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Performances of VSI Multivariate Control Charts with Accumulate-Combine Approach

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Abstract

Performances of variable sampling interval(VSI) multivariate control charts with accumulate-combine approach for monitoring mean vector of p related quality variables were investigated. Shewhart control chart is also proposed to compare the performances of CUSUM and EWMA charts. Numerical comparisons show that performances of CUSUM and EWMA charts are more efficient than Shewhart chart for small or moderate shifts, and VSI chart is more efficient than fixed sampling interval(FSI) chart. We also found that performances of the CUSUM or EWMA chart with accumulate-combine approach are substantially efficient than those of Shewhart chart.

Keywords : Accumulate-combine approach, ANSS, ANSW, ATS

1. Introduction

The traditional practice in using a control chart to monitor a process is to take sample from the process at FSI. Recent years, applications of VSI procedure have become quite frequent. VSI procedures vary the sampling interval as a function of what is observed from the process can detect process changes faster than FSI charts. For the VSI chart, Reynolds(1989) showed that the use of two sampling intervals spaced as apart as possible is optimal.

One disadvantage of VSI scheme is that frequent switching between different sampling intervals requires more cost and effort to administer the process than corresponding FSI scheme. Amin and Letsinger(1991) described general procedures for VSI scheme and presented that the average number of switches to signal

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(ANSW) of the CUSUM and EWMA procedures exists far fewer than the Shewhart procedure.

The Shewhart chart, although simple to understand and apply, uses only the informations in the current sample and is thus relatively inefficient in detecting small shifts of the process. CUSUM or EWMA charts are often used instead of Shewhart when detection of small shifts in a process is important. Multivariate EWMA and CUSUM charts can takes two ways to use the past sample information, such as combine-accumulate approach and accumulate-combine approach. Crosier(1988) and Pignatiello and Runger(1990) considered new multivariate CUSUM control schemes that accumulate past sample information for each process parameter and then form a univariate CUSUM statistic from the multivariate data.

A multivariate EWMA control chart for monitoring mean vector of a multivariate normal process using accmulate-combine approach was presented by Lowry et. al.(1992). By simulation, they showed that the performances of the multivariate EWMA procedure performs better than the multivariate CUSUM procedures of Crosier(1988) and Pignatiello and Runger(1990), and it performs roughly the same if small shift in the mean vector has occurred. Vargas et al.(2004) presented a comparative study of the performance of CUSUM and EWMA charts in order to detect small changes of process average.

In this paper, we investigate the performances of the VSI multivariate charts with accumulate-combine approach for monitoring $\underline{\mu}$ of multivariate normal process. By numerical computation, VSI procedures have been shown to be more efficient when compared to the corresponding FSI procedures with respect to the ATS. And we also found that performances of EWMA and CUSUM charts based on accumulate-combine feature are more effective than Shewhart charts.

2. Constructing Sample Statistics

Assume that the $p\,(p \geq 2)$ quality variables of interest are $\underline{X}=(X_1,\cdots,X_p)'$ and we take a sequence of random vectors $\underline{X}_1, \underline{X}_2, \cdots$ where $\underline{X}_i=(\underline{X}'_{i1},\cdots,\underline{X}'_{\in})'$ is a sample of observations at the sampling time $i\,(i=1,2,\cdots)$ and $\underline{X}_{ij}=(X_{ij1},\cdots,X_{ijp})'$. It will be also assumed that the distribution of successive observation vectors are $N_p(\underline{\mu},\Sigma)$.

Let $\underline{\theta}_0=(\underline{\mu}_0,\Sigma_0)$ be the known target process parameters for $\underline{\theta}=(\underline{\mu},\Sigma)$ of p related quality variables. For simplicity, it is assumed that $\underline{\mu}_0=\underline{0}'$, all diagonal and off-diagonal elements of Σ_0 are 1 and 0.3, respectively.

2.1 Sample Statistic for Shewhart Chart

Since the general statistical quality control chart can be considered as a repetitive tests of significance, we can obtain multivariate control statistic for monitoring μ by using the likelihood ratio test(LRT) statistic for testing $H_0 : \mu = \mu_0$ vs $H_1 : \mu \neq \mu_0$ where Σ_0 is known. By calculation, we can obtain LRT statistic as

$$Z_i^2 = n(\overline{X}_i - \mu_0)' \Sigma_0^{-1} (\overline{X}_i - \mu_0). \tag{2.1}$$

Thus, the sample statistic Z_i^2 can be used as the control statistic for monitoring μ of p quality variables. Alt(1982) described various types of multivariate Shewhart type T^2 charts based on Hotelling's $T^2 = n(\overline{X}_i - \mu_0)' S^{-1} (\overline{X}_i - \mu_0)$ statistic and provided recommendations for implementation where S is the covariance matrix of the sample.

2.2 Sample Statistic for CUSUM Chart

The CUSUM chart is a good alternative to the Shewhart chart when small or moderate shifts are important. This chart is maintained by taking samples and plotting a cumulative sum of differences between sample statistic and the target value in time order on the chart.

Pignatiello and Runger(1990) proposed a multivariate CUSUM chart based on accumulate-combine approach for controlling μ of $N_p(\mu, \Sigma)$ process. This chart, called MC1, is based on the following vectors of cumulative sum :

$$\underline{D}_i = \sum_{j=i-l_i+1}^{i} (\overline{\underline{X}}_j - \underline{\mu}_0)$$

and

$$MC1_i = \max\{0, (n\underline{D}_i' \Sigma_0^{-1} \underline{D}_i)^{1/2} - k l_i\}, \tag{2.2}$$

where reference value $k > 0$ and $i = 1, 2, \cdots$,

$$l_i = \begin{cases} l_{i-1} + 1 & \text{if } MC1_{i-1} > 0 \\ 1 & \text{otherwise} \end{cases}$$

2.3 Sample Statistic for EWMA Chart

The EWMA control chart is based on an exponentially weighted moving average of the current and past sample information. In EWMA chart, the more recent observations are assigned more weights and the older observations are assigned less weights.

Lowry et al.(1992) proposed an MEWMA chart for μ with accumulate-combine technique. They asserted that MEWMA chart for μ is a more straightforward generalization of the corresponding univariate procedure than the multivariate CUSUM statistics in (2.2). The vectors of EWMA's are defined as

$$\underline{Y}_i = (I-\Lambda)\underline{Y}_{i-1} + \Lambda \underline{\overline{X}}_i \tag{2.3}$$

$i=1,2,3,\cdots$ where $\underline{Y}_0 = \underline{\mu}_0$ and $\Lambda = diag(\lambda_1, \lambda_2, \cdots, \lambda_p), 0 < \lambda_j \leq 1$ $(j=1,2,\cdots,p)$. Equation (2.3) can be rewritten by repeated substitution as

$$\underline{Y}_i = \sum_{k=1}^{i} \Lambda (I-\Lambda)^{i-k} \underline{\overline{X}}_k + (I-\Lambda)^i \underline{\mu}_0. \tag{2.4}$$

Then the dispersion matrix of $\underline{Y}_i$ can be obtained as

$$\begin{aligned} \Sigma_{\underline{Y}_i} &= \sum_{k=1}^{i} Var(\Lambda (I-\Lambda)^{i-k} \underline{\overline{X}}_k) \\ &= \sum_{k=1}^{i} [\Lambda (I-\Lambda)^{i-k} \Sigma (I-\Lambda)^{i-k} \Lambda]/n, \end{aligned} \tag{2.5}$$

where Σ is a known covariance matrix.

3. FSI Control Charts

In FSI chart, $t_{i+1} - t_i$, the length of sampling interval between sampling times, is constant for all $i (i=0,1,\cdots)$. Therefore, the expected time to signal is simply the product of the average run length(ARL) and the length of sampling interval. Hence, the ARL in FSI chart has the same definition as the ANSS in VSI chart.

3.1 FSI Shewhart Chart

Multivariate Shewhart chart for $\underline{\mu}$ based on the sample statistics Z_i^2 in (2.1) signals whenever

$$Z_i^2 \geq h. \tag{3.1}$$

If the process is in-control, the statistic Z_i^2 has a chi-squared distribution with p degrees of freedom when $\underline{\mu} = \underline{\mu}_0$ and $\Sigma = \Sigma_0$. When the process has shifted to $\underline{\mu}$ from the target $\underline{\mu}_0$, Z_i^2 has a noncentral chi-square distribution with p degrees of freedom and noncentrality parameter $\tau^2 = n(\underline{\mu} - \underline{\mu}_0)' \Sigma_0^{-1} (\underline{\mu} - \underline{\mu}_0)$.

The percentage point of Z_i^2 can be obtained from the chi-square distribution when the process are in-control or target $\underline{\mu}_0$ has changed. UCL(upper control limit) h can be obtained from chi-square distributions to satisfy a desired ANSS. And, ANSS of this chart can be calculated as $1/P$ where P denotes the probability that χ^2 exceeds the UCL. The on-target value of P is determined from the probability that χ^2 exceeds the UCL under the central

$\chi^2(p)$ distribution while the off-target value of P is determined from the probability that χ^2 exceeds the UCL under the noncentral $\chi^2(p)$ distribution

3.2 FSI CUSUM Chart

This multivariate CUSUM chart based on (2.1) signals when

$$MC1_i > h_1 \tag{3.2}$$

where $h_1 > 0$. Pignatiello and Runger (1990) found that an $MC1$ chart based on the accumulate-combine approach has a superior ARL performance than a multivariate CUSUM chart based on the combine-accumulate approach.

3.3 FSI EWMA Chart

MEWMA chart for means signals whenever

$$T_i^2 = (\underline{Y}_i - \underline{\mu}_0)' \Sigma^{-1}_{\underline{Y}_i} (\underline{Y}_i - \underline{\mu}_0) > h_2,$$

where $h_2(>0)$ is chosen to achieve a specified in-control ANSS. If there is no distinct reason to weight past observations differently for the p quality variables being monitored, we can let $\lambda_1 = \cdots = \lambda_p = \lambda$. Then, MEWMA vectors can be written as

$$\underline{Y}_i = (1-\lambda)\underline{Y}_{i-1} + \lambda \overline{\underline{X}}_i \tag{3.3}$$

$i = 1, 2, \cdots$, and the dispersion matrix of $\underline{Y}_i$ is given by

$$\Sigma_{\underline{Y}_i} = \frac{\lambda}{2-\lambda}[1-(1-\lambda)^{2i}]\frac{\Sigma}{n}. \tag{3.4}$$

Lowry et al.(1992) also showed that the distribution of T_i^2 depends on $\underline{\mu}$ and Σ only through the noncentrality parameter τ as

$$\tau = [n(\underline{\mu} - \underline{\mu}_0)' \Sigma^{-1} (\underline{\mu} - \underline{\mu}_0)]^{1/2}$$

and smaller values of λ are more effective in detecting the smaller shifts for $\underline{\mu}$. Since it is difficult to obtain the joint distribution of T_i^2, we obtain the parameters h_1 and performances of this chart by simulation.

4. VSI Control Charts

In VSI procedures, the sampling times are random variables and $t_{i+1} - t_i$ is a function of chart statistic and depends on the past sample informations $X_1, X_2, \cdots, X_i$. Therefore in VSI charts, the time required for the chart to signal is not a constant multiple of the run length. To evaluate the performance of a VSI control chart, it is necessary to obtain time and number of samples separately. Therefore, we use ATS and ANSS for evaluating and comparing the properties of the FSI and VSI charts.

But, frequent switching between the different sampling intervals d_1 and d_2 can be a complicating factor in the application of control charts with VSI procedures. Therefore, it is necessary to define the number of switches(NSW) as the number of switches made from the start of the process until the chart signals, and let the average number of switches (ANSW) be the expected value of the NSW. The ANSW can be obtained as follows

$$ANSW = (ANSS - d_0) \cdot P(\text{switch}) \tag{4.1}$$

And, the probability of switch is given by

$$P(\text{switch}) = P(d_1) \cdot P(d_2 \mid d_1) + P(d_2) \cdot P(d_1 \mid d_2) \tag{4.2}$$

where $P(d_i)$ is the probability of using sampling interval d_i, and $P(d_i \mid d_j)$ is the conditional probability of using sampling interval d_i in the current sample given that the sampling interval d_j $(d_i \neq d_j)$ was used in the previous sample.

4.1 VSI Shewhart Chart

For VSI Shewhart chart based on Z_i^2 in (2.1), suppose that the sampling interval ;

$$d_1 \text{ is used when } Z_i^2 \in (g, h],$$
$$d_2 \text{ is used when } Z_i^2 \in (0, g],$$

where $g_{Z^2} \leq h_{Z^2}$ and $d_1 < d_2$.

If the process is in-control, the LRT statistic Z_i^2 has a chi-squared distribution with p degrees of freedom when $\mu = \mu_0$ and $\Sigma = \Sigma_0$. Hence, the design parameters g and h can be obtained to satisfy a desired ATS and ANSS. When the process has shifted to μ from the target μ_0, Z_i^2 has a noncentral chi-square distribution with p degrees of freedom and noncentrality parameter τ^2. The ANSW values of VSI Shewhart charts are evaluated by Markov chain method.

4.2 VSI CUSUM Chart

For VSI CUSUM chart based on $MC1$, suppose that the sampling interval ;

$$d_1 \text{ is used when } MC1_i \in (g_1, h_1],$$
$$d_2 \text{ is used when } MC1_i \in (0, g_1],$$

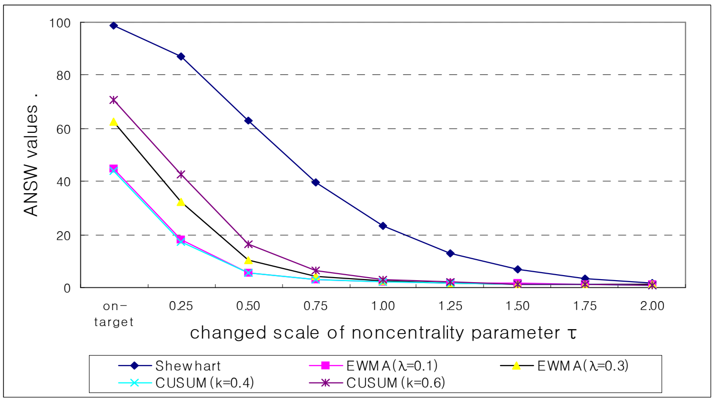
where $g_{MC1} \le h_{MC1}$ and $d_1 < d_2$. The numerical performances of the chart in (3.2), and the design parameters g_{MC1} and h_{MC1} were obtained to satisfy a desired ATS and ANSS by simulation.

4.3 VSI EWMA Chart

And for VSI EWMA chart based on $\underline{Y}_i$ in (2.4), suppose the sampling interval ;

$$d_1 \text{ is used when } T_i^2 \in (g_2, h_2],$$
$$d_2 \text{ is used when } T_i^2 \in (0, g_2],$$

where $g_2 \le h_2$ and $d_1 < d_2$. Since it is difficult to obtain the exact distribution of chart statistic $\underline{Y}_i$, process parameters g_2, h_2 can be obtained to satisfy a specified



<Figure 1> ANSW values for small shift($p=3$)

<Table 1> Performances of Shewhart control chart

scale of shift	$p=2$			$p=3$			$p=4$		
	ANSS	ATS	ANSW	ANSS	ATS	ANSW	ANSS	ATS	ANSW
in-control	200.0	200.0	98.5	200.0	200.0	98.5	200.0	200.0	98.5
τ=0.25	171.0	167.8	84.0	177.4	174.5	87.2	181.0	178.5	89.0
τ=0.5	115.5	107.3	55.9	129.2	121.2	62.8	138.1	130.5	67.3
τ=0.75	70.4	59.7	32.7	84.1	73.0	39.7	94.4	83.1	44.9
τ=0.1	41.9	31.5	17.9	52.4	40.8	23.2	61.0	48.7	27.5
τ=1.5	15.8	8.8	4.7	20.4	12.1	6.8	24.6	15.2	8.8
τ=2.0	6.9	3.1	1.0	8.8	4.0	1.6	10.6	5.0	2.3
τ=2.5	3.5	1.6	0.2	4.4	1.9	0.3	5.2	2.2	0.5
τ=3.0	2.2	1.2	0.0	2.6	1.3	0.0	2.9	1.4	0.1
τ=3.5	1.5	1.1	0.0	1.7	1.1	0.0	1.9	1.1	0.0
τ=4.0	1.2	1.0	0.0	1.3	1.0	0.0	1.4	1.1	0.0

<Table 2> Performances of CUSUM control charts

scale of shift	p=2						p=4					
	k=0.4			k=0.6			k=0.4			k=0.6		
	ANSS	ATS	ANSW	ANSS	ATS	ANSW	ANSS	ATS	ANSW	ANSS	ATS	ANSW
in-control	200.0	200.0	52.7	200.0	200.0	72.1	200.0	200.0	39.9	200.0	200.0	55.7
τ=0.25	81.2	70.5	19.8	99.4	91.3	34.0	92.3	80.6	17.6	116.3	107.3	31.2
τ=0.5	27.8	18.3	6.0	35.1	25.8	10.5	31.8	21.0	5.6	41.7	31.0	10.1
τ=0.75	14.3	7.9	3.0	16.1	9.4	4.2	16.4	9.1	3.0	18.9	10.9	4.1
τ=0.1	9.4	4.7	2.1	9.5	4.8	2.4	10.8	5.7	2.2	10.9	5.4	2.4
τ=1.5	5.5	2.6	1.5	5.0	2.3	1.4	6.5	3.3	1.7	5.8	2.6	1.5
τ=2.0	4.0	1.8	1.2	3.5	1.6	1.2	4.7	2.4	1.5	4.1	1.8	1.3
τ=2.5	3.2	1.4	1.1	2.7	1.3	1.0	3.8	1.8	1.3	3.2	1.4	1.1
τ=3.0	2.6	1.3	1.0	2.3	1.2	1.0	3.2	1.5	1.2	2.6	1.3	1.0
τ=3.5	2.3	1.2	1.0	2.0	1.1	0.9	2.8	1.3	1.1	2.3	1.2	1.0
τ=4.0	2.1	1.1	1.0	1.8	1.1	0.7	2.4	1.2	1.0	2.1	1.1	1.0

<Table 3> Performances of EWMA control charts

scale of shift	p=2						p=4					
	λ=0.1			λ=0.3			λ=0.1			λ=0.3		
	ANSS	ATS	ANSW	ANSS	ATS	ANSW	ANSS	ATS	ANSW	ANSS	ATS	ANSW
in-control	200.0	200.0	44.9	200.0	200.0	72.4	200.0	200.0	43.4	200.0	200.0	70.1
τ=0.25	73.7	59.1	15.7	111.5	101.7	38.8	89.4	74.5	18.8	132.7	123.3	45.5
τ=0.5	25.1	14.5	5.0	42.8	31.0	13.3	31.5	18.7	6.2	58.3	44.6	18.6
τ=0.75	12.5	6.2	2.6	19.2	10.7	5.2	15.6	7.8	3.2	26.4	15.7	7.5
τ=0.1	7.7	3.6	1.9	10.6	5.0	2.7	9.5	4.5	2.2	14.1	6.9	3.6
τ=1.5	4.0	2.0	1.2	4.8	2.2	1.4	4.9	2.3	1.4	6.0	2.7	1.6
τ=2.0	2.6	1.4	0.9	3.0	1.5	1.0	3.1	1.6	1.0	3.6	1.7	1.1
τ=2.5	1.9	1.2	0.6	2.1	1.2	0.7	2.2	1.3	0.8	2.5	1.3	0.9
τ=3.0	1.5	1.1	0.4	1.6	1.1	0.5	1.7	1.1	0.6	1.9	1.1	0.7
τ=3.5	1.3	1.0	0.2	1.3	1.0	0.3	1.4	1.1	0.4	1.5	1.1	0.5
τ=4.0	1.1	1.0	0.1	1.2	1.0	0.2	1.2	1.0	0.2	1.3	1.0	0.3

ATS and ANSS. And when the process is in-control or out-of-control states, the values ANSS, ATS, ANSW can be evaluated by simulation.

5. Computational Results and Concluding Remarks

In evaluating the usefulness of the VSI feature, it is usual to compare the performances of the VSI procedure to the same procedure using FSI. In this article, the numerical results were obtained when the ANSS and ATS of the in-control state was approximately equal to 200.0, d_0=1 and the sample size for each variable was five for $p=2\sim4$. After the reference values and smoothing constants of the proposed charts have been determined, the UCL $h's$ and $g's$ of the proposed charts were calculated by Markov chains with the number of transient states $r=100$ or simulation with 10,000 iterations.

When the process is in-control or mean vector of the quality variables has changed, the performances of the proposed charts are presented in Table 1 ~ 3. ANSS, ATS and ANSW values in tables were also obtained by Markov chains with the number of transient states $r=100$ or simulation with 10,000 runs.

In our computation, VSI charts have been shown to be more efficient when compared to the corresponding FSI charts with respect to the ATS. When small or moderate changes in the process have occurred, the ANSS, ATS and ANSW values for CUSUM or EWMA procedure with accumulate-combine approach are substantially less than those of Shewhart procedure.

Numerical results for various reference values show that large reference values are efficient for large shifts from the target value and vice versa in multivariate CUSUM charts in terms of ANSS, ATS and ANSW. Our numerical results also show that the $MC1$ chart appears to be a good control charting device for detecting small or moderate shifts in the mean vector of a multivariate normal process.

As illustrated in table, smaller values of λ are more effective in detecting small shifts in the mean vector in EWMA charts. Our numerical results also show that multivariate EWMA chart based on accumulate-combine approach can be recommended for any scale of shifts in mean vector of the process. The optimal selection of λ depends on the size of the shift in the mean vector to be detected quickly. And, it may be possible to improve the performance of the MC1 chart at selected off-target conditions with alternate choices of k.

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