{Z_{a_j:n}}-\sum_{j=1}^{s}Z_{a_j:n}\right.\\&\left.+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(Z_{a_j:n})-f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\right]\\=&\,0.\end{aligned} \tag{2.4}$$

Since these likelihood equations are very complicated, the equations (2.3) and (2.4) do not admit explicit solutions for σ and θ. So we need some approximate likelihood equations which give explicit solutions.

Han and Kang (2006) obtained some approximations by Taylor series expansion as follows;

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}Z_{a_1:n}\simeq \alpha_1+\beta_1 Z_{a_1:n} \tag{2.5}$$

$$\frac{f(Z_{a_j:n})Z_{a_j:n}-f(Z_{a_{j-1}:n})Z_{a_{j-1}:n}}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\simeq \alpha_{1j}+\beta_{1j} Z_{a_j:n}+\gamma_{1j} Z_{a_{j-1}:n} \tag{2.6}$$

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}\simeq \alpha_2+\beta_2 Z_{a_1:n} \tag{2.7}$$

$$\frac{f(Z_{a_j:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\simeq \alpha_{2j}+\beta_{2j} Z_{a_j:n}+\gamma_{2j} Z_{a_{j-1}:n} \tag{2.8}$$

$$\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\simeq \alpha_{3j}+\beta_{3j} Z_{a_j:n}+\gamma_{3j} Z_{a_{j-1}:n} \tag{2.9}$$

$$\frac{1}{Z_{a_j:n}}\simeq \kappa_j+\delta_j Z_{a_j:n} \tag{2.10}$$

$$\frac{f(Z_{a_j:n})-f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\simeq \alpha_{4j}+\beta_{4j} Z_{a_j:n}+\gamma_{4j} Z_{a_j:n}, \tag{2.11}$$

where

$$\xi_i=F^{-1}(p_i)=\left[-2\ln(1-p_i)\right]^{1/2}$$

$$p_i=\frac{i}{n+1}, \quad q_i=1-p_i$$

$$\alpha_1=\frac{-\xi_{a_1}^2}{p_{a_1}}\left[f'(\xi_{a_1})-\frac{f^2(\xi_{a_1})}{p_{a_1}}\right]$$

$$\beta_1=\frac{1}{p_{a_1}}\left[f(\xi_{a_1})+\xi_{a_1}f'(\xi_{a_1})-\frac{f^2(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\alpha_{1j}=K^2-\frac{\xi_{a_j}^2 f'(\xi_{a_j})-\xi_{a_{j-1}}^2 f'(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}$$

$$\beta_{1j}=\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[(1-K)f(\xi_{a_j})+\xi_{a_j}f'(\xi_{a_j})\right]$$

$$\gamma_{1j}=-\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[(1-K)f(\xi_{a_{j-1}})+\xi_{a_{j-1}}f'(\xi_{a_{j-1}})\right]$$

$$K=\frac{\xi_{a_j}f(\xi_{a_j})-\xi_{a_{j-1}}f(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}$$

$$\alpha_2=\frac{1}{p_{a_1}}\left[f(\xi_{a_1})-\xi_{a_1}f'(\xi_{a_1})+\frac{f^2(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\beta_2=\frac{1}{p_{a_1}}\left[f'(\xi_{a_1})-\frac{f^2(\xi_{a_1})}{p_{a_1}}\right]$$

$$\alpha_{2j}=\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[(1+K)f(\xi_{a_j})-\xi_{a_j}f'(\xi_{a_j})\right]$$

$$\beta_{2j}=\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[f'(\xi_{a_j})-\frac{f^2(\xi_{a_j})}{p_{a_j}-p_{a_{j-1}}}\right]$$

$$\gamma_{2j}=\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{(p_{a_j}-p_{a_{j-1}})^2}$$

$$\alpha_{3j}=\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[(1+K)f(\xi_{a_{j-1}})-\xi_{a_{j-1}}f'(\xi_{a_{j-1}})\right]$$

$$\beta_{3j}=-\frac{f(\xi_{a_j})f(\xi_{a_{j-1}})}{[p_{a_j}-p_{a_{j-1}}]^2}=-\gamma_{2j}$$

$$\gamma_{3j}=\frac{1}{p_{a_j}-p_{a_{j-1}}}\left[f'(\xi_{a_{j-1}})+\frac{f^2(\xi_{a_{j-1}})}{p_{a_j}-p_{a_{j-1}}}\right]$$

$$\kappa_j=2/\xi_{a_j} \quad \delta_j=-1/\xi_{a_j}^2 \quad \alpha_{4j}=\alpha_{2j}-\alpha_{3j}$$

$$\beta_{4j}=\beta_{2j}-\beta_{3j}, \quad \text{and} \quad \gamma_{4j}=\gamma_{2j}-\gamma_{3j}.$$

Now making use of the approximate expressions in (2.5), (2.6), (2.7), (2.10), and (2.11), we may approximate the likelihood equations of (2.3) and (2.4) as follows;

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\Bigg[2s+(a_1-1)(\alpha_1+\beta_1 Z_{a_1:n})-(n-a_s)Z_{a_s:n}^2-\sum_{j=1}^{s} Z_{a_j:n}^2 + \sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}+\beta_{1j} Z_{a_j:n}+\gamma_{1j} Z_{a_{j-1}:n})\Bigg] = 0 \tag{2.12}$$

and

$$\frac{\partial \ln L}{\partial \theta} \simeq -\frac{1}{\sigma}\Bigg[(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})-(n-a_s)Z_{a_s:n}+\sum_{j=1}^{s}(\kappa_j+\delta_j Z_{a_j:n}) - \sum_{j=1}^{s} Z_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{4j}+\beta_{4j} Z_{a_j:n}+\gamma_{4j} Z_{a_{j-1}:n})\Bigg] = 0. \tag{2.13}$$

Upon solving the equations (2.12) and (2.13) for σ and θ we derive the AMLEs of σ and θ as follows;

$$\hat{\sigma}_1 = \frac{-B_1+\sqrt{B_1^2-4A_1C_1}}{2A_1} \tag{2.14}$$

and

$$\hat{\theta}_1 = M_1\hat{\sigma}_1 + M_2, \tag{2.15}$$

where

$$A = (a_1-1)\alpha_2+\sum_{j=1}^{s}\kappa_j+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\alpha_{4j}$$

$$B = (a_1-1)\beta_2 X_{a_1:n}-(n-a_s)X_{a_s:n}+\sum_{j=1}^{s}\delta_j X_{a_j:n}-\sum_{j=1}^{s}X_{a_j:n} + \sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{4j} X_{a_j:n}+\gamma_{4j} X_{a_{j-1}:n})$$

$$C = (a_1-1)\beta_2-(n-a_s)+\sum_{j=1}^{s}\delta_j-s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{4j}+\gamma_{4j})$$

$$M_1 = \frac{A}{C},\quad M_2 = \frac{B}{C}$$

$$A_1 = 2s+(a_1-1)(\alpha_1-\beta_1 M_1)-(n-a_s+s)M_1^{\,2} + \sum_{j=2}^{s}(a_j-a_{j-1}-1)\big(\alpha_{1j}-\beta_{1j}M_1-\gamma_{1j}M_1\big)$$

$$B_1 = (a_1-1)\beta_1(X_{a_1:n}-M_2)+2(n-a_s)M_1(X_{a_s:n}-M_2)+2M_1\sum_{j=1}^{s}(X_{a_j:n}-M_2) + \sum_{j=2}^{s}(a_j-a_{j-1}-1)\big[\beta_{1j}(X_{a_j:n}-M_2)+\gamma_{1j}(X_{a_{j-1}:n}-M_2)\big]$$

$$C_1 = -(n-a_s)(X_{a_s:n}-M_2)^2-\sum_{j=1}^{s}(X_{a_j:n}-M_2)^2.$$

Second, making use of the approximate expressions in (2.6), (2.7), (2.10), and (2.11), we may approximate the likelihood equation of (2.3) as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &\simeq -\frac{1}{\sigma}\Bigg[2s+(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})\,Z_{a_1:n}-(n-a_s)Z^2_{a_s:n}-\sum_{j=1}^{s} Z^2_{a_j:n}\\ &\qquad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{1j}+\beta_{1j}\,Z_{a_j:n}+\gamma_{1j}\,Z_{a_{j-1}:n})\Bigg]\\ &=0.\end{aligned}\tag{2.16}$$

Upon solving the equations (2.16) and (2.13) for σ and θ we derive the AMLEs of σ and θ as follows;

$$\hat{\sigma}_2=\frac{-B_2+\sqrt{B_2^2-4A_2C_2}}{2A_2}\tag{2.17}$$

and

$$\hat{\theta}_2=M_1\hat{\sigma}_2+M_2,\tag{2.18}$$

where

$$A_2=2s-(a_1-1)M_1(\alpha_2-\beta_2M_1)-(n-a_s+s)M_1{}^2+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[\alpha_{1j}-\beta_{1j}M_1-\gamma_{1j}M_1\right]$$

$$\begin{aligned}B_2=\,&(a_1-1)(\alpha_2-2\beta_2M_1)(X_{a_1:n}-M_2)+2(n-a_s)M_1(X_{a_s:n}-M_2)\\ &+2M_1\sum_{j=1}^{s}(X_{a_j:n}-M_2)+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[\beta_{1j}(X_{a_j:n}-M_2)+\gamma_{1j}(X_{a_{j-1}:n}-M_2)\right]\end{aligned}$$

$$C_2=(a_1-1)\beta_2(X_{a_1:n}-M_2)^2-(n-a_s)(X_{a_s:n}-M_2)^2-\sum_{j=1}^{s}(X_{a_j:n}-M_2)^2.$$

Third, making use of the approximate expressions in (2.5), (2.7), (2.8), (2.9), (2.10), and (2.11), we may approximate the likelihood equation of (2.3) as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &\simeq -\frac{1}{\sigma}\Bigg[2s+(a_1-1)(\alpha_1+\beta_1 Z_{a_1:n})-(n-a_s)Z^2_{a_s:n}-\sum_{j=1}^{s} Z^2_{a_j:n}\\ &\qquad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)\big\{(\alpha_{2j}+\beta_{2j}\,Z_{a_j:n}+\gamma_{2j}\,Z_{a_{j-1}:n})\,Z_{a_j:n}\\ &\qquad -(\alpha_{3j}+\beta_{3j}\,Z_{a_j:n}+\gamma_{3j}\,Z_{a_{j-1}:n})\,Z_{a_{j-1}:n}\big\}\Bigg]\\ &=0.\end{aligned}\tag{2.19}$$

Upon solving the equations (2.19) and (2.13) for σ and θ we derive the AMLEs of σ and θ as follows;

$$\hat{\sigma}_3=\frac{-B_3+\sqrt{B_3^2-4A_3C_3}}{2A_3}\tag{2.20}$$

and

$$\hat{\theta}_3=M_1\hat{\sigma}_3+M_2,\tag{2.21}$$

where

$$A_3=2s+(a_1-1)(\alpha_1-\beta_1M_1)-(n-a_s+s)M_1{}^2+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[(\beta_{4j}+\gamma_{4j})M_1{}^2-\alpha_{4j}M_1\right]$$

$$B_3 = (a_1-1)\beta_1(X_{a_1:n}-M_2)+2(n-a_s)M_1(X_{a_s:n}-M_2)+2M_1\sum_{j=1}^{s}(X_{a_j:n}-M_2)$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[(\alpha_{2j}-2\beta_{2j}M_1-\gamma_{2j}M_1+\beta_{3j}M_1)(X_{a_j:n}-M_2)\right.$$
$$\left.-(\alpha_{3j}-\beta_{3j}M_1+\gamma_{2j}M_1-2\gamma_{3j}M_1)(X_{a_{j-1}:n}-M_2)\right]$$
$$C_3 = -(n-a_s)(X_{a_s:n}-M_2)^2-\sum_{j=1}^{s}(X_{a_j:n}-M_2)^2$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[\beta_{2j}(X_{a_j:n}-M_2)^2+(\gamma_{2j}-\beta_{3j})(X_{a_j:n}-M_2)(X_{a_{j-1}:n}-M_2)\right.$$
$$\left.-\gamma_{3j}(X_{a_{j-1}:n}-M_2)^2\right].$$

Fourth, making use of the approximate expressions in (2.7), (2.8), (2.9), (2.10), and (2.11), we may approximate the likelihood equation of (2.3) as follows;

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\left[2s+(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})Z_{a_1:n}-(n-a_s)Z_{a_s:n}^2-\sum_{j=1}^{s}Z_{a_j:n}^2\right.$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left\{(\alpha_{2j}+\beta_{2j}Z_{a_j:n}+\gamma_{2j}Z_{a_{j-1}:n})Z_{a_j:n}\right. \quad (2.22)$$
$$\left.\left.-(\alpha_{3j}+\beta_{3j}Z_{a_j:n}+\gamma_{3j}Z_{a_{j-1}:n})Z_{a_{j-1}:n}\right\}\right]$$
$$=0.$$

Upon solving the equations (2.22) and (2.13) for σ and θ we derive the AMLEs of σ and θ as follows;

$$\hat{\sigma}_4 = \frac{-B_4+\sqrt{B_4^2-4A_4C_4}}{2A_4} \quad (2.23)$$

and

$$\hat{\theta}_4 = M_1\hat{\sigma}_4+M_2, \quad (2.24)$$

where

$$A_4 = 2s-(a_1-1)M_1(\alpha_2-\beta_2M_1)-(n-a_s+s)M_1^{\,2}$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[(\beta_{4j}+\gamma_{4j})M_1^{\,2}-\alpha_{4j}M_1\right]$$
$$B_4 = (a_1-1)(\alpha_2-2\beta_2M_1)(X_{a_1:n}-M_2)+2(n-a_s)M_1(X_{a_s:n}-M_2)$$
$$+2M_1\sum_{j=1}^{s}(X_{a_j:n}-M_2)+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[(\alpha_{2j}-2\beta_{2j}M_1-\gamma_{2j}M_1+\beta_{3j}M_1)\right.$$
$$\left.\times(X_{a_j:n}-M_2)-(\alpha_{3j}-\beta_{3j}M_1+\gamma_{2j}M_1-2\gamma_{3j}M_1)(X_{a_{j-1}:n}-M_2)\right]$$

$$C_4 = (a_1-1)\beta_2(X_{a_1:n}-M_2)^2-(n-a_s)(X_{a_s:n}-M_2)^2-\sum_{j=1}^{s}(X_{a_j:n}-M_2)^2$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\left[\beta_{2j}(X_{a_j:n}-M_2)^2+(\gamma_{2j}-\beta_{3j})(X_{a_j:n}-M_2)(X_{a_{j-1}:n}-M_2)\right.$$
$$\left.-\gamma_{3j}(X_{a_{j-1}:n}-M_2)^2\right].$$

It is difficult to find the moments of all proposed estimators. So, we simulate the MSEs of all proposed estimators through Monte Carlo simulation method. The simulation procedure is repeated 10,000 times for the sample size $n=20,40$ and various choices of censoring ($m=n-s$ is the number of unobserved or missing data) under multiply Type-II censored samples. These values are given in Tables 1 and 2.

From Table 1, the estimator $\widehat{\sigma_4}$ is more efficient than the other estimators of the scale parameter σ in the sense of the MSE, and $\widehat{\sigma_2}$ is generally more efficient than the estimators $\widehat{\sigma_1}$ and $\widehat{\sigma_3}$.

From Table 2, the estimator $\widehat{\theta_4}$ that use the estimator $\widehat{\sigma_4}$ is more efficient than the other estimators of the location parameter θ in the sense of the MSE, and $\widehat{\theta_2}$ that use the estimator $\widehat{\sigma_2}$ is generally more efficient than the estimators $\widehat{\theta_1}$ and $\widehat{\theta_3}$. So we can recommend the proposed estimators $\widehat{\sigma_4}$ and $\widehat{\theta_4}$ of the scale and location parameters in the two-parameter Rayleigh distribution.

As expected, the MSE of all estimators decreases as sample size n increases. For fixed sample size, the MSE increases generally as m increases.

<Table 1> The relative MSEs for the estimators of the scale parameter σ

n	m	a_j	$\widehat{\sigma_1}$	$\widehat{\sigma_2}$	$\widehat{\sigma_3}$	$\widehat{\sigma_4}$
20	0	1～20	0.028502	0.028502	0.028502	0.028502
	2	1～18	0.032998	0.032998	0.032998	0.032998
		3～20	0.030774	0.030442	0.030774	0.030442
		2～19	0.030818	0.030302	0.030818	0.030302
	4	2～17	0.036191	0.035532	0.036191	0.035532
		4～19	0.036932	0.036638	0.036932	0.036638
		3～18	0.036259	0.035852	0.036259	0.035852
		2～4 7～14 16～20	0.028474	0.028018	0.027184	0.026956
	5	3～17	0.039857	0.039386	0.039857	0.039386
		4～18	0.040200	0.039875	0.040200	0.039875
		2～6 10～19	0.030863	0.030346	0.029875	0.029514
	6	4～17	0.044489	0.044115	0.044489	0.044115
		1 2 6～9 12～15 17～20	0.028333	0.028333	0.024955	0.024955
40	0	1～40	0.012746	0.012746	0.012746	0.012746
	2	1～38	0.013717	0.013717	0.013717	0.013717
		3～40	0.013202	0.013096	0.013202	0.013096
		2～39	0.013047	0.012890	0.013047	0.012890
	4	2～37	0.013964	0.013792	0.013964	0.013792
		4～39	0.014505	0.014425	0.014505	0.014425
		3～38	0.014248	0.014133	0.014248	0.014133
		2～4 7～14 16～40	0.012566	0.012416	0.012211	0.012123
	5	3～37	0.014717	0.014598	0.014717	0.014598
		4～38	0.015108	0.015026	0.015108	0.015026
		2～6 10～19 21～40	0.012573	0.012422	0.012200	0.012107
	6	4～37	0.015625	0.015540	0.015625	0.015540
		1 2 6～9 12～15 17～40	0.012629	0.012629	0.011458	0.011458

<Table 2> The relative MSEs for the estimators of the location parameter θ

n	m	a_j	$\widehat{\theta_1}$	$\widehat{\theta_2}$	$\widehat{\theta_3}$	$\widehat{\theta_4}$
20	0	1～20	0.042813	0.042813	0.042813	0.042813
	2	1～18	0.044991	0.044991	0.044991	0.044991
		3～20	0.048540	0.047970	0.048540	0.047970
		2～19	0.043407	0.042632	0.043407	0.042632
	4	2～17	0.046007	0.045115	0.046007	0.045115
		4～19	0.059168	0.058652	0.059168	0.058652
		3～18	0.052017	0.051373	0.052017	0.051373
		2～4 7～14 16～20	0.042145	0.041428	0.039136	0.038663
	5	3～17	0.053998	0.053301	0.053998	0.053301
		4～18	0.061601	0.061052	0.061601	0.061052
		2～6 10～19	0.043465	0.042694	0.041409	0.040798
	6	4～17	0.064405	0.063810	0.064405	0.063810
		1 2 6～9 12～15 17～20	0.042021	0.042021	0.035126	0.035126

<Table 2> (continued)

n	m	a_j	$\widehat{\theta_1}$	$\widehat{\theta_2}$	$\widehat{\theta_3}$	$\widehat{\theta_4}$
40	0	1~40	0.017474	0.017474	0.017474	0.017474
	2	1~38	0.017915	0.017915	0.017915	0.017915
		3~40	0.018181	0.018018	0.018181	0.018018
		2~39	0.016897	0.016682	0.016897	0.016682
	4	2~37	0.017281	0.017055	0.017281	0.017055
		4~39	0.020317	0.020183	0.020317	0.020183
		3~38	0.018716	0.018546	0.018716	0.018546
		2~4 7~14 16~40	0.016657	0.016451	0.015763	0.015612
	5	3~37	0.018917	0.018744	0.018917	0.018744
		4~38	0.020668	0.020531	0.020668	0.020531
		2~6 10~19 21~40	0.016653	0.016446	0.015780	0.015624
	6	4~37	0.020917	0.020778	0.020917	0.020778
		1 2 6~9 12~15 17~40	0.017167	0.017167	0.015025	0.015025

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