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Compound Linear Test Plan for 3-level Constant Stress Tests

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Abstract

Several accelerated life test plans use tests at only two levels of stress and thus, have practical limitations. They highly depend upon the assumption of a linear relationship between stress and time-to-failure and use only two extreme stresses that can cause irrelevant failure modes. Thus 3-level stress plans are preferable. When the lifetime distribution of test unit is exponential with mean lifetime Θ_i at stress x_i , $i = 0, 1, 2, 3$, we derive the optimum quadratic plan under the assumption that a quadratic relationship exists between stress and log(mean lifetime), and propose the compound linear plans, as an alternative to the optimum quadratic plan. The proposed compound linear plan is better than two other compromise plans for constant stress testing and nearly as good as the optimum quadratic plan, and has the advantage of simplicity.

Keywords : Accelerated life test, Compound linear plan, Constant stress

1. Introduction

In many reliability studies, it may require a long testing times because the lifetimes of test units under the use stress tend to be long for extremely reliable units. As a common approach, the accelerated life tests(ALTs) are widely used to shorten the lifetimes of test units. Accelerated life testing quickly yields information on the life distribution of test unit. Using data from accelerated conditions, a model is fitted and then extrapolated to make inferences on the lifetimes, the reliability, failure rates, etc. under the use stress. Widely used methods of applying stress to test units are the constant stress test and the step

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stress test.

In constant stress testing, a test unit is subjected to a fixed stress and observed until it fails or is removed(censored). Meeker(1984) and Meeker and Nelson(1975) considered the design for Type I censored constant stress accelerated life tests and gave the design the optimum test conditions and sample allocation. Nelson(1980, 1983, 1990) presented the cumulative exposure model analyzing the data from the step stress ALTs and studied the design to determine the optimum stress change time. Bai, Kim and Lee(1989) and Miller and Nelson(1983) obtained the stress change time which minimizes the asymptotic variance of maximum likelihood estimate of the log scale parameter at the design condition. Bai and Chung(1992) studied two optimum designs and compared the performances of two-step stress and constant stress partially ALTs under the tampered random variable model proposed DeGroot and Goel(1979). Khamis and Higgins(1996) derived optimum three-step stress test and evaluated compound linear plan when the lifetime of test unit for any stress is exponential. Khamis(1997a) studied the optimum designs for two-step stress and constant stress ALTs under Weibull models. Khamis(1997b) also studied the M-step stress test with K-stress variables for exponential distribution.

In this paper, we propose the compound linear plans, as an alternative to the optimum quadratic plan, and compare the compound linear plan with the optimum quadratic plan and two other compromise plan proposed by Meeker(1984), and referred by Khamis and Higgins(1996), using 3 stress levels for constant stress test under the assumption that a quadratic relationship exists between stress and log(mean lifetime).

2. Optimum Quadratic 3-level Constant Stress Plan

Testing is done with 3-level constant stress and the life distribution of the test unit for any stress is assumed to be exponential with mean life θ_i at stress x_i , $i=0,1,2,3$. For the 3-level constant stress ALTs, n_i, $i=1,2,3$ units randomly chosen from $n(=n_1+n_2+n_3)$ test units are put on each stress level, and they are run until failure occurs. Suppose that there are stresses $x_0<x_1<x_2<x_3$. In the presentation of our results and without loss of generality, we use the

$$y_i=\frac{x_i-x_0}{x_m-x_0}, \qquad i=1,2,3.$$

The log(mean lifetime)($\log\theta_i$) at stress y_i is assumed to be given by

$$\log\theta_i=\beta_0+\beta_1 y_i+\beta_2 y_i^2, \qquad i=1,2,3. \tag{1}$$

The probability density function(p.d.f) for distribution under the constant stress ALTs at each stress level can be written as

$$f_i(t) = \frac{1}{\Theta_i}\exp\left(-\frac{t}{\Theta_i}\right), \qquad 0 \le t < \infty.$$

The lifetimes of test units are independent and identically distributed. The likelihood function from observations $T_{ij} = t_{ij}$ of test units at stress y_i, $i = 1,2,3$, $j = 1,2,\cdots,n_i$ is

$$L(\Theta_1, \Theta_2, \Theta_3) = \prod_{i=1}^{3}\prod_{j=1}^{n_{ui}}\left(\frac{1}{\Theta_i}\exp\left(-\frac{t_{ij}}{\Theta_i}\right)\right), \tag{2}$$

where n_i = the number of units failed at stress y_i.

Substituting (1) for Θ_i, $i = 1,2,3$ in (2), the log-likelihood function is given with unknown parameters $\beta_0, \beta_1, \beta_2$ as follows;

$$\begin{aligned}\log L(\beta_0,\beta_1,\beta_2) = & -n\beta_0 - (n_1y_1 + n_2y_2 + n_3y_3)\beta_1 - (n_1y_1^2 + n_2y_2^2 + n_3y_3^2)\beta_2 \\ & - U_1\exp(-\beta_0 - \beta_1y_1 - \beta_2y_1^2) - U_2\exp(-\beta_0 - \beta_1y_2 - \beta_2y_2^2) \\ & - U_3\exp(-\beta_0 - \beta_1y_3 - \beta_2y_3^2),\end{aligned} \tag{3}$$

where $U_i = \sum_{j=1}^{n_i} t_{ij}$ and is the total test time at stress y_i, $i = 1,2,3$.

Maximum likelihood estimators(MLEs) for the model parameters $\beta_0, \beta_1, \beta_2$ can be obtained by solving the following equations using the Newton Raphson method.

$$\begin{aligned}\frac{\partial \log L(\beta_0,\beta_1,\beta_2)}{\partial \beta_0} &= -\sum_{i=1}^{3} n_i + \sum_{i=1}^{3} U_i\exp(-\beta_0 - \beta_1y_i - \beta_2y_i^2) = 0, \\ \frac{\partial \log L(\beta_0,\beta_1,\beta_2)}{\partial \beta_1} &= -\sum_{i=1}^{3} n_iy_i + \sum_{i=1}^{3} U_iy_i\exp(-\beta_0 - \beta_1y_i - \beta_2y_i^2) = 0 \\ \frac{\partial \log L(\beta_0,\beta_1,\beta_2)}{\partial \beta_2} &= -\sum_{i=1}^{3} n_iy_i^2 + \sum_{i=1}^{3} U_iy_i^2\exp(-\beta_0 - \beta_1y_i - \beta_2y_i^2) = 0\end{aligned}$$

The Fisher information matrix is obtained by taking the expected value of the second partial and mixed partial derivatives of $\log L(\beta_0, \beta_1, \beta_2)$ with respect to $\beta_0, \beta_1, \beta_2$.

$$\begin{aligned}\frac{\partial \log L(\beta_0,\beta_1,\beta_2)}{\partial \beta_0^2} &= -\sum_{i=1}^{3} U_i\exp(-\beta_0 - \beta_1y_i - \beta_2y_i^2), \\ \frac{\partial \log L(\beta_0,\beta_1,\beta_2)}{\partial \beta_0 \partial \beta_s} &= -\sum_{i=1}^{3} U_iy_i^s\exp(-\beta_0 - \beta_1y_i - \beta_2y_i^2),\end{aligned}$$

$$\frac{\partial \log L(\beta_0,\beta_1,\beta_2)}{\partial \beta_s^2} = -\sum_{i=1}^{3} U_i y_i^{2s} \exp(-\beta_0-\beta_1 y_i-\beta_2 y_i^2),$$
$$\frac{\partial \log L(\beta_0,\beta_1,\beta_2)}{\partial \beta_s \partial \beta_t} = -\sum_{i=1}^{3} U_i y_i^{s} y_i^{t} \exp(-\beta_0-\beta_1 y_i-\beta_2 y_i^2),$$

where $s \neq t = 1,2,3$.

It can be seen that the Fisher information matrix is

$$F = n\begin{pmatrix} \sum_{i=1}^{3}\phi_i, & \sum_{i=1}^{3}\phi_i y_i, & \sum_{i=1}^{3}\phi_i y_i^2 \\ \sum_{i=1}^{3}\phi_i y_i, & \sum_{i=1}^{3}\phi_i y_i^2, & \sum_{i=1}^{3}\phi_i y_i^3 \\ \sum_{i=1}^{3}\phi_i y_i^2, & \sum_{i=1}^{3}\phi_i y_i^3, & \sum_{i=1}^{3}\phi_i y_i^4 \end{pmatrix}, \tag{4}$$

where $\phi_i = \frac{n_i}{n}$ at stress y_i, $i=1,2$, $j=1,2,\cdots,n_i$, and $\phi_3 = 1-\phi_1-\phi_2$.
Then the asymptotic variance multiplied by sample size, $nAVR$, of the MLEs of the log(mean lifetime) at the use stress y_0 is then given by

$$nAVR = n(1,y_0,y_0^2)F^{-1}(1,y_0,y_0^2)^T$$

where T means vector transpose.
The asymptotic variance of log(mean lifetime) at use stress y_0 is given by

$$nAVR = \frac{d_1^2}{\phi_1^*} + \frac{d_2^2}{\phi_2^*} + \frac{d_3^2}{1-\phi_1^*-\phi_2^*}, \tag{5}$$

where

$$d_1^2 = \left[\frac{(y_0-y_2)(y_0-y_3)}{(y_1-y_2)(y_1-y_3)}\right]^2, d_2^2 = \left[\frac{(y_0-y_1)(y_0-y_3)}{(y_1-y_2)(y_2-y_3)}\right]^2, d_3^2 = \left[\frac{(y_0-y_1)(y_0-y_2)}{(y_1-y_3)(y_2-y_3)}\right]^2.$$

By differentiating (5) with respect to ϕ_i, $i=1,2$ and equating to zero, the optimum sample proportions ϕ_i^*, $i=1,2$ to be allocated at stress y_i, $i=1,2$ can be found by numerical methods, which minimize the asymptotic variance.

3. Compound Linear Plan

In this section, we propose compound linear test plan using 3 stress levels for constant stress test with a log-quadratic stress model and compare the compound linear plan with the optimum quadratic plan and other compromise plans. Several compromise test plans have been proposed and studied by Meeker(1984), Khamis and Higgins(1996). These compromise test plans maintain relatively high efficiency in comparison to the optimum plans.

3.1 Compound Linear Plan

Our compound linear plan uses the optimum simple(2-level) constant stress plan twice. That is, instead of searching optimum sample proportions ϕ^*_i, $i=1,2$ to minimize $nAVQ$ in (5) for 3-level constant stress test with a log-quadratic model, we use the optimum linear plan to finding the optimum sample proportions.

1) First, choose sample proportion, ϕ_1 to minimize the asymptotic variance in simple constant stress test with stresses y_1, y_2. Then sample proportion ϕ_1 using the optimum linear plan is given by

$$\phi_1 = \frac{1+\xi}{1+2\xi}, \text{ where } \xi = \frac{y_1 - y_0}{y_2 - y_1}.$$

If ϕ_1 has a value greater than 0.5, then $\phi^*_1 = \phi_1 \times 0.65$. Otherwise, $\phi^*_1 = \phi_1$.

2) Let ϕ'_1 be sample proportion to minimize the asymptotic variance in simple constant stress test with stresses y_2, y_3. Then sample proportion ϕ'_1 using the optimum linear plan is given by

$$\phi'_1 = \frac{1+\xi'}{1+2\xi'}, \text{ where } \xi' = \frac{y_2 - y_0}{y_3 - y_2}.$$

3) The optimum proportion to be allocated at stress y_2 is given by $\phi^*_2 = 1 - \phi'_1$. Thus, $\phi^*_3 = 1 - \phi^*_1 - \phi^*_2$.

We compare the compound linear plan with two other compromise plans developed by Meeker(1984), Khamis and Higgins(1996) for constant stress testing.

3.2 Plan with $\phi_2 = 0.1$

Meeker(1985) suggested the choosing ϕ^*_1 to minimize the asymptotic variance of log(mean lifetime) at use stress under the linear model with $\phi_2 = 0.1$. Then $\phi^*_3 = 1 - \phi^*_1 - \phi_2$.

3.3 4:2:1 Plan

The compromise plan by Khamis and Higgins(1996) for 3-step stress testing is used. This is to allocate test units to stresses y_1, y_2, y_3 in the proportion 4:2:1. This plan did well relatively over a range of test situations.

4. Comparison

The efficiencies to compare compromise plans with the optimum quadratic plan are defined as follows.

$$eff(\phi_1, \phi_2) = \frac{AVQ}{AVC_i}, \quad i = 1, 2, 3, \tag{6}$$

where AVQ is the minimum AVR for the optimum quadratic plan and AVC_i, $i = 1, 2, 3$ is the minimum AVR for the compound linear plan and two other compromise plans.

The optimum sample proportions for the optimum quadratic plan and compound linear plan, two other compromise plans are given in Table 1. The efficiencies in (6) are computed for various y_1 and y_2 and the results are given in Table 2. They are ordered from largest to smallest according to their efficiencies.

From Table 1, one can see that the sample proportions at stress y_1 are decreasing from about 68% to 26% and the sample proportions at stress y_2 are increasing from about 25% to 50% for the optimum quadratic plan as stresses y_1 and y_2 are increasing. The proposed compound linear plan are very similar for over all testing situations. On the other hand, sample proportions for the 4:2:1 plan are a little differences at low stresses, but sample proportions at high stresses are remarkable differences, and the most samples are allocated only two extreme stresses y_1 and y_3 at the plan with $\phi_2 = 0.1$.

<Table 1> Optimum Sample Proportions for Compromise Plans and the Optimum Quadratic Test Plan($y_0 = 0, y_3 = 1$)

		Φ_1^*				Φ_2^*				Φ_3^*			
y_1	y_2	OptQ	CLin	ϕ_2=0.1	4:2:1	OptQ	CLin	ϕ_2=0.1	4:2:1	OptQ	CLin	ϕ_2=0.1	4:2:1
0.100	0.550	.68323	.55000	.81818	.57143	.24845	.35484	.10000	.28571	.06832	.09516	.08182	.14286
0.150	0.575	.59818	.51552	.78261	.57143	.31209	.36508	.10000	.28571	.08973	.11940	.11739	.14286
0.200	0.600	.53571	.48750	.75000	.57143	.35714	.37500	.10000	.28571	.10714	.13750	.15000	.14286
0.250	0.625	.48780	.46429	.72000	.57143	.39024	.38462	.10000	.28571	.12195	.15110	.18000	.14286
0.350	0.675	.41893	.42805	.66667	.57143	.43445	.40299	.10000	.28571	.14663	.16897	.23333	.14286
0.550	0.775	.33677	.38019	.58065	.57143	.47800	.43662	.10000	.28571	.18523	.18319	.31935	.14286
0.750	0.875	.28866	.35000	.51429	.57143	.49485	.46667	.10000	.28571	.21649	.18333	.38571	.14286
0.900	0.950	.26352	.33378	.47368	.57143	.49931	.48718	.10000	.28571	.23717	.17904	.42632	.14286

OptQ : optimum quadratic plan, CLin : compound linear plan

This efficiency can be used to compare the proposed compound linear plan with the quadratic optimum plan and two other compromise plans. That is, if a plan is near the optimum quadratic plan, then the efficiency approaches 1.

From Table 2, we can see that the efficiencies for the proposed compound linear plan are near 1 for over all testing situations. This indicates that the proposed compound linear plan is nearly as good as the optimum quadratic plan and is better than two other compromise plans when the quadratic relationship exists between stress and lifetime.

<Table 2> Comparison of Efficiencies for Compromise Plans and the Optimum Quadratic Test Plan($y_0=0, y_3=1$)

y_1	y_2	CLin	4:2:1	$\Phi_2=0.1$
0.100	0.550	.93306	.93842	.80331
0.150	0.575	.97246	.97709	.66675
0.200	0.600	.98783	.97180	.57647
0.250	0.625	.99315	.94918	.51652
0.350	0.675	.99443	.89428	.44587
0.550	0.775	.99118	.80754	.38646
0.750	0.875	.98188	.75134	.36600
0.900	0.950	.96715	.72054	.36080

The proposed compound linear plan has the advantage of simplicity in searching optimum sample proportions Φ_1^* and Φ_2^* to be allocated stresses y_1 and y_2 compared with the optimum quadratic plan. For the optimum quadratic plan, the optimum sample proportions Φ_1^* and Φ_2^* can be found by numerical methods, after differentiating the asymptotic variance of log(mean lifetime) in (5) with respect to Φ_1 and Φ_2 and equating to zero. But the proposed compound linear plan can search the optimum sample proportions, Φ_1^* and Φ_2^*

with comparative ease by the optimum linear plan because it uses the optimum simple constant stress plan twice.

References

1. Bai, D. S., Kim, M. S. and Lee, S. H. (1989). Optimum simple step-stress accelerated life tests with censoring, *IEEE Transactions on Reliability*, 38, 528-532.
2. Bai, D. S. and Chung, S. W. (1992). Optimal design of partially accelerated life-test for exponential distribution under Type-I censoring, *IEEE Transactions on Reliability*, 41, 400-406.
3. DeGroot, M. H. and Goel, P. K. (1979). Bayesian estimation and optimal designs in partially accelerated life testing, *Naval Research Logistics Quarterly*, 26, 223-235.
4. Khamis, I. H. (1997a). Comparison between constant and step-stress tests for Weibull models, *International Journal of Quality & Reliability Management*, 14, 74-81.
5. Khamis, I. H.(1997b). Optimum M-step. step-stress design with k stress variables, *Communications in Statistics, Computation and Simulation*, 26, 1301-1313.
6. Khamis, I. H. and Higgins, J. J. (1996). Optimum 3-step step-stress tests, *IEEE Transactions on Reliability*, 45, 341-345.
7. Meeker, W. Q. (1984). A comparison of accelerated life test plans for Weibull and lognormal distributions and Type I censoring, *Technometrics*, 26, 157-171.
8. Meeker, W. Q. and Nelson, W. (1975). Optimum accelerated life tests for the Weibull and Extreme value distribution, *IEEE Transactions on Reliability*, 24, 321-332.
9. Nelson, W. (1980). Accelerated life testing step-stress models and data analysis, *IEEE Transactions on Reliability*, 29, 103-108.
10. Nelson, W. (1990). *Accelerated Testing : Statistical Models, Test Plans, and Data Analysis*, John Wiley & Sons.
11. Nelson, W. and Miller, R. (1983). Optimum simple step-stress plans for accelerated testing, *IEEE Transactions on Reliability*, 32, 59-65.

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