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A Note on the Minimal Variability Weighting Function Problem

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Abstract

Recently, Liu (2005) proposed a special type of weighting function under a given preference index level with the minimal variability similar to the minimal variability OWA operator weights problem proposed by Fullèr and Majlender (2003). He solved this problem using a result of classical optimal control theory. In this note, we give a direct elementary proof of this problem without using any known results.

Keywords : Fuzzy sets, Minimal variability, OWA operator, Preference index level, Weighting function

1. Introduction

An OWA operator [9,11,12] of dimension n is a mapping $F: R^n \rightarrow R$ that has an associated weighting vector $W = (w_1, \cdots, w_n)^T$ having the properties $w_1 + \cdots + w_n = 1, 0 \le w_i \le 1$, $i = 1, \cdots, n$, and such that

$$F(a_1, \cdots, a_n) = \sum_{i=1}^{n} w_i b_i,$$

where b_j is the jth largest element of the collection of the aggregated objects $\{a_1, \cdots, a_n\}$.

In Yager(1988), Yager introduced a measure of "orness" associated with the weighting vector W of an OWA operator, defined as

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$$orness(W) = \sum_{i=1}^{n} \frac{n-1}{n-i} w_{i.}$$

This measure characterizes the degree to which the aggregation is like an or operation. He also introduced the measure of "dispersion" of the aggregation, defined as

$$disp(W) = -\sum_{i=1}^{n} w_i \ln w_{i.}$$

This measure measures the degree to which W takes into account all information in the aggregation.

An important issue is determining a special class of OWA operators that take into account maximum information available in the aggregation for a given level of compensation. The following approaches, suggested by O'Hagan(1988), determine a special class of OWA operators having maximal entropy of the OWA operator weights for a given level of orness. This is based on solving the following mathematical program problem:

$$\begin{aligned} &\text{Maximize} &&: disp(W) = -\frac{1}{n}\sum_{i=1}^{n} w_i \ln w_{i.} \\ &\text{subject to} &&: orness(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \le \alpha \le 1, \\ & &&\quad w_1 + \cdots + w_n = 1, \quad 0 \le w_i, \quad i = 1, \cdots, n, \end{aligned}$$

which was solved analytically in Fullèr and Majlender(2001).

Another approach, considered by Fullèr and Majlender(2003), is to determine a class of OWA operators having minimal variability weights for any given level of orness, where the measure of variance is defined as the average squared distance between the associated weights and their mean value. This approach is based on the following constrained mathematical programming problem:

$$\begin{aligned} &\text{Minimize} &&: \sum w_i^2 \\ &\text{subject to} &&: orness(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \le \alpha \le 1, \\ & &&\quad w_1 + \cdots + w_n = 1, \quad 0 \le w_i, \quad i = 1, \cdots, n, \end{aligned}$$

which was solved analytically and derived the exact minimal variability OWA weights for any level of orness using the Kuhn-Tucker second-order sufficiency conditions for optimality.

Yager(2004) provided an extension of the OWA operator to the case of a continuous valued interval rather than a finite set of values. This introduces the idea of attitudinal-based expected value associated with a continuous random variable.

The interval approximation of a fuzzy number is considered as an aggregation of different level sets with the weighting function in the unit interval. The weighting function characterizes the aggregation of interval cuts of a fuzzy number, which can represent the decision maker's preferences in the processes. Recently, Liu(2006) extended the concept of the weighting function proposed by Fullér and Majlender(2003) without the monotonic increasing assumption in the interval approximation of the fuzzy number context. Similar to the maximum entropy OWA operator Fullér and Majlender(1988,2003), Liu(2006) proposed a special type of weighting function under a given preference index level with the maximum entropy principle.

The Shannon entropy of the weighting function $f(r)$ is

$$H_f = -\int_0^1 f(r)\ln f(r)\,dr$$

with $\int_0^1 f(r)\,dr = 1, \quad f(r) \geq 0.$

The maximum entropy weighting function problem with a given preference index value is

$$\begin{aligned} &\text{Maximize} && -\int_0^1 f(r)\ln f(r)\,dr \\ &\text{subject to} && \int_0^1 rf(r)\,dr = \alpha, \quad 0 < \alpha < 1, \\ & && \int_0^1 f(r)\,dr = 1, \quad f(r) \geq 0 \end{aligned}$$

$E_f = \int_0^1 rf(r)\,dr$ is called the preference index value of the weighting function f.

Recently, Liu(2005) proposed a special type of weighting function under a given preference index level with minimal variability, similar to minimal variability OWA operator weights problem proposed by Fullér and Majlender(2003).

The minimal variability weighting function problem with a given preference index level is

$$\begin{aligned} &\text{Minimize} && D_f = \int_0^1 f^2(r)\,dr \\ &\text{subject to} && \int_0^1 rf(r)\,dr = \alpha, \quad 0 < \alpha < 1, \\ & && \int_0^1 f(r)\,dr = 1, \quad f(r) \geq 0. \end{aligned} \tag{1}$$

He solved above problem (1) using a result of classical optimal control theory (Sethi(1981)) and drive the minimal variability weighting function for any given level of preference index. In this note, we give a direct elementary proof of this problem without using any known results.

2. Obtaining minimal variability weighting functions

Without loss of generality, we can assume that $\alpha\in(0,0.5]$, since if a weighting function $f^*(r)$ is optimal to problem (1) for some given level of preference $\alpha\in(0,0.5]$ then $f^*(1-r)$ is optimal to the problem (1) for a given level of preference $1-\alpha$. Indeed, since $D_f=D_{f^R}$, $\int_0^1 f(r)\,dr=\int_0^1 f^R(r)\,dr$ and $E_{f^R}=1-E_f$, where $f^R(r)=f(1-r)$ hence for $\alpha>0.5$, we can consider problem (1) for the level of preference with index $1-\alpha$, and then take the reverse of that optimal solution.

We first consider the following weighting function for given α such that $1/3\le\alpha\le 1/2$

$$f^*(r)=\begin{cases}(12\alpha-6)r+4-6\alpha, & \text{if } r\in[0,1]\\ 0 & \text{elsewhere,}\end{cases} \tag{2}$$

and for $0<\alpha\le 1/3$,

$$f^*(r)=\begin{cases}-2r/9\alpha^2+2/3\alpha, & \text{if } r\in[0,3\alpha]\\ 0 & \text{elsewhere.}\end{cases} \tag{3}$$

Then we can easily check that these weighting functions are feasible for problem (1).

We will show now that f^*, defined by (2) and (3), is the unique optimal solution for a given α.

Let nonnegative function f satisfy $1=\int_0^1 f(r)\,dr$ and $\int_0^1 rf(r)\,dr=\alpha$. Without loss of generality, we may assume that $\{r: f^*(r)=0\}=[t,1]$ for some $t\in(0,1]$ and $f^*(r)=a^*r+b^*$, $r\in[0,t]$. We put $f(r)=f^*(r)+g(r)$, $r\in[0,1]$. Then, noting that $f(r)=g(r)$, $r\in[t,1]$ we have

$$\int_0^t g(r)\,dr+\int_t^1 f(r)\,dr=\int_0^1 g(r)\,dr=0, \tag{4}$$

since $1=\int_0^1 f(r)\,dr=\int_0^1 f^*(r)\,dr+\int_0^1 g(r)\,dr=1+\int_0^1 g(r)\,dr$. We also have

$$\int_0^t rg(r)\,dr+\int_t^1 rf(r)\,dr=\int_0^1 rg(r)\,dr=0, \tag{5}$$

since $₩ $\alpha = \int_0^1 rf(r)\,dr = \int_0^1 rf^{*(r)dr+} \int_0^1 rg(r)\,dr = \alpha + \int_0^1 rg(r)\,dr$. We now show that

$$\int_0^1 r^2 f(r)\,dr \geq \int_0^1 r^2 f^*(r)\,dr.$$

It is because

$$\begin{aligned}
&\int_0^1 f^2(r)\,dr - \int_0^1 f^{*2}(r)\,dr \\
&= \int_0^1 (f^*(r) + g(r))^2\,dr - \int_0^1 f^{*2}(r)\,dr \\
&= 2\int_0^1 f^*(r)g(r)\,dr + \int_0^1 g(r)^2\,dr \\
&= 2\int_0^t (a^* r + b^*)g(r)\,dr + \int_0^1 g(r)^2\,dr \\
&= 2a^* \int_0^t rg(r)\,dr + 2b^* \int_0^t g(r)\,dr + \int_0^1 g(r)^2\,dr \\
&= 2a^*\left(-\int_t^1 rf(r)\,dr\right) + 2b^*\left(-\int_t^1 f(r)\,dr\right) + \int_0^1 g(r)^2\,dr \\
&= -2\int_t^1 (a^* r + b^*)f(r)\,dr + \int_0^1 g(r)^2\,dr \\
&\geq \int_0^1 g(r)^2\,dr \geq 0,
\end{aligned}$$

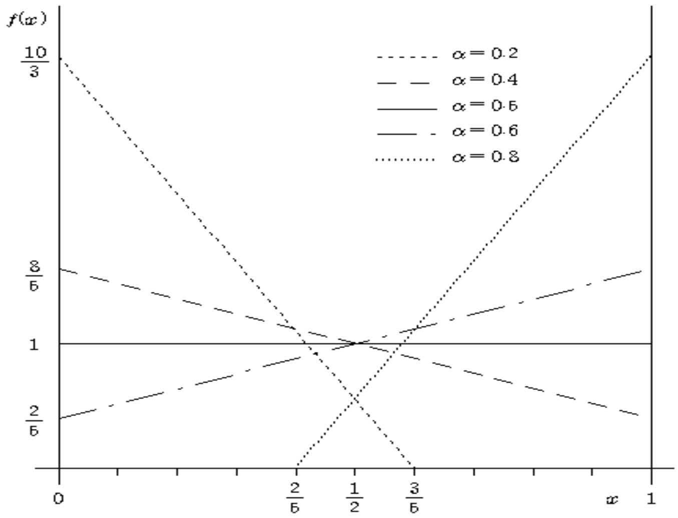
where the fifth equality comes from (4) and (5), and the first inequality comes from the fact that $a^* r + b^* \leq 0$ for $r \in [t,1]$ and the equality holds if and only if $f^* = f$ which completes the proof.

The shape of $f^*(r)$ defined by (2) and (3) for different levels of preference index E_f are plotted in Fig. 1.

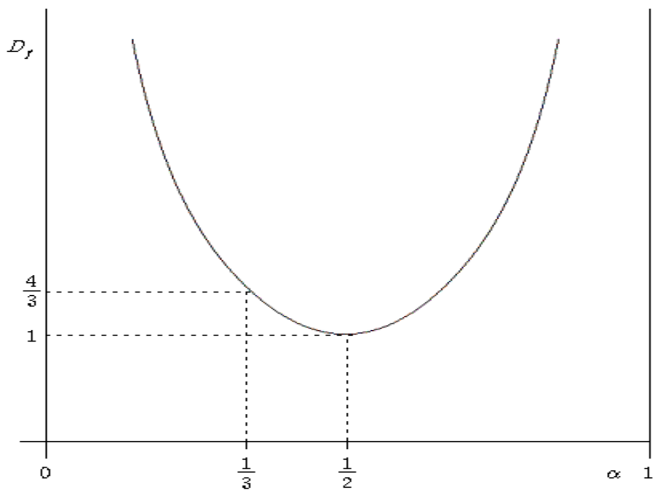
The value D_{f^*} for $0 < \alpha < 1$ is

$$D_{f^*}(\alpha) = \begin{cases} 4/(9\alpha) & \text{if } \alpha \in (0, 1/3], \\ 12\alpha^2 - 12\alpha + 4, & \text{if } \alpha \in [1/3, 2/3], \\ 4/9(1-\alpha) & \text{if } \alpha \in [1/3, 1), \end{cases}$$

The relationship between D_{f^*} and α is plotted in Fig.2. From Fig.2, we can obtain that the value of the minimal variability weighting function f^*, D_{f^*} is decreasing when $E_f < 1/2$, and the value of the minimal variability weighting function f^* is increasing when $E_f > 1/2$, and D_{f^*} reaches its minimum when $E_f = 1/2$.



<Figure 1> The shape of $f(r)$ with different E_f



<Figure 2> The minimal variability D_f changes with α.

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