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Jackknife Estimation in a Truncated Exponential Distribution with an Uniform Outlier

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Abstract

We shall propose ML, ordinary jackknife and biased reducing estimators of the parameter in the right truncated exponential distribution with an unidentified uniform outlier when the truncated point is unknown and their biases and MSE's are compared numerically each other in the small sample sizes.

Keywords : Efficiency, Generalized and ordinary jackknife estimator, Outlier

1. Introduction

The problem of estimating parameters in the presence of contaminated observations, which possibly occur as outlier has been studied extensively. Outliers may arise in a variety of life testing situation ; (i) the timing mechanism for one of the items fails, yielding an overestimated lifetime ; (ii) one of the items is accidently subjected to excessively low or incorrectly measured stress, yielding an excessively long life ; (iii) a failed item is replaced one or more items but the replacement is inadvertently overlooked. Here we shall consider problems of estimation for the parameter in a right truncated exponential distribution with an unidentified uniform outlier.

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Gather(1986) found the efficient estimator for the mean of an exponential model with a single outlier. Jeevanand & Nail(1994) and Woo & Ali(1996) considered parametric estimations for two parameter exponential model in the presence of unidentified outliers. Woo & Lee(1999) studied estimations for a uniform scale parameter in the presence of an unidentified outlier. Ryu, Lee & kim(2005) considered problems for estimation of the scale parameter and right-tail probability in an exponential distribution with a unidentified Pareto outlier. Ryu, Lee and Park(2005) considered unified estimates for parameter changes in an exponential model with a Pareto outlier.

In this paper, we shall propose ML, ordinary jackknife and biased reducing estimators for the parameter in the right truncated exponential distribution with the presence of an unidentified uniform outlier when the truncated point is unknown. Also, the biases and MSE's for their estimators compare numerically in the small sample size. And the ML and moment estimators of mean parameter in the right truncated exponential distribution will be compared by their biases and MSE's.

2. Estimations for Parameter

We consider the right truncated exponential distribution with density function

$$f(x;\Theta)=\frac{1}{1-e^{-\Theta}}\,e^{-x},\quad 0<x<\Theta. \tag{2.1}$$

where the truncated point Θ is unknown, denoted by $TEXP(\Theta)$. The mean and variance for the right truncated exponential distribution are $1-\Theta e^{-\Theta}/(1-e^{\Theta})$ and $1-\Theta^2 e^{-\Theta}/(1-e^{\Theta})^2$.

The truncated exponential distribution has an increasing failure rate, ideally suited for use as a survival distribution for biological and industrial data.

For values of Θ that are corresponding to small amounts of truncation, the hazard rate function increases very slowly up to a certain time and then asymptotically climbs to infinity at Θ.

The general right truncated exponential pdf of X,

$$f(x;\eta,T)=\frac{\eta e^{-\eta x}}{1-e^{-\eta T}},0<x<T,$$

(see Hannon & Dahiya(1999)) becomes the pdf (2.1) when $Y=\eta\cdot X$, if the parameter η is known.

Balakrishnan(1994) considered order statistics from non-identical right-truncated exponential random variables and application. Hannon & Dahiya(1999) studied problems for estimation of scale parameter and truncation point for the right truncated exponential distribution when the truncated point is unknown. Woo(2003) considered jackknife estimations for the truncated point in the right truncated exponential distribution

We shall consider the following situation : Suppose $X_1, \cdots, X_n$ are independent random variables such that all but one of them are from $TEXP(\Theta)$ and one remaining random variable is from $UNIF(\Theta)$ where UNIF denotes an uniform distribution with the scale parameter Θ.

Let $X_{(1)}, \cdots, X_{(n)}$ be the corresponding order statistics for $X_1, \cdots, X_n$. Then from the factorization Theorem in Rohatgi(1976), $X_{(n)}$ is the sufficient statistic for Θ.

From the permanent theory(Vaught et al(1972)), the density function of $X_{(i)}$, $1 \le i \le n$, can be obtained as follow :

$$\begin{aligned} f_i(x) = \ & C(n,i) \frac{1}{\Theta \cdot \tau(\Theta)^{n-1}} \{ (i-1) x e^{-x} (1-e^{-x})^{i-2} (e^{-x} - e^{-\Theta})^{n-i} \\ & + (1-e^{-x})^{i-1} (e^{-x} - e^{-\Theta})^{n-i} \\ & + (n-i)(\Theta - x) e^{-x} (1-e^{-x})^{i-1} (e^{-x} - e^{-\Theta})^{n-i-1}, \quad 0 < x < \Theta, \end{aligned} \tag{2.2}$$

where $C(n,i) = (n-1)!/((i-1)! \cdot (n-i)!)$ and $\tau(\Theta) = 1 - e^{-\Theta}$.
Especially if $i = n$, then

$$f_n(x) = \frac{1}{\Theta \cdot \tau(\Theta)^{n-1}} \{ (n-1) x e^{-x} (1-e^{-x})^{n-2} - (1-e^{-x})^{n-1} \}, \quad 0 < x < \Theta.$$

And the joint density function for $X_{(i)}$ and $X_{(j)}$, $i < j \,; 1 \cdots, n$, is

$$\begin{aligned} f_{i,j}(x) \ = & \ C(n,i,j) \frac{1}{\Theta \cdot \tau(\Theta)^{n-1}} \\ \cdot \ [\ & (i-1) x e^{-x-y} (1-e^{-x})^{i-2} (e^{-y} - e^{-x})^{j-i-1} (e^{y} - e^{\Theta})^{n-j} \\ & + e^{-y} (1-e^{-x})^{i-1} (e^{-y} - e^{-x})^{j-i-1} (e^{y} - e^{\Theta})^{n-j} \\ & + (j-i-1)(y-x) e^{-x-y} (1-e^{-x})^{i-1} (e^{-y} - e^{-x})^{j-i-2} (e^{y} - e^{\Theta})^{n-j} \\ & + e^{-x} (1-e^{-x})^{i-1} (e^{-y} - e^{-x})^{j-i-1} (e^{y} - e^{\Theta})^{n-j} \\ & + (n-j)(\Theta - x) e^{-x-y} (1-e^{-x})^{i-1} (e^{-y} - e^{-x})^{j-i-1} (e^{y} - e^{\Theta})^{n-j-1}], \end{aligned} \tag{2.3}$$

where $C(n,i,j) = \dfrac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}$.

From the formulas 3.381(1) & 8.352(1) in Gradsheyn & Ryzhik(1965), we can show the following indefinite integral by finite sums.

Let $I(p,q,r\,;\Theta)\equiv\int_0^{\Theta} x^p(1-e^{-x})^q e^{-rx}dx$, where p, q, r are non-negative integers.
Then, for $r\neq 0$,

$$I(p,q,r;\Theta)=p!\cdot\sum_{k=0}^{q}(-1)^k\binom{q}{k}(k+r)^{-p-1}(1-e^{-(k+r)\Theta}\cdot\sum_{m=0}^{p}\frac{(k+r)^m}{m!}\Theta^m)$$

and

$$I(p,q,0\,;\Theta)=\frac{1}{p}\Theta^{p+1}+p!\cdot\sum_{k=1}^{q}(-1)^k\binom{q}{k}k^{-p-1}(1-e^{-k\Theta}\cdot\sum_{m=0}^{p}\frac{k^m}{m!}\Theta^m).$$

Here we shall consider parametric estimation for the parameter Θ in the right truncated exponential distribution with an unidentified uniform outlier. From the likelihood function of the density function (2.1), the ML estimator for the parameter Θ is $\widehat{\Theta}_1=X_{(n)}$.

From the result (2.2), we can obtain the expectation and variance for ML estimator $\widehat{\Theta}_1$ for Θ as following :

$$E[\widehat{\Theta}_1]=\frac{1}{\Theta\cdot\tau(\Theta)^{n-1}}\{(n-1)I(2,n-2,1\,;\Theta)+I(1,n-1,0\,;\Theta)\},\tag{2.4}$$

and

$$Var[\widehat{\Theta}_1]=\frac{1}{\Theta\cdot\tau(\Theta)^{n-1}}\{(n-1)I(3,n-2,1\,;\Theta)+I(2,n-1,0\,;\Theta)\}$$
$$-\frac{1}{\Theta^2\cdot\tau(\Theta)^{2(n-1)}}\{(n-1)I(2,n-2,1\,;\Theta)+I(1,n-1,0\,;\Theta)\}^2.$$

By definition of the ordinary jackknife technique(Gray & Schucany(1972)), we can obtain the ordinary jackknife estimator $J(\widehat{\Theta}_1)$ for $\widehat{\Theta}_1$ as follows ;

$$J(\widehat{\Theta}_1)=\frac{2n-1}{n}X_{(n)}-\frac{n-1}{n}X_{(n-1)}.$$

From the results (2.2) and (2.3), the first and second moments for ordinary jackknife estimator $J(\widehat{\Theta}_1)$ are given by

$$E[J(\widehat{\Theta}_1)] = \frac{2n-1}{n\cdot\Theta\cdot\tau(\Theta)^{n-1}}\{(n-1)I(2,n-2,1;\Theta)+I(1,n-1,0;\Theta)\}$$
$$-\frac{(n-1)^2}{n\cdot\Theta\cdot\tau(\Theta)^{n-1}}\{(n-2)I(2,n-3,2;\Theta)+(\Theta+1)I(1,n-2,1;\Theta)$$
$$-[(n-2)I(2,n-3,1;\Theta)+I(1,n-2,0;\Theta)]\cdot e^{\Theta}-I(2,n-2,1;\Theta)\},$$

and

$$E[J^2(\widehat{\Theta}_1)] = \frac{(2n-1)^2}{n^2\cdot\Theta\cdot\tau(\Theta)^{n-1}}\{(n-1)I(3,n-2,1;\Theta)+I(2,n-1,0;\Theta)\}$$
$$+\frac{(n-1)^2}{n^2\cdot\Theta\cdot\tau(\Theta)^{n-1}}\{(n-2)I(3,n-3,2;\Theta)+(\Theta+1)I(2,n-2,1;\Theta)$$
$$-[(n-2)I(3,n-3,1;\Theta)+I(2,n-2,0;\Theta)]\cdot e^{-\Theta} \tag{2.5}$$
$$-I(3,n-2,1;\Theta)\}$$
$$-\frac{2(n-1)^2(2n-1)}{n^2\cdot\Theta\cdot\tau(\Theta)^{n-1}}\{(n-2)[I(2,n-3,2;\Theta)-(1+\Theta)e^{-\Theta}I(1,n-3,1;\Theta)$$
$$+I(1,n-3,2;\Theta)]+I(2,n-2,1\,\Theta)+I(1,n-2,1;\Theta)$$
$$-(1+\Theta)e^{-\Theta}I(1,n-2,0;\Theta)+\frac{\Theta^2}{2}I(1,n-2,1;\Theta)-\frac{1}{2}I(3,n-2,1;\Theta)\}.$$

Hannon & Dahiya(1999) proposed a bias reducing estimator for the parameter Θ in the right truncated exponential distribution as follows ;

$$G(\widehat{\Theta}_1) = 2X_{(n)} - X_{(n-1)},$$

which is the generalized jackknife estimator for Θ.

From the results (2.2) and (2.3), the first and second moments for the bias reducing estimator $\widehat{\Theta}_2$ are given by

(2.6)

From the results (2.4) through (2.6), Table 1 shows the numerical values of biases and MSE's for proposed three estimators of the parameter Θ in the right truncated exponential distribution with an unidentified uniform outlier for the sample size n=10(5)30, the truncation point $\Theta = 1$.

From Table 1, the bias reducing estimator is more efficient than other esrimators of the parameter in the right truncated exponential distribution with an unidentified uniform outlier in a sense of MSE and bias when $\Theta = 1$.

<Table 1> Bias and MSE's for the proposed estimators for the parameter Θ

n	$\widehat{\Theta}_1$		$J(\widehat{\Theta}_1)$		$G(\widehat{\Theta}_1)$	
	Bias	MSE	Bias	MSE	Bias	MSE
10	0.13194	0.02966	0.02360	0.01538	0.01160	0.01298
15	0.09470	0.01592	0.01254	0.01277	0.00667	0.01013
20	0.07406	0.00995	0.00782	0.01053	0.00433	0.00097
25	0.06085	0.00682	0.00535	0.00094	0.00304	0.00085
30	0.05153	0.00472	0.00425	0.00075	0.00262	0.00062

Now, we consider estimation for mean $\mu \equiv 1-\Theta e^{-\Theta}/(1-e^{\Theta})$ in a right truncated exponential distribution with an unidentified uniform outlier when the truncated point is unknown. Then the ML estimator for the parameter μ is

$$\widehat{\mu}_1 = 1 - \frac{X_{(n)} e^{-X_{(n)}}}{(1-e^{-X_{(n)}})}.$$

From the result (2.2), we can obtain the expectation and variance for ML estimator $\widehat{\mu}_1$ for μ as following :

$$E[\widehat{\mu}_1] = 1 - \frac{1}{\Theta \cdot \tau(\Theta)^{n-1}} \{(n-1)I(2,n-3,2;\Theta) + I(1,n-2,1;\Theta)\},$$

$$Var[\widehat{\mu}_1] = \frac{1}{\Theta \cdot \tau(\Theta)^{n-1}} \{(n-1)I(3,n-4,3;\Theta) + I(2,n-3,2;\Theta)\} \qquad (2.7)$$

$$- \frac{1}{\Theta^2 \cdot \tau(\Theta)^{2(n-1)}} \{(n-1)I(2,n-3,2;\Theta) + I(1,n-2,1;\Theta)\}^2.$$

And the moment estimator for the parameter μ is $\widehat{\mu}_2 = \overline{X}$ which is the sample mean. The expectation and variance for moment estimator $\widehat{\mu}_2$ for μ are given by

$$E[\widehat{\mu}_2] = \frac{1}{n}\{(n-1)(1-\frac{\Theta e^{-\Theta}}{\tau(\Theta)})+\frac{\Theta}{2}\},$$
$$Var[\widehat{\mu}_2] = \frac{1}{n^2}\{(n-1)(1-\frac{\Theta^2 e^{-\Theta}}{\tau(\Theta)^2})+\frac{\Theta^2}{12}\}. \qquad (2.8)$$

Since the mean parameter $\mu = \mu(\Theta) = 1 - \frac{\Theta e^{-\Theta}}{1-e^{-\Theta}}$, $\Theta>0$, $\lim_{\Theta\to 0}\mu(\Theta)=1$ and $\frac{d}{d\Theta}\mu(\Theta) = \frac{e^{-\Theta}(1-\Theta-e^{-\Theta})}{(1-e^{-\Theta})^2}$ is negative, and hence $\mu(\Theta)$ is a monotone decreasing function of Θ. Therefore, inference on Θ is equivalence to inference on $\mu(\Theta)$, and so it's sufficient for us to estimate Θ instead of estimating $\mu(\Theta)$.

From the results (2.7) and (2.8), Table 2 shows the numerical values of biases and MSE's for ML and moment estimators of the mean parameter μ in the right truncated exponential distribution with an unidentified uniform outlier for the sample size n=10(5)30, the truncation point $\Theta=1$.

From the Table 1, the ML estimator is more efficient than moment estimator of the parameter in the right truncated exponential distribution with an unidentified uniform outlier in a sense of MSE and bias when $\Theta=1$.

<Table 2> Bias and MSE's for proposed estimators for mean μ

n	$\widehat{\mu}_1$		$\widehat{\mu}_2$	
	Bias	MSE	Bias	MSE
10	0.04686	0.00388	0.09180	0.08549
15	0.03329	0.00202	0.06120	0.05664
20	0.02584	0.00124	0.04590	0.04234
25	0.02113	0.00084	0.03672	0.03381
30	0.01788	0.00059	0.03060	0.02813

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