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# The Fractional Bayes Factor Approach to the Bayesian Testing of the Weibull Shape Parameter

Young Joon Cha[1] · Kil Ho Cho[2] · Jang Sik Cho[3]

Abstract

The techniques for selecting and evaluating prior distributions are studied over recent years which the primary emphasis is on noninformative priors. But, noninformative priors are typically improper so that such priors are defined only up to arbitrary constants which affect the values of Bayes factors. In this paper, we consider the Bayesian hypotheses testing for the Weibull shape parameter based on fractional Bayes factor which is to remove the arbitrariness of improper priors. Also we present a numerical example to further illustrate our results.

***Keywords*** : Bayesian testing, Fractional Bayes Factor, Weibull shape parameter

## 1. Introduction

Hypotheses testing for the shape and scale parameters of the weibull model has been researched from a frequentist viewpoint. In particular, testing of the shape parameter often means that the hazard function is monotone increasing or monotone decreasing or constant. Although there exist numerous researches that present the a frequentist viewpoint, we focus attention on the Bayesian testing technique for the Weibull shape parameter. The methodology for selecting and evaluating prior distributions was studied in recent years which the primary emphasis is on noninformative priors. But, noninformative priors are typically

1) Professor, Department of Information Statistics, Andong National University, Andong, 760-749, Korea.
2) Professor, Department of Statistics, Kyungpook National University, Daegu, 702-701, Korea.
3) (Corresponding Author) Associate Professor, Department of Informational Statistics, Kyungsung University, Busan, 608-736, Korea.
E-Mail : jscho@ks.ac.kr

improper so that such priors are defined only up to arbitrary constants which affect the values of Bayes factors.

San Martini and Spezzaferri(1984) have suggested the predictive model selection criterion to compensate for that arbitrariness. Berger and Pericchi(1996a, 1996b) introduced a new model selection and hypotheses testing criterion, called the intrinsic Bayes factors which are to eliminate the arbitrariness of improper priors. Also Berger and Pericchi(1998) proposed accurate and stable Bayesian model selection for the median intrinsic Bayes factor. Lingham and Sivaganesan(1997) suggested testing hypotheses about the power law process under failure truncation using the intrinsic Bayes factors. Kang, Kim and Cho(1999) and Cho, Kim and Kang(1999) discussed the Bayesian testing based on intrinsic Bayes factor approach. O'Hagan(1995) introduced the fractional Bayes factor(FBF) which is used to a portion of the likelihood to remove the arbitrariness. Cho and Cho(2006) proposed Bayesian testing procedures for the exponential models based on FBF.

In this paper, we suggest the Bayesian testing methodology for the shape parameter in the Weibull model using the FBF which is to remove the arbitrariness of improper priors. Also we take a numerical example to further illustrate our results.

## 2. FBF Criterion

Let $H_1,\cdots,H_N$ be hypotheses under consideration. And let $\boldsymbol{X}=(X_1,\cdots,X_n)$ be a random sample from a population which is probability density function $f_i(\boldsymbol{x}\mid\Theta_i)$ under hypotheses $H_i$, $i=1,\cdots,N$. And let $\pi_i(\Theta_i)$ and $p_i$ be the prior distribution and the prior probabilities of the hypothesis $H_i$, respectively. Then the posterior probability that the hypothesis $H_i$ is true is given as

$$P(H_i\mid \boldsymbol{x})=\left(\sum_{j=1}^{q}\frac{p_j}{p_i}B_{ji}\right)^{-1}, \tag{1}$$

where $B_{ji}$, the Bayes factor of $H_j$ to $H_i$, is defined by

$$B_{ji}=\frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})}, \tag{2}$$

where $m_i(\boldsymbol{x})=\int_{\Theta_i} f(\boldsymbol{x}\mid\Theta_i)\pi_i(\Theta_i)d\Theta_i$. The posterior probabilities in (1) are then used to select the most plausible hypothesis.

If one use some noninformative priors $\pi_i^N(\Theta_i)$, then $\pi_i^N(\Theta_i)$ is usually written as $\pi_i^N(\Theta_i)\propto h_i(\Theta_i)$, where $h_i$ is a function whose integral over the parameter space diverges. Formally, we can write $\pi_i^N(\Theta_i)=c_i h_i(\Theta_i)$, although the normalizing constant $c_i$ does not exist, but treating it as an unspecified constant. then (2) becomes

$$B_{ji}^{N}=\frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})}=\frac{\int_{\Theta_j} f(\boldsymbol{x}\mid\Theta_j)\pi_j^N(\Theta_j)d\Theta_j}{\int_{\Theta_i} f(\boldsymbol{x}\mid\Theta_i)\pi_i^N(\Theta_i)d\Theta_i}. \tag{3}$$

Hence, the corresponding Bayes factor $B_{ji}^N$, is indeterminate. To solve this problem, O'Hagan(1995) proposed the FBF for Bayesian testing. The FBF of model $H_j$ to model $H_i$ is

$$B_{ji}^{F}=\frac{q_j(b,\boldsymbol{x})}{q_i(b,\boldsymbol{x})}, \tag{4}$$

where $q_i(b,\boldsymbol{x})=\frac{m_i(\boldsymbol{x})}{m_i^b(\boldsymbol{x})}=\frac{\int_{\Theta_i} f_i(\boldsymbol{x}\mid\Theta_i)\pi_i^N(\Theta_i)d\Theta_i}{\int_{\Theta_i} f_i^b(\boldsymbol{x}\mid\Theta_i)\pi_i^N(\Theta_i)d\Theta_i}$ and $f_i(\boldsymbol{x}\mid\Theta_i)$ is the likelihood function and b specifies a fraction of likelihood which is to be used as a prior density. One frequently suggested choice is b = m/n, where m is the size of the minimal training sample.

## 3. Bayesian Testing Based on FBF

Let the random variable $X$ have the weibull model with scale and shape parameters $\alpha$ and $\beta$, respectively. Then probability density function is given as

$$f(x:\alpha,\beta) = \frac{\beta}{\alpha}\left(\frac{x}{\alpha}\right)^{\beta-1}\cdot\exp\left(-\left(\frac{x}{\alpha}\right)^{\beta}\right), \tag{5}$$

where $x\geq 0$, $\alpha>0$, and $\beta>0$.

In this section, we are interested in the Bayesian testing of the shape parameters in two weibull model based on FBF. That is, we want to test the hypotheses whether the hazard function of weibull model over time is constant or not, that is, $H_1:\beta=1$ and $H_2:\beta\neq 1$.

Let's consider sample of size $n$ from two weibull populations with parameters $\Theta=(\alpha,\beta)$. Then observed sample consists of the failure times $x_1,\cdots,x_n$. To test the hypothesis of the shape parameter based on FBF, we need to compute (4). By Kang, Kim and Cho(1999), the reference priors for $H_1:\beta=1$, v.s. $H_2:\beta\neq 1$ are respectively given by

$$\pi_1^N(\Theta_1) = \frac{1}{\alpha}\cdot I(0<\alpha<\infty) \tag{6}$$

and

$$\pi_2^N(\Theta_2) = \frac{1}{\alpha\beta}\cdot I(0<\alpha<\infty)\cdot I(0<\beta<\infty) \tag{7}$$

where $I(A)$ means the indicator function of A for any set A. Now we derive the marginals with respect to above the reference priors. Since likelihood function under $H_1 : \beta=1$ is $f_1(\boldsymbol{x} : \Theta_1) = \left(\frac{1}{\alpha}\right)^n \cdot \exp\left(-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)\right)$, $m_1(\boldsymbol{x})$ under $H_1 : \beta=1$ is given by

$$m_1(\boldsymbol{x}) = \int_0^\infty \left(\frac{1}{\alpha}\right)^n \exp\left(-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)\right) \cdot \frac{1}{\alpha}\, d\alpha = \Gamma(n) \cdot \left(\sum_{i=1}^{n}\right)^{-n} \tag{8}$$

Also $m_1^b(\boldsymbol{x})$ under $H_1 : \beta=1$ is given by

$$m_1^b(\boldsymbol{x}) = \int_0^\infty \left(\frac{1}{\alpha}\right)^{bn} \exp\left(-b\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)\right) \cdot \frac{1}{\alpha}\, d\alpha = \Gamma(bn) \cdot b^{-bn}\left(\sum_{i=1}^{n}\right)^{-bn} \tag{9}$$

Hence, $q_1(b, \boldsymbol{x})$ under $H_1 : \beta=1$ by (8) and (9) is given by

$$q_1(b, \boldsymbol{x}) = \frac{\int_{\Theta_1} f_1(\boldsymbol{x} \mid \Theta_1)\pi_i^N(\Theta_1)d\Theta_1}{\int_{\Theta_1} f_1^b(\boldsymbol{x} \mid \Theta_1)\pi_i^N(\Theta_1)d\Theta_1} = \frac{\Gamma(n) \cdot \left(\sum_{i=1}^{n} x_i\right)^{n(b-1)}}{\Gamma(bn)b^{-bn}} \tag{10}$$

On the other side, since the likelihood function under $H_2 : \beta\neq1$ is

$$f_2(\boldsymbol{x} : \Theta_2) = \left(\frac{\beta}{\alpha}\right)^n \cdot \prod_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta-1} \cdot \exp\left(-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right) \tag{11}$$

$m_2(\boldsymbol{x})$ under $H_2 : \beta\neq1$ is given by

$$\begin{aligned} m_2(\boldsymbol{x}) &= \int_0^\infty \int_0^\infty \left(\frac{\beta}{\alpha}\right)^n \cdot \prod_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta-1} \cdot \exp\left(-\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right) \cdot \frac{1}{\alpha\beta}\, d\alpha\, d\beta \\ &= \Gamma(n) \cdot \left(\prod_{i=1}^{n} x_i\right)^{-1} \cdot S_1(\boldsymbol{x}), \end{aligned} \tag{12}$$

where $S_1(\boldsymbol{x}) = \int_0^\infty \beta^{n-2}\left(\prod_{i=1}^{n} x_i\right)^{\beta}\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{-n} d\beta$. Also $m_2^b(\boldsymbol{x})$ under $H_2 : \beta\neq1$ is given by

$$\begin{aligned} m_2^b(\boldsymbol{x}) &= \int_0^\infty \int_0^\infty \left(\frac{\beta}{\alpha}\right)^{bn} \cdot \prod_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{b(\beta-1)} \cdot \exp\left(-b\sum_{i=1}^{n}\left(\frac{x_i}{\alpha}\right)^{\beta}\right) \cdot \frac{1}{\alpha\beta}\, d\alpha\, d\beta \\ &= \Gamma(bn) \cdot b^{-bn}\left(\prod_{i=1}^{n} x_i\right)^{-b} \cdot S_2(\boldsymbol{x}), \end{aligned} \tag{13}$$

where $S_2(\boldsymbol{x}) = \int_0^\infty \beta^{bn-2}\left(\prod_{i=1}^{n} x_i\right)^{b\beta}\left(\sum_{i=1}^{n} x_i^{\beta}\right)^{-bn} d\beta$. Hence, the $q_2(b, \boldsymbol{x})$ under $H_2 : \beta\neq1$ by (12) and (13) is given by

$$q_2(b, \boldsymbol{x}) = \frac{m_2(\boldsymbol{x})}{m_2^b(\boldsymbol{x})} = \frac{\Gamma(n)\left(\prod_{i=1}^{n} x_i\right)^{b-1} \cdot S_1(\boldsymbol{x})}{\Gamma(bn)b^{-bn} \cdot S_2(\boldsymbol{x})}. \tag{14}$$

From (8) and (10), the FBF of $H_2$ to $H_1$ is given by

$$B_{21}^F = \frac{q_2(b, \boldsymbol{x})}{q_1(b, \boldsymbol{x})} = \left(\prod_{i=1}^{n} x_i\right)^{b-1}\left(\sum_{i=1}^{n} x_i\right)^{n(1-b)} \cdot \frac{S_1(\boldsymbol{x})}{S_2(\boldsymbol{x})} \tag{15}$$

Using these FBF, we can calculate the posterior probability for hypothesis $H_i$, $i=1,2$. Thus, we can select the hypothesis with highest posterior probability in hypotheses $H_i$ based on FBF.

## 4. Illustrative Example and Conclusion

In this section, an example is presented to illustrate for the test $H_1 : \beta=1$ v.s. $H_2 : \beta\neq1$. We take the prior probability of $H_i$ being true, $p_i=0.5$, $i=1,2$.

***Example*** : The following data are time to breakdown of a type of electrical insulating fluid subject to a constant voltage stress(Nelson(1970)).

| | |
|---|---|
| **30 KV** | 7.74, 17.05, 20.46, 21.02, 22.66, 43.40, 47.30, 139.07, 144.12, 175.88, 194.90 |
| **32 KV** | 0.27, 0.40, 0.69, 0.79, 2.75, 3.91, 9.88, 13.95, 15.93, 27.80, 53.24, 82.85, 89.29, 100.58, 215.10 |

We compute the FBF's and posterior probabilities of the test $H_1 : \beta=1$, v.s. $H_2 : \beta\neq1$ for voltage breakdown data in table 1. From table 1, there are strong evidence for $H_1$ in terms of the posterior probability for 30KV since $P(H_2 \mid \boldsymbol{x})$ is 0.1384. But there are strong evidence for $H_2$ in terms of the posterior probability for 32KV since $P(H_2 \mid \boldsymbol{x})$ is 0.9285.

<Table 1> FBF and the posterior probability for $H_1 : \beta=1$ v.s. $H_2 : \beta\neq1$

| Hypothesis | 30 KV | | 32 KV | |
|---|---|---|---|---|
| | $B_{21}^F$ | $P(H_2 \mid \boldsymbol{x})$ | $B_{21}^F$ | $P(H_2 \mid \boldsymbol{x})$ |
| $H_1$ v.s. $H_2$ | 0.1607 | 0.1384 | 12.9925 | 0.9285 |

In conclusion, FBF is completely automatic Bayes factors in that they are based only on the data and noninformative priors. FBF methodology can be easily applied to nonnested as well as to irregular problems. Also they can be applied in general when the samples come from any model.

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