Journal of the Korean
Data & Information Science Society
2006, Vol. 17, No. 3, pp. 933 ~943

Estimation for the Rayleigh Distribution with Known Parameter under Multiply Type-II Censoring

Jun-Tae Han[1] · Suk-Bok Kang[2]

Abstract

For multiply Type-II censored samples from two-parameter Rayleigh distribution, we derive some approximate maximum likelihood estimators of parameter in the Rayleigh distribution when the other parameter is known. We also compare the proposed estimators in the sense of the mean squared error for various censored samples.

Keywords : Approximate maximum likelihood estimator, Multiply Type-II censored sample, Rayleigh distribution

1. Introduction

The random variable X has the Rayleigh distribution if it has a probability density function (pdf) of the form

$$f(x;\theta,\sigma)=\frac{x-\theta}{\sigma^2}\exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\},\quad x\geq\theta,\ \sigma>0 \tag{1.1}$$

and the cumulative distribution function (cdf) of the form

$$F(x;\theta,\sigma)=1-\exp\left\{-\frac{(x-\theta)^2}{2\sigma^2}\right\},\quad x\geq\theta,\ \sigma>0, \tag{1.2}$$

where θ and σ are the location and the scale parameters, respectively.

The Rayleigh distribution is very useful in communication engineering and the Rayleigh model would be especially suitable for life testing experiments of components which age with time.

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1) Graduate student, Department of Statistics, Yeungnam University, Gyongsan, 712-749, Korea

2) Corresponding Author : Professor, Department of Statistics, Yeungnam University, Gyongsan, 712-749, Korea
E-mail : sbkang@yu.ac.kr

Balakrishnan (1989) purposed the approximate maximum likelihood estimator (AMLE) of the scale parameter of the Rayleigh distribution with censoring. Cohen (1991) obtained the maximum likelihood estimators (MLEs) of the scale parameter in singly right censored and truncated samples from the Rayleigh distribution.

Multiply Type-II censoring is a generalization of Type-II censoring. Kong and Fei (1996) discussed the limit theorems for the maximum likelihood estimate under general multiply Type-II censoring. Upadhyay et al. (1996) considered the estimation for the exponential distribution under multiply Type-II censoring.

Recently, Kang (2005) derived the approximate maximum likelihood estimators of the scale parameter and location parameter of the extreme value distribution based on multiply Type-II censored samples. Kang and Lee (2005) derived the AMLEs of the scale and location parameters of the two-parameter exponential distribution based on multiply Type-II censored samples. They also obtained the moments of the proposed estimators. Kang and Park (2005) derived the approximate maximum likelihood estimators for the location and scale parameters that are explicit function of order statistics in the exponentiated exponential distribution. They also compared the proposed estimators in the sense of the mean squared error (MSE) for various censored samples.

In this paper, we derive some AMLEs of the scale parameter σ in the two-parameter Rayleigh distribution when the location parameter θ is known and also derive an AMLE of the location parameter θ when the scale parameter σ is known under multiply Type-II censoring by the approximate maximum likelihood estimation method. We also compare the proposed estimators in the sense of the MSE for various censored samples.

2. Approximate Maximum Likelihood Estimators

Let us assume that the following multiply Type-II censored sample from a sample of size n is

$$X_{a_1:n} < X_{a_2:n} < \cdots < X_{a_s:n}, \tag{2.1}$$

where $1 \le a_1 < a_2 < \cdots < a_s \le n$.

Let $a_0 = 0$, $a_{s+1} = n+1$, $F(x_{a_0:n}) = 0$, $F(x_{a_{s+1}:n}) = 1$, then the likelihood function based on the multiply Type-II censored sample (2.1) is given by

$$L = n! \prod_{j=1}^{s} f(x_{a_j:n}) \prod_{j=1}^{s+1} \frac{[F(x_{a_j:n}) - F(x_{a_{j-1}:n})]^{a_j - a_{j-1} - 1}}{(a_j - a_{j-1} - 1)!}. \tag{2.2}$$

By putting $Z_{i:n} = (X_{i:n} - \theta)/\sigma$, the likelihood function can be rewritten as

$$L = \frac{1}{\sigma^s} \frac{n!}{\prod_{j=1}^{s+1} (a_j - a_{j-1} - 1)!} \left[F(Z_{a_1:n}) \right]^{a_1 - 1} \left[1 - F(Z_{a_s:n}) \right]^{n - a_s}$$
$$\times \prod_{j=1}^{s} f(Z_{a_j:n}) \left[\prod_{j=2}^{s} \left[F(Z_{a_j:n}) - F(Z_{a_{j-1}:n}) \right]^{a_j - a_{j-1} - 1} \right], \tag{2.3}$$

where $f(z) = ze^{-z^2/2}$ and $F(z) = 1 - e^{-z^2/2}$ are the pdf and the cdf of the standard Rayleigh distribution, respectively.

2.1 AMLEs of the Scale Parameter When the Location Parameter is Known

We can derive the AMLEs of the scale parameter σ when the location parameter θ is known.

From the equation (2.3), $f(z) = z[1 - F(z)]$, and $f'(z) = 1 - F(z) - zf(z)$, we obtain the likelihood equations as follows;

$$\frac{\partial \ln L}{\partial \sigma} = -\frac{1}{\sigma} \left[2s + (a_1 - 1) \frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} - (n - a_s) Z^2_{a_s:n} - \sum_{j=1}^{s} Z^2_{a_j:n} \right.$$
$$\left. + \sum_{j=2}^{s} (a_j - a_{j-1} - 1) \frac{f(Z_{a_j:n}) Z_{a_j:n} - f(Z_{a_{j-1}:n}) Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \right] \tag{2.1.1}$$
$$= 0 .$$

Since the likelihood equation is very complicated, the equation (2.1.1) does not admit an explicit solution for σ. So we need some approximate likelihood equations which give explicit solutions.

Let

$$\xi_i = F^{-1}(p_i) = \left[-2 \ln (1 - p_i) \right]^{1/2},$$

where $p_i = \dfrac{i}{n+1}$, $q_i = 1 - p_i$.

First, we can approximate the functions by Taylor series expansion as follows;

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} Z_{a_1:n} \simeq \alpha_1 + \beta_1 Z_{a_1:n} \tag{2.1.2}$$

and

$$\frac{f(Z_{a_j:n}) Z_{a_j:n} - f(Z_{a_{j-1}:n}) Z_{a_{j-1}:n}}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n}, \tag{2.1.3}$$

where

$$\alpha_1 = -\frac{\xi_{a_1}^2}{p_{a_1}} \left[f'(\xi_{a_1}) - \frac{f^2(\xi_{a_1})}{p_{a_1}} \right]$$

$$\beta_1 = \frac{1}{p_{a_1}}\left[f(\xi_{a_1}) + \xi_{a_1} f'(\xi_{a_1}) - \frac{f^2(\xi_{a_1})}{p_{a_1}}\xi_{a_1}\right]$$

$$\alpha_{1j} = K^2 - \frac{\xi_{a_j}{}^2 f'(\xi_{a_j}) - \xi_{a_{j-1}}{}^2 f'(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}$$

$$\beta_{1j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1-K) f(\xi_{a_j}) + \xi_{a_j} f'(\xi_{a_j}) \right]$$

$$\gamma_{1j} = -\frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1-K) f(\xi_{a_{j-1}}) + \xi_{a_{j-1}} f'(\xi_{a_{j-1}}) \right]$$

$$K = \frac{\xi_{a_j} f(\xi_{a_j}) - \xi_{a_{j-1}} f(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}}.$$

By substituting the equations (2.1.2) and (2.1.3) into the equation (2.1.1), we can approximate the likelihood equation for σ as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \sigma} &\simeq -\frac{1}{\sigma}\Big[2s + (a_1 - 1)(\alpha_1 + \beta_1 Z_{a_1:n}) - (n - a_s) Z^2_{a_s:n} - \sum_{j=1}^{s} Z^2_{a_j:n} \\ &\quad + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n})\Big] \qquad (2.1.4) \\ &= 0 .\end{aligned}$$

Upon solving the equation (2.1.4) for σ, we can derive an estimator of σ as follows;

$$\hat{\sigma}_{10} = \frac{-B_{10} + \sqrt{B_{10}^2 - 4A_{10}C_{10}}}{2A_{10}}, \qquad (2.1.5)$$

where

$$A_{10} = 2s + (a_1 - 1)\alpha_1 + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)\alpha_{1j}$$

$$\begin{aligned} B_{10} &= (a_1 - 1)\beta_1 X_{a_1:n} + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_{1j} X_{a_j:n} + \gamma_{1j} X_{a_{j-1}:n}) \\ &\quad - \Big[(a_1 - 1)\beta_1 + \sum_{j=2}^{s}(a_j - a_{j-1} - 1)(\beta_{1j} + \gamma_{1j})\Big]\theta \end{aligned}$$

$$C_{10} = -(n - a_s)(X_{a_s:n} - \theta)^2 - \sum_{j=1}^{s}(X_{a_j:n} - \theta)^2.$$

Second, we can approximate the likelihood equation (2.1.1) by the equations (2.1.3) and

$$\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})} \simeq \alpha_2 + \beta_2 Z_{a_1:n}, \qquad (2.1.6)$$

where

$$\alpha_2 = \frac{1}{p_{a_1}}\left[f(\xi_{a_1}) - \xi_{a_1} f'(\xi_{a_1}) + \frac{f^2(\xi_{a_1})}{p_{a_1}} \xi_{a_1} \right]$$

$$\beta_2 = \frac{1}{p_{a_1}}\left[f'(\xi_{a_1}) - \frac{f^2(\xi_{a_1})}{p_{a_1}} \right].$$

Then we can obtain another approximate likelihood equation for σ as follows;

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\Big[2s + (a_1 - 1)(\alpha_2 + \beta_2 Z_{a_1:n}) Z_{a_1:n} - (n - a_s) Z^2_{a_s:n} - \sum_{j=1}^{s} Z^2_{a_j:n} + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{1j} + \beta_{1j} Z_{a_j:n} + \gamma_{1j} Z_{a_{j-1}:n}) \Big] = 0 . \quad (2.1.7)$$

Upon solving the equation (2.1.7) for σ, we can derive another estimator of σ as follows;

$$\hat{\sigma}_{20} = \frac{-B_{20} + \sqrt{B^2_{20} - 4A_{20}C_{20}}}{2A_{20}}, \quad (2.1.8)$$

where

$$A_{20} = 2s + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\alpha_{1j}$$

$$B_{20} = (a_1 - 1)\alpha_2 X_{a_1:n} + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_{1j} X_{a_j:n} + \gamma_{1j} X_{a_{j-1}:n}) - \left[(a_1 - 1)\alpha_2 + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\beta_{1j} + \gamma_{1j}) \right]\Theta$$

$$C_{20} = (a_1 - 1)\beta_2 (X_{a_1:n} - \Theta)^2 - (n - a_s)(X_{a_s:n} - \Theta)^2 - \sum_{j=1}^{s} (X_{a_j:n} - \Theta)^2.$$

Third, we can also approximate the likelihood equation (2.1.1) by the equations (2.1.2),

$$\frac{f(Z_{a_j:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{2j} + \beta_{2j} Z_{a_j:n} + \gamma_{2j} Z_{a_{j-1}:n} \quad (2.1.9)$$

and

$$\frac{f(Z_{a_{j-1}:n})}{F(Z_{a_j:n}) - F(Z_{a_{j-1}:n})} \simeq \alpha_{3j} + \beta_{3j} Z_{a_j:n} + \gamma_{3j} Z_{a_{j-1}:n}, \quad (2.1.10)$$

where

$$\alpha_{2j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[(1 + K) f(\xi_{a_j}) - \xi_{a_j} f'(\xi_{a_j}) \right]$$

$$\beta_{2j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[f'(\xi_{a_j}) - \frac{f^2(\xi_{a_j})}{p_{a_j} - p_{a_{j-1}}} \right]$$

$$\gamma_{2j} = \frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{[p_{a_j} - p_{a_{j-1}}]^2}$$

$$\alpha_{3j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}[(1+K) f(\xi_{a_{j-1}}) - \xi_{a_{j-1}} f'(\xi_{a_{j-1}})]$$

$$\beta_{3j} = -\frac{f(\xi_{a_j}) f(\xi_{a_{j-1}})}{[p_{a_j} - p_{a_{j-1}}]^2} = -\gamma_{2j}$$

$$\gamma_{3j} = \frac{1}{p_{a_j} - p_{a_{j-1}}}\left[f'(\xi_{a_{j-1}}) + \frac{f^2(\xi_{a_{j-1}})}{p_{a_j} - p_{a_{j-1}}} \right].$$

We can obtain the other approximate likelihood equation for σ as follows;

$$\begin{aligned} \frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\Big[& 2s + (a_1 - 1)(\alpha_1 + \beta_1 Z_{a_1:n}) - (n - a_s) Z^2_{a_s:n} - \sum_{j=1}^{s} Z^2_{a_j:n} \\ & + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)\{(\alpha_{2j} + \beta_{2j} Z_{a_j:n} + \gamma_{2j} Z_{a_{j-1}:n}) Z_{a_j:n} \\ & - (\alpha_{3j} + \beta_{3j} Z_{a_j:n} + \gamma_{3j} Z_{a_{j-1}:n}) Z_{a_{j-1}:n}\} \Big] \\ = 0 . \end{aligned} \tag{2.1.11}$$

Upon solving the equation (2.1.11) for σ, we can derive the other estimator of σ as follows;

$$\hat{\sigma}_{30} = \frac{-B_{30} + \sqrt{B_{30}^2 - 4A_{30}C_{30}}}{2A_{30}}, \tag{2.1.12}$$

where

$$A_{30} = 2s + (a_1 - 1)\alpha_1$$

$$\begin{aligned} B_{30} = & (a_1 - 1)\beta_1 X_{a_1:n} + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{2j} X_{a_j:n} - \alpha_{3j} X_{a_{j-1}:n}) \\ & - \left[(a_1 - 1)\beta_1 + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)(\alpha_{2j} - \alpha_{3j}) \right] \Theta \end{aligned}$$

$$\begin{aligned} C_{30} = & -(n - a_s)(X_{a_s:n} - \Theta)^2 - \sum_{j=1}^{s} (X_{a_j:n} - \Theta)^2 + \sum_{j=2}^{s} (a_j - a_{j-1} - 1)[\beta_{2j}(X_{a_j:n} - \Theta)^2 \\ & + 2\gamma_{2j}(X_{a_j:n} - \Theta)(X_{a_{j-1}:n} - \Theta) - \gamma_{3j}(X_{a_{j-1}:n} - \Theta)^2]. \end{aligned}$$

Lastly, we can approximate the likelihood equation (2.1.1) by the equations (2.1.6), (2.1.9), and (2.1.10). Then we can obtain other approximate likelihood equation for σ as follows;

$$\frac{\partial \ln L}{\partial \sigma} \simeq -\frac{1}{\sigma}\Big[2s+(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})Z_{a_1:n}-(n-a_s)Z^2_{a_s:n}-\sum_{j=1}^{s} Z^2_{a_j:n}$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\{(\alpha_{2j}+\beta_{2j}Z_{a_j:n}+\gamma_{2j}Z_{a_{j-1}:n})Z_{a_j:n} \tag{2.1.13}$$
$$-(\alpha_{3j}+\beta_{3j}Z_{a_j:n}+\gamma_{3j}Z_{a_{j-1}:n})Z_{a_{j-1}:n}\}\Big]$$
$$=0.$$

Upon solving the equation (2.1.13) for σ, we can derive other estimator of σ as follows;

$$\hat{\sigma}_{40}=\frac{-B_{40}+\sqrt{B_{40}^2-8sC_{40}}}{4s}, \tag{2.1.14}$$

where

$$B_{40}=(a_1-1)\alpha_2 X_{a_1:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{2j}X_{a_j:n}-\alpha_{3j}X_{a_{j-1}:n})$$
$$-\Big[(a_1-1)\alpha_2+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{2j}-\alpha_{3j})\Big]\theta$$
$$C_{40}=(a_1-1)\beta_2(X_{a_1:n}-\theta)^2-(n-a_s)(X_{a_s:n}-\theta)^2-\sum_{j=1}^{s}(X_{a_j:n}-\theta)^2$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\big[\beta_{2j}(X_{a_j:n}-\theta)^2+2\gamma_{2j}(X_{a_j:n}-\theta)(X_{a_{j-1}:n}-\theta)$$
$$-\gamma_{3j}(X_{a_{j-1}:n}-\theta)^2\big].$$

2.2 AMLE of the Location Parameter When the Scale Parameter is Known

In this section, we consider the two-parameter Rayleigh distribution with the density function (1.1) when the scale parameter is known.

From the equation (2.3), the likelihood equation for θ is obtained as

$$\frac{\partial \ln L}{\partial \theta}=-\frac{1}{\sigma}\Big[(a_1-1)\frac{f(Z_{a_1:n})}{F(Z_{a_1:n})}-(n-a_s)Z_{a_s:n}+\sum_{j=1}^{s}\frac{1}{Z_{a_j:n}}-\sum_{j=1}^{s}Z_{a_j:n}$$
$$+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\frac{f(Z_{a_j:n})-f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})}\Big] \tag{2.2.1}$$
$$=0.$$

The equation (2.2.1) does not admit an explicit solution for θ. So we need an approximate likelihood equation that give explicit solution. We can obtain some approximations by Taylor series expansion as follows;

$$\frac{1}{Z_{a_j:n}}\simeq \kappa_j+\delta_j Z_{a_j:n} \tag{2.2.2}$$

$$\frac{f(Z_{a_j:n})-f(Z_{a_{j-1}:n})}{F(Z_{a_j:n})-F(Z_{a_{j-1}:n})} \simeq \alpha_{4j}+\beta_{4j}Z_{a_j:n}+\gamma_{4j}Z_{a_j:n}, \tag{2.2.3}$$

where

$$\kappa_j=2/\xi_{a_j},\ \ \delta_j=-1/\xi^2_{a_j}\ \ \alpha_{4j}=\alpha_{2j}-\alpha_{3j},\ \ \beta_{4j}=\beta_{2j}-\beta_{3j},\ \text{and}\ \ \gamma_{4j}=\gamma_{2j}-\gamma_{3j}.$$

By substituting the equations (2.1.6), (2.2.2) and (2.2.3) into the equation (2.2.1), we obtain an approximate likelihood equation for θ as follows;

$$\begin{aligned}\frac{\partial \ln L}{\partial \theta} &\simeq -\frac{1}{\sigma}\Big[(a_1-1)(\alpha_2+\beta_2 Z_{a_1:n})-(n-a_s)Z_{a_s:n}+\sum_{j=1}^{s}(\kappa_j+\delta_j Z_{a_j:n}) \\ &\quad -\sum_{j=1}^{s}Z_{a_j:n}+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\alpha_{4j}+\beta_{4j}Z_{a_j:n}+\gamma_{4j}Z_{a_{j-1}:n})\Big] \\ &=0\,.\end{aligned} \tag{2.2.4}$$

Upon solving the equation (2.2.4) for θ, we can derive an estimator of θ as follows;

$$\hat{\theta}=\frac{A}{C}\sigma+\frac{B}{C}, \tag{2.2.5}$$

where

$$\begin{aligned}A &= (a_1-1)\alpha_2+\sum_{j=1}^{s}\kappa_j+\sum_{j=2}^{s}(a_j-a_{j-1}-1)\alpha_{4j} \\ B &= (a_1-1)\beta_2 X_{a_1:n}-(n-a_s)X_{a_s:n}+\sum_{j=1}^{s}\delta_j X_{a_j:n}-\sum_{j=1}^{s}X_{a_j:n} \\ &\quad +\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{4j}X_{a_j:n}+\gamma_{4j}X_{a_{j-1}:n}) \\ C &= (a_1-1)\beta_2-(n-a_s)+\sum_{j=1}^{s}\delta_j-s+\sum_{j=2}^{s}(a_j-a_{j-1}-1)(\beta_{4j}+\gamma_{4j}).\end{aligned}$$

It's difficult to find the moments of all proposed estimators. So we simulate the MSEs of the proposed estimators through the Monte Carlo simulation method. The simulation procedure is repeated 10,000 times for the sample size $n=20(10)50$ and various choices of censoring ($m=n-s$ is the number of unobserved or missing data). These values are given in Table 1.

From Table 1, the estimator $\hat{\sigma}_{10}$ is more efficient than the other estimators and $\hat{\sigma}_{30}$ is generally more efficient than the estimators $\hat{\sigma}_{20}$ and $\hat{\sigma}_{40}$ in the sense of MSE when location parameter θ is known.

<Table 1> The relative MSEs of the proposed estimators

n	m	a_j	σ is known	θ is known			
			$\hat{\theta}$	$\hat{\sigma}_{10}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{30}$	$\hat{\sigma}_{40}$
20	0	1~20	0.018751	0.012306	0.012306	0.012306	0.012306
	2	1~18	0.018822	0.013586	0.013586	0.013586	0.013586
		3~20	0.019408	0.012310	0.012398	0.012310	0.012398
		2~19	0.018275	0.012967	0.013035	0.012967	0.013035
	4	2~17	0.018399	0.014464	0.014545	0.014464	0.014545
		4~19	0.020783	0.012976	0.013089	0.012976	0.013089
		3~18	0.019491	0.013588	0.013696	0.013588	0.013696
		2~4 7~14 16~20	0.018238	0.012314	0.012364	0.012781	0.012922
	5	3~17	0.019594	0.014467	0.014586	0.014467	0.014586
		4~18	0.020827	0.013594	0.013716	0.013594	0.013716
		2~6 10~19	0.018298	0.012981	0.013044	0.013286	0.013410
	6	4~17	0.020945	0.014473	0.014604	0.014473	0.014604
		1 2 6~9 12~15 17~20	0.018519	0.012334	0.012334	0.013349	0.013349
30	0	1~30	0.011831	0.008344	0.008344	0.008344	0.008344
	2	1~28	0.011845	0.008610	0.008610	0.008610	0.008610
		3~30	0.011847	0.008342	0.008388	0.008342	0.008388
		2~29	0.011410	0.008610	0.008644	0.008610	0.008644
	4	2~27	0.011870	0.009196	0.009196	0.009196	0.009196
		4~29	0.011435	0.009196	0.009236	0.009196	0.009236
		3~28	0.012546	0.008610	0.008672	0.008610	0.008672
		2~4 7~14 16~30	0.011896	0.008947	0.009001	0.008947	0.009001
	5	3~27	0.011366	0.008339	0.008369	0.008607	0.008680
		4~28	0.011893	0.009193	0.009251	0.009193	0.009251
		2~6 10~19 21~30	0.012576	0.008949	0.009017	0.008949	0.009017
	6	4~27	0.011395	0.008341	0.008371	0.008641	0.008713
		1 2 6~9 12~15 17~30	0.012573	0.009195	0.009266	0.009195	0.009266

<Table 1> (continued)

n	m	a_j	σ is known	Θ is known			
			$\hat{\Theta}$	$\hat{\sigma}_{10}$	$\hat{\sigma}_{20}$	$\hat{\sigma}_{30}$	$\hat{\sigma}_{40}$
40	0	1～40	0.008352	0.006032	0.006032	0.006032	0.006032
	2	1～38	0.008358	0.006331	0.006331	0.006331	0.006331
		3～40	0.008306	0.006032	0.006059	0.006032	0.006059
		2～39	0.008010	0.006181	0.006198	0.006181	0.006198
	4	2～37	0.008026	0.006481	0.006501	0.006481	0.006501
		4～39	0.008653	0.006180	0.006211	0.006180	0.006211
		3～38	0.008316	0.006331	0.006363	0.006331	0.006363
		2～4 7～14 16～40	0.008001	0.006031	0.006047	0.006217	0.006257
	5	3～37	0.008330	0.006481	0.006515	0.006481	0.006515
		4～38	0.008657	0.006330	0.006365	0.006330	0.006365
		2～6 10～19 21～40	0.007992	0.006029	0.006046	0.006205	0.006245
	6	4～37	0.008672	0.006480	0.006516	0.006480	0.006516
		1 2 6～9 12～15 17～40	0.008270	0.006034	0.006034	0.006360	0.006360
50	0	1～50	0.006593	0.004936	0.004936	0.004936	0.004936
	2	1～48	0.006597	0.005134	0.005134	0.005134	0.005134
		3～50	0.006592	0.004936	0.004952	0.004936	0.004952
		2～49	0.006340	0.005031	0.005042	0.005031	0.005042
	4	2～47	0.006348	0.005232	0.005244	0.005232	0.005244
		4～49	0.006824	0.005031	0.005046	0.005031	0.005046
		3～48	0.006599	0.005134	0.005152	0.005134	0.005152
		2～4 7～14 16～50	0.006334	0.004936	0.004946	0.005028	0.005054
	5	3～47	0.006604	0.005232	0.005249	0.005232	0.005249
		4～48	0.006830	0.005134	0.005150	0.005134	0.005150
		2～6 10～19 21～50	0.006338	0.004936	0.004946	0.005064	0.005090
	6	4～47	0.006836	0.005232	0.005248	0.005232	0.005248
		1 2 6～9 12～15 17～50	0.006505	0.004936	0.004936	0.005125	0.005125

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[received date : Jun. 2006, accepted date : Jul. 2006]