Journal of the Korean
Data & Information Science Society
2006, Vol. 17, No. 3, pp. 953 ~961

Multivariate Poisson Distribution Generated via Reduction from Independent Poisson Variates

Daehak Kim[1) ・ Heong Chul Jeong[2)

Abstract

Let's say that we are given a k number of random variables following Poisson distribution that are individually dependent and which forms multivariate Poisson distribution. We particularly dealt with a method of creating random numbers that satisfies the covariance matrix, where the elements of covariance matrix are parameters forming a multivariate Poisson distribution. To create such random numbers, we propose a new algorithm based on the method reducing the number of parameter set and deal with its relationship to the Park et al.(1996) algorithm used in creating multivariate Bernoulli random numbers.

Keywords : Dependent structure, Multivariate Poisson distribution, Random number, Variance-covariance matrix

1. Introduction

A Poisson distribution can generate in various fields such as counts of rare generation rates and also is a distribution that is highly utilized in fields such as biology and insurance. Situations of fatal traffic accidents or contagious diseases all follow the Poisson distribution.

Now let's say that a k number of variables $X_1, \cdots, X_k$ each follow the Poisson distribution of $P(\lambda_i), i=1,\cdots,k$ and that they each form complex correlation structures. If a discrete distribution like Poisson distribution forms a multivariate structure, it can create significant amount of problems in statistical inferences such as probability calculation, estimation, and generation

1) Department of statistical information, Catholic University of Daegu, Kyungbuk, 712-702, Korea
E-mail : dhkim@cu.ac.kr

1) Department of applied statistics, Suwon University, Kyunggi, Korea
E-mail : jhc@suwon.ac.kr

of random numbers due to numerous amounts of parameters. Tsiamyrtzis and Karlis(2004) have proposed an efficient algorithm which can calculate the probability of k-variate Poisson distribution, namely the multivariate Poisson distribution.

However, the most preferential requirement is an accurate definition of multivariate Poisson distribution and its parameters. Krummenauer(1998a,b) studied the multivariate Poisson distribution and Kim et al.(2006) considered an algotithm for the generation of multivarate poisson random deviates by solving linear systems which satisfies given variance covariance matrix

In this paper, we propose a new algorithm of numerical analysis which can find solutions to linear equations, and we will explain its relationship with an algorithm used for the creation of multivariate Bernoulli random numbers conducted by Park et al.(1996).

Section 2 introduces every parameters that can be considered in multivariate Poisson distributions and section 3 introduces a method to create random numbers from a given parameter matrix. Section 4 explains the proposed algorithms which can create random numbers satisfying given variance-covariance matrix and section 5 shows an example of the generation of random numbers when $k=4$. Lastly, section 6 illustrates the conclusion and discussions.

2. Multivariate Poisson Distribution

Let $B_1,\cdots,B_k$ be a Bernoulli random variables with success probabilities $p_i, i=1,\cdots,k$. Now let's label the set having natural numbers $1,2,\cdots,k$ as its elements as $K=\{1,2,\cdots,k\}$ and then let's consider $p_I=P(B_i=1\ \forall i\in I)$ upon the subset of set K, $I\subseteq K(I\neq\varnothing)$. Let's Say that expressions such as $p_{i_1,i_2,\cdots,i_l}=p_{\{i_1,i_2,\cdots,i_l\}}=p_{i_1 i_2\cdots i_l}$ upon the random subset $\{i_1,\cdots,i_l\}(l\leq k)$ of set K are of the same meaning. Let's also put every permutations $j_1,j_2,\cdots,j_l$ of $\{i_1,\cdots,i_l\}(l\leq k)$ as $p_{i_1,i_2,\cdots,i_l}=p_{j_1,j_2,\cdots,j_l}$. We then have the following expression of parameter set

$$P=\{p_I \mid \forall I\subseteq K, I\neq\varnothing\}$$

upon k-variate Bernoulli distribution (Krummenauer, 1998a).

If we now put $\boldsymbol{X}^{(1)},\cdots,\boldsymbol{X}^{(n)}$ as variables following the independent k-variate Bernoulli distribution, we can say that $\boldsymbol{X}^{(1)}+\cdots+\boldsymbol{X}^{(n)}$ follows k-variate binomial distribution, call it multivariate binomial distribution, and express it as $MVB_k(n,\boldsymbol{p})$ (Krummenauer, 1998a). At this moment, $\boldsymbol{p}=(p_1,\cdots,p_k)$. Upon the set of parameters $\Lambda=\{\lambda_I \mid \forall I\subseteq K, I\neq\varnothing\}$, if $n\boldsymbol{p}\to\Lambda$,

$MVB_k(n, \boldsymbol{p})$ will converge in probability into multivariate Poisson distribution, $MVP(\Lambda)$, in which the marginal distributions follow the Poisson distribution, $P(\lambda_i), i=1,\cdots,k$.

Using such expression will allow us to say that among the element λ_I of parameter set Λ upon $I \subseteq K (I \neq \varnothing)$, λ_{i_1} indicates the parameter of marginal distribution of each Poisson distribution, λ_{i_1,i_2} indicates the covariance relationship of the two variables, and $\lambda_{i_1,i_2,\cdots,i_k}$ indicates the covariance relationship of k number of Poisson variables.

Dwass and Teicher(1956) expressed a k variate Poisson distribution into a 3 variate reduction method.

Let's take a look at the general structure of k-variate Poisson distribution by naturally expanding on the characteristics of bivariate Poisson distribution. If we set the following expressions, then we can say that $X_1,\cdots,X_k$ follows multivariate Poisson distribution with $k+1$ number of independent Poisson variables

$$\begin{aligned} X_1(\lambda_1) &= Y_1(\mu_1) + Y_0(\mu_0) = Y_1(\lambda_1 - \lambda_{ab}) + Y_0(\lambda_{ab}) \\ X_2(\lambda_2) &= Y_2(\mu_2) + Y_0(\mu_0) = Y_2(\lambda_2 - \lambda_{ab}) + Y_0(\lambda_{ab}) \\ &\cdots \\ X_k(\lambda_k) &= Y_k(\mu_k) + Y_0(\mu_0) = Y_k(\lambda_k - \lambda_{ab}) + Y_0(\lambda_{ab}) \end{aligned} \tag{1}$$

where, $Y_i, i=0,\cdots,k$ carries the Poisson parameter $\mu_i, i=0,\cdots,k$ and they each are independent Poisson variables. Here, the correlation coefficient range of two variables (X_i, X_j) is determined through $0 \le \rho_{ij} < \frac{\min(\sqrt{\lambda_i}, \sqrt{\lambda_j})}{\max(\sqrt{\lambda_i}, \sqrt{\lambda_j})}$ and we can say that it holds a non-negative correlation structure for all variables.

But according to the above expression using $k+1$ number of parameters, we cannot say that it is a general expression because it cannot express every parameters of multivariate Poisson distribution. We can refer to Loukas and Kemp(1983), and Karlis(2003) for the method of expressing the structure of a general multivariate Poisson distribution. Let's generalize formula (2) and express it as a matrix form of $\boldsymbol{A} = [A_1, \cdots, A_k]$. Here, A_i holds the dimension of $k \times \binom{k}{i}$ and it is the matrix of a structure where accurately i number of $\mathbf{1}$s and $k-i$ number of 0s exist. Now, the multivariate Poisson distribution can be expressed as

$$X_{k \times 1} = A_{k \times (2^k - 1)} Y_{(2^k - 1) \times 1} \tag{2}$$

upon the independent 2^k-1 number of Poisson variables $Y=\{Y_I \mid \forall I\subseteq K, I\neq\varnothing\}$ and $\boldsymbol{X}=(X_1,\cdots,X_k)'$(Karlis, 2003).

3. Generation of Multivariate Poisson Random Numbers

Let's take a look at the method and limits of creating multivariate Poisson random numbers under the $\boldsymbol{X}=\boldsymbol{A}\,\boldsymbol{Y}$ structure of the form (3). Let's put all parameter sets of Y_I as $M=\{\mu_I\in[0,\infty), I\subseteq K, I\neq\varnothing\}$. If we are given the appropriate parameter set Λ, the following formula is expressed as $\Lambda_{(2^k-1)\times 1}=\boldsymbol{B}_{(2^k-1)\times(2^k-1)}\boldsymbol{M}_{(2^k-1)\times 1}$ and therefore we can calculate the entire parameter μ_I of Y_I from the relations of $M=\boldsymbol{B}^{-1}\Lambda$ by

$$\lambda_I=\sum_{J\subseteq K, J\supseteq I}\mu_J, \forall I\subseteq K, I\neq\varnothing \cdot \tag{3}$$

Here, $\boldsymbol{B}$ is the design matrix and its size is $(2^k-1)\times(2^k-1)$.

Now if we say that Θ_I is a random number created at Poisson variable $P(\mu_I)$, we can create multivariate Poisson random numbers following the parameter Λ through the following formula.

$$X_i=\sum_{I\subseteq K, i\in I}\Theta_I, i=1,\cdots,k \tag{4}$$

In other words, if entire parameters $\Lambda=\{\lambda_I \mid \forall I\subseteq K, I\neq\varnothing\}$ of multivariate Poisson distribution are given, accordingly the parameter μ_I of variable Y_I can be determined which allows the creation of multivariate Poisson random numbers. Krummenaure (1998a) mentioned of a calculation of the parameter μ_I through a backward substitution process and not using the design matrix $\boldsymbol{B}$ in formula (4). But the biggest problem of this method is that researchers must predefine every parameters, $\Lambda=\{\lambda_I \mid \forall I\subseteq K, I\neq\varnothing\}$.

Karlis(2003) mentioned that the assumption of every parameter in multivariate Poisson distribution is very exhaustive and that there exists a significant amount of difficulty in calculation. When a researcher tries to create random numbers following the multivariate Poisson distribution, it is deemed unnecessary and burdensome on the fact of defining 3 or more variables of common parameter values exceeding the correlation structure. In a general case, if a researcher defines the correlation structure (variation-covariation) of the two variables and the average parameter of each marginal distribution similar to the method of creating random numbers of multivariate normal distribution, cases are highly required where random number generation satisfies such matters.

4. Proposed Algorithm for the Generation of Random Numbers

In this section, we introduces a proposed algorithm which create random numbers that satisfies the conditions in a situation where only the average (variation) $\lambda_i, i=1,\cdots,k$ of each Poisson distribution and $Cov(X_i, X_j)=\lambda_{ij}$ are given. Such a case is where variation-covariation matrix $\Sigma=\{\sigma_{ij}\}$ between k number of variables is given and parameters which expresses the relationship of three or more variates (ex. λ_{ijk}) are not given. Thereby the number of given parameters is $k(k+1)/2$. However, we express the known parameter Λ_1 and the unknown parameter Λ_2 within the matrix expression $\Lambda = \boldsymbol{B}M$ as the following.

$$\begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix} \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}.$$

Here, $\Lambda_1=(\lambda_1,\cdots\lambda_k,\lambda_{12},\cdots,\lambda_{(k-1)k})'$ and $\Lambda_2=\{\lambda_J | J\subseteq K, J\supseteq\{i_1,i_2\}\}$ where $\{i_1,i_2\}\subseteq K$. Now we can find the appropriate $\mu_I \geq 0$ that satisfies the corresponding linear equation with only the information of the known Λ_1, which can generate multivariate Poisson random numbers maintaining the given variance-covariance relationship. Of course, the number of μ_I to induce is 2^k-1 and thus we can easily see that it is a linear equation of numerous amounts of solutions satisfying $\mu_I \geq 0$. Upon such matters, we will propose the following algorithm to calculate μ_I in linear equation (3).

[Algorithms for the derivation of $k(k+1)/2$ number of μ_I]

[Step 0] Calculate the variation-covariation matrix $\{\sigma_{ij}^{(0)}\}=\Sigma^{(0)}$ upon $\lambda_{ij}=\rho_{ij}\sqrt{\lambda_i}\sqrt{\lambda_j}$ from the average (variation) $\lambda_i, i=1,\cdots,k$ and correlation coefficient ρ_{ij}.

[Step 1] Set $\beta_l=\{\sigma_{rs}: \min\{\sigma_{ij}^{(0)}\}, 1\leq i\leq j\leq k\}$. Now by setting $\beta_l=\sigma_{rs}$, $T_l=\{r,s\}$, T_l becomes the index set designating the location of the elements holding $\min\{\sigma_{ij}\}$. Let's also define the index set $J_l=\{i | a_{ij}>0, j=i,\cdots,k, i=1,\cdots,k\}$. Now set $S_l=T_l\cup J_l$ and allocate $\mu_{S_l}=\beta_l$. Here, the initial $S_1=\{1,2,\cdots,k\}$ and $l=1$. We can also see that $\mu_{S_1}=\min\{\sigma_{ij}^{(0)}\}$.

[Step 2] For the set $D=\{i_1,i_2\}$ where $i_1,i_2\in\{1,2,\cdots,k\}$, calculate the following.

$$\{\sigma_D^{(l)}\}=\{\sigma_D^{(l-1)}\}-\mu_{S_l},\quad \text{if } D\subseteq S_l$$
$$\{\sigma_D^{(l)}\}=\{\sigma_D^{(l-1)}\},\quad \text{if } D\not\subseteq S_l$$

[Step 3] Calculate $\beta_{l+1}=\{\sigma_{rs}:\min\{\sigma_{ij}^{(l)}\},1\le i\le j\le k\}$ to determine the set S_{l+1}. If $n(T_{l+1})=1$, this means that $r=s$ and is determined by $S_{l+1}=\{r\}$. Here, allocate into $\mu_{S_{l+1}}=\beta_{l+1}$.

[Step 4] Set $l\leftarrow l+1$ and repeat steps 2 and 3 until every $\{\sigma_{i,j}^{(l)}\}=0$.

[Step 5] For $S=\{S_1,S_2,\cdots,S_{k(k+1)/2}\}$, derive through $\mu_{J\subseteq\{1,2,\cdots,k\}-S}=0$.

According to the algorithm above, the entire parameter μ_I is determined along with the parameter set $\lambda_I=\sum_{J\subseteq\{1,\cdots,k\},I\subseteq J}\mu_J\ \forall\ I\subseteq\{1,\cdots,k\}$ where $\Lambda=\{\lambda_I|\ \varnothing\neq I\subseteq\{1,\cdots,k\}\}$. The key of this second algorithm upon the covariation parameter λ_{ij} and average (variance) parameter λ_i of two variables is to only determine the equivalent number of parameters $\mu_i, i\in S$ and set the remaining non selected parameters $\mu_i, i\not\in S$ as 0. In the end, the number of selected parameters μ_I among the entire 2^k-1 number of parameters is $k(k+1)/2$.

5. Example

Now let's look at the case where $k=4$. If we express the probability variables (X_1,X_2,X_3,X_4), we want to generate into a general multivariate Poisson distribution, it becomes the following.

$$X_1=Y_1+Y_{12}+Y_{13}+Y_{14}+Y_{123}+Y_{134}+Y_{124}+Y_{1234}$$
$$X_2=Y_2+Y_{12}+Y_{23}+Y_{24}+Y_{123}+Y_{234}+Y_{124}+Y_{1234}$$
$$X_3=Y_3+Y_{13}+Y_{23}+Y_{34}+Y_{123}+Y_{134}+Y_{234}+Y_{1234}$$
$$X_4=Y_4+Y_{14}+Y_{24}+Y_{34}+Y_{134}+Y_{234}+Y_{124}+Y_{1234}$$

Now upon $\forall I\subseteq\{1,2,3,4\}$, $\lambda_I=\sum_{J\subseteq I}\mu_J, \forall\ I\subseteq K$ becomes the following.

$\lambda_{1234}=\mu_{1234}$

$\lambda_{123}=\mu_{123}+\mu_{1234}$, $\lambda_{134}=\mu_{134}+\mu_{1234}$, $\lambda_{124}=\mu_{124}+\mu_{1234}$, $\lambda_{234}=\mu_{234}+\mu_{1234}$

$\lambda_{12}=\mu_{12}+\mu_{123}+\mu_{124}+\mu_{1234}$, $\lambda_{13}=\mu_{13}+\mu_{123}+\mu_{134}+\mu_{1234}$

$\lambda_{23}=\mu_{23}+\mu_{123}+\mu_{234}+\mu_{1234}$, $\lambda_{14}=\mu_{14}+\mu_{124}+\mu_{234}+\mu_{1234}$

$\lambda_{24}=\mu_{24}+\mu_{124}+\mu_{234}+\mu_{1234}$, $\lambda_{34}=\mu_{34}+\mu_{134}+\mu_{234}+\mu_{1234}$

$\lambda_{1}=\mu_{1}+\mu_{12}+\mu_{13}+\mu_{14}+\mu_{123}+\mu_{134}+\mu_{124}+\mu_{1234}$

$\lambda_{2}=\mu_{2}+\mu_{12}+\mu_{23}+\mu_{24}+\mu_{123}+\mu_{234}+\mu_{124}+\mu_{1234}$

$\lambda_{3}=\mu_{3}+\mu_{13}+\mu_{23}+\mu_{34}+\mu_{123}+\mu_{134}+\mu_{234}+\mu_{1234}$

$\lambda_{4}=\mu_{4}+\mu_{14}+\mu_{24}+\mu_{34}+\mu_{134}+\mu_{234}+\mu_{124}+\mu_{1234}$

Now let's create a random number of 4-variate Poisson distribution in the structure of $\lambda_1=1, \lambda_2=2, \lambda_3=3, \lambda_4=4, \rho_{12}=0.4, \rho_{13}=0.3, \rho_{14}=0.2, \rho_{23}=0.6, \rho_{24}=0.4$ and $\rho_{3,4}=0.7$. We must check whether the given correlation structure satisfies $0\le\rho_{ij}<\frac{\min(\sqrt{\lambda_{i,}}\sqrt{\lambda_j})}{\max(\sqrt{\lambda_{i,}}\sqrt{\lambda_j})}$. Upon $\lambda_{ij}=\rho_{ij}\sqrt{\lambda_i}\sqrt{\lambda_j}$, the variation-covariation matrix on the given correlation structure is the following.

$$\{\sigma_{ij}^{(0)}\}=\Sigma^{(0)} = \begin{bmatrix} 1 & 0.566 & 0.520 & 0.400 \\ & 2 & 1.470 & 1.131 \\ & & 3 & 2.424 \\ & & & 4 \end{bmatrix}$$

Firstly, in order to apply the corresponding algorithm, it is wise to conduct pivoting so that the variation-covariation matrix will move to the top of variables of small distributions.

[Calculation by proposed algorithm]

1) Due to the algorithm, the initial values of $\beta_1=0.4$, $S_1=\{1,2,3,4\}$, $u_{1234}=0.4$ are set.

2) Continue the algorithm as the following.

$$\begin{bmatrix} 0.600 & 0.166 & 0.120 & 0.000 \\ & 1.6 & 1.07 & 0.731 \\ & & 2.6 & 2.024 \\ & & & 3.6 \end{bmatrix} \begin{bmatrix} 0.480 & 0.046 & 0.000 & 0.000 \\ & 1.48 & 0.95 & 0.731 \\ & & 2.48 & 2.024 \\ & & & 3.6 \end{bmatrix} \begin{bmatrix} 0.434 & 0.000 & 0.000 & 0.000 \\ & 1.434 & 0.95 & 0.731 \\ & & 2.48 & 2.024 \\ & & & 3.6 \end{bmatrix}$$

$\beta_2 = 0.12 \quad S_2 = \{1,2,3\}$ $\quad$ $\beta_3 = 0.046 \quad S_3 = \{1,2\}$ $\quad$ $\beta_4 = 0.434 \quad S_4 = \{1,1\}$

$\mu_{123} = 0.12$ $\quad$ $\mu_{12} = 0.046$ $\quad$ $\mu_1 = 0.434$

$$\begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ & 1.434 & 0.95 & 0.731 \\ & & 2.48 & 2.024 \\ & & & 3.6 \end{bmatrix} \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ & 0.703 & 0.219 & 0.000 \\ & & 1.749 & 1.293 \\ & & & 2.869 \end{bmatrix} \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ & 0.484 & 0.000 & 0.000 \\ & & 1.53 & 1.293 \\ & & & 2.869 \end{bmatrix}$$

$\beta_5 = 0.731 \quad S_5 = \{2,3,4\}$ $\quad$ $\beta_6 = 0.219 \quad S_6 = \{2,3\}$ $\quad$ $\beta_7 = 0.484 \quad S_7 = \{2,2\}$

$\mu_{234} = 0.731$ $\quad$ $\mu_{23} = 0.219$ $\quad$ $\mu_2 = 0.484$

$$\begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ & 0.000 & 0.000 & 0.000 \\ & & 1.53 & 1.293 \\ & & & 2.869 \end{bmatrix} \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ & 0.000 & 0.000 & 0.000 \\ & & 0.237 & 0.000 \\ & & & 1.576 \end{bmatrix} \begin{bmatrix} 0.000 & 0.000 & 0.000 & 0.000 \\ & 0.000 & 0.000 & 0.000 \\ & & 0.000 & 0.000 \\ & & & 1.576 \end{bmatrix}$$

$\beta_8 = 1.293 \quad S_8 = \{3,4\}$ $\quad$ $\beta_9 = 0.237 \quad S_9 = \{3,3\}$ $\quad$ $\beta_{10} = 1.576 \quad S_{10} = \{4,4\}$

$\mu_{34} = 1.293$ $\quad$ $\mu_3 = 0.237$ $\quad$ $\mu_4 = 1.576$

According to the sequence above, a total of 10 parameter sets $k(k+1)/2$ $S = \{\{1\},\{2\},\{3\},\{4\},\{1,2\},\{2,3\},\{3,4\},\{1,2,3\},\{2,3,4\},\{1,2,3,4\})\}$ are derived and the remaining non-selected parameters are set as $\mu_{J \subseteq \{1,2,\cdots,k\} - S} = 0$. The entire parameters μ_I has been derived which allows the creation of multivariate Poisson random numbers by formula (3.2).

In this example, the μ_I acquired from the results of proposed algorithm are identical to the result of Kim et al(2006). This shows that if the initial value is set appropriately in proposed algorithm, the calculation of μ_I is fairly simple.

6. Conclusion and Discussion

We handled a method to create random numbers from a general multivariate Poisson distribution mentioned by Krummenauer(1998a) and Karlis(2003). We proposed a new algorithm to find the solutions of the linear equation in (3). The generated random number was based on a variation-covariation matrix and due to the characteristic of parameters of a Poisson distribution that variance-covariance matrixes are all greater than 0, we could see that there exists a positive correlation between each variables. Therefore, we were able to see that multivariate Poisson random numbers holding negative correlations cannot radically be created according to this algorithm.

Park *et al.*(1996) have used invertibility with Poisson distribution to create multivariate Bernoulli random numbers. They converted $X_i, i=1,\cdots,k$, which were derived from formula (3, into $Z_i = I_{\{0\}}(X_i), i=1,\cdots,k$ and created multivariate Bernoulli random numbers. Here, $I_{\{0\}}$ is an index function in which it becomes 1 if the generated Poisson random number is 0 and 0 when it isn't. The algorithm they discussed was compared with algorithm 2 handled in this research to conclude that it equals the method of finding $k(k+1)/2$ number of μ_I. This research has however been accessed in the angle of 2^k-1 number of parameters that is determined through a general Poisson model. In comparison with the general solution of continuously solving linear equations, this research differs by the fact that it discussed the algorithmic aspect of selecting the given number and the equivalent number of parameters from the variation-covariation matrix amongst the countless number of solutions.

References

1. Dwass, M. and Teicher, H. (1956). On infinitely divisible random vectors, *Annals of Mathematical Statistics*, 27, 461-470.
2. Karlis, D. (2003). An EM algorithm for multivariate Poisson distribution and related models, *Journal of Applied Statistics*, 30, 1, 63-77.
3. Kim, D, Jeong, H.C. and Jung, B.C. (2006). On the Multivariate Poisson Distribution with Specific Covariance matrix, *Journal of the Data & Information Science Society,* 17, 161-171
4. Krummenauer, F. (1998a). Efficient simulation of multivariate binomial and Poisson distributions, *Biometrical Journal*, 40, 7, 823-832.
5. Krummenauer, F. (1998b). Limit theorems for multivariate discrete distributions, *Metrika* 47, 47-69.
6. Loukas, S., and Kemp, C.D. (1983). On computer sampling from trivariate and multivariate discrete distribution, *Journal of Statistical Computations and Simulation*, 17, 113-123.
7. Park, C.G, Park, T. and Shin, D.W. (1996). A simple method for generating correlated binary variates, *The American Statistician*, 50, 4, 306-310.
8. Tsiamyrtzis, P. and Karlis, D. (2004). Strategies for efficient computation of multivariate Poisson probabilities, *Communications in Statistics-Simulation and Computations*, 33, 2, 271-292.

[received date : Jun. 2006, accepted date : Aug. 2006]