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Bayesian Multiple Comparisons for the Ratio of the Failure Rates in Two Components System

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Abstract

In this paper, we consider multiple comparisons for the ratio of the failure rates in two components system that the lifetimes of the components have independent exponential distributions. Also we suggest Bayesian multiple comparisons procedure based on fractional Bayes factor when noninformative priors are applied for the parameters. Finally, we give numerical examples to illustrate our procedure.

Keywords : Bayesian testing, Exponential model, Fractional Bayes factor, Multiple comparisons, Noninformative priors, Posterior probability

1. Introduction

The exponential model has been widely used as a model in areas ranging from studies on the lifetime. So we consider two components system that the lifetimes of two components have independent exponential distribution. In this paper, we focus on testing for the ratio of failure rates in two components system. For testing equality of the ratios, classical tests such as approximate test are widely used. But the test of equality of the ratios more than three systems relies on likelihood ratio test statistic. And classical tests only decide whether the null hypothesis, commonly the equality of the ratios, will be rejected or not. But Bayesian approach to resolve the multiple comparisons problem selects the best model with the highest posterior probability.

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In this paper, we focus on Bayesian multiple comparisons for the ratios of the failure rates of K systems based on Bayes factor. In many cases, noninformative priors are required because of limited information and time constrains. But, since noninformative priors are typically improper, the priors are only up to arbitrary constants which affect the values of Bayes factors.

Berger and Pericchi(1996) introduced the intrinsic Bayes factor(IBF) using a data splitting idea, which would eliminate the arbitrariness of improper priors. Kim and Ko(2005) considered testing the equality of two inverse Gaussian populations based on IBF. On the other side, O'Hagan(1995) proposed the fractional Bayes factor(FBF) to remove the arbitrariness by using a portion of the likelihood. Kim, Kang and Lee(2006) considered the Bayesian multiple comparison procedure of binomial populations.

In this paper, we consider multiple comparisons for the ratio of the failure rates in K two components systems that the lifetimes of the components in each system have independent exponential models. And we suggest the Bayesian multiple comparisons procedure based on FBF when noninformative priors are applied for the failure rates. Finally, we give some numerical examples to illustrate our procedure.

2. Preliminaries

Let $M_1, \cdots, M_N$ be models under consideration. And let X and Y be random variables with joint probability density function $f_i(x, y \mid \theta_i)$, under model M_i, $i=1,\cdots,N$. The parameter vectors θ_i are unknown. Let $\pi_i(\theta_i)$ be the prior distribution of model M_i, and let p_i be the prior probabilities of model M_i. Then the posterior probability that the model M_i is true is given as

$$P(M_i \mid \mathbf{x}, \mathbf{y}) = \left(\sum_{j=1}^{N} \frac{p_j}{p_i} B_{ji} \right)^{-1},$$

where B_{ij} is the Bayes factor of model M_j to model M_i defined by

$$B_{ji} = \frac{m_j(\mathbf{x}, \mathbf{y})}{m_i(\mathbf{x}, \mathbf{y})} = \frac{\int_{\Theta_j} f_j(\mathbf{x}, \mathbf{y} \mid \theta_j) \pi_j(\theta_j) d\theta_j}{\int_{\Theta_i} f_i(\mathbf{x}, \mathbf{y} \mid \theta_i) \pi_i(\theta_i) d\theta_i}, \qquad (1)$$

where $\mathbf{x} = (x_{i1}, \cdots, x_{in_{i1}})$ and $\mathbf{y} = (y_{i1}, \cdots, y_{in_{i2}})$ under model M_i, $i=1,\cdots,N$. The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Usually, one can use the noninformative prior, often improper, for parameters. Let π_i^N be the noninformative prior for model M_i. The use of improper priors $\pi_i^N(\cdot)$ in (1) causes the B_{ji} to contain arbitrary constants.

To solve this problem, O'Hagan(1995) proposed the FBF for Bayesian testing and model selection problem as follow. If we use a noninformative prior $\pi_i^N(\theta_i)$ under M_i, then equation (1) becomes

$$B_{ji}^N = \frac{m_j^N(x,\ y)}{m_i^N(x,\ y)} = \frac{\int_{\Theta_j} f_j(x,\ y \mid \theta_j)\pi_j^N(\theta_j)d\theta_j}{\int_{\Theta_i} f_i(x,\ y \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}. \qquad (2)$$

Then the FBF of model M_j versus model M_i is

$$B_{ji}^F = \frac{q_j(b,\ x,\ y)}{q_i(b,\ x,\ y)}, \qquad (3)$$

where $q_i(b,\ x,\ y) = \dfrac{\int_{\Theta_i} f_i(x,\ y \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}{\int_{\Theta_i} f_i^b(x,\ y \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}$ and b specify a fraction of the likelihood which is to be used as a prior density.

Let's consider K two components systems that the distributions of the lifetimes have parameters $\Theta = (\theta_1, \cdots, \theta_K)$.

The multiple comparisons of K ratios of the failure rates are to make inferences concerning relationships among the θ_i's.

Let $\Omega = \{(\theta_1, \theta_2, \cdots, \theta_K):\ \theta_i \in R,\ i = 1, 2, \cdots, K\}$ be the K-dimensional parameter space. Equality and inequality relationships among the θ_i's induce statistical hypotheses that subsets of Θ, i.e., M_1: $\Omega_1 = \{\theta_i : \theta_1 \neq \theta_2 = \cdots = \theta_K\}$ and so on up to M_N: $\Omega_N = \{\theta_i : \theta_1 \neq \theta_2 \neq \cdots \neq \theta_K\}$. The hypotheses $M_r : \Omega_r$, $r = 1, 2, \cdots, N$, are disjoint, and $\bigcup_{r=1}^N \Omega_r = \Omega$.

Each hypothesis can be classified $r(r = 1, \cdots, K)$ distinct groups. Let $\theta_1^*, \cdots, \theta_r^*$ denote the set of distinct θ_i's, where r is the number of distinct elements in the vector Ω. We need to define the configuration notation.

Definition 1 (*Configuration*). The configuration $S = \{S_1, \cdots, S_K\}$ determines a classification of θ into r distinct groups or clusters. Write K_j for the set of indices of parameters in group j, $K_j = \{i: S_i = j\}$. Let $n_{K_j} = \{n_i \mid i \in K_j\}$ be the index set of observations and θ^*_j be the common parameter value for K_j.

There is a one to one correspondence between hypotheses and configurations. Therefore the Bayes factor for multiple comparisons can be easily compute by this configuration notations.

Now we will develop the Bayesian multiple comparisons procedure based on FBF. Suppose that a model classified r distinct groups. Let $\boldsymbol{x}_i = (x_{i1}, \cdots, x_{in_{i1}})$ and $\boldsymbol{y}_i = (y_{i1}, \cdots, y_{in_{i2}})$ be a $n_{i1} \times 1$ and $n_{i2} \times 1$ vectors of independent observations on θ_i. Then the likelihood function is given by

$$L(\theta^*_1, \cdots, \theta^*_r \mid \boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{r} \left[\prod_{\{i: i \in K_i\}} f_i(\boldsymbol{x}_i, \boldsymbol{y}_i \mid \theta_i) \right]. \tag{4}$$

Hence, the element of the FBF by O'Hagan (1995) is given as

$$q(b, \boldsymbol{x}, \boldsymbol{y}) = \frac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} L(\theta^*_1, \cdots, \theta^*_r \mid \boldsymbol{x}, \boldsymbol{y}) \cdot \pi^N_r\left(\theta^*_1, \cdots, \theta^*_r\right) d\theta^*_1 \cdots d\theta^*_r}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} L^b(\theta^*_1, \cdots, \theta^*_r \mid \boldsymbol{x}, \boldsymbol{y}) \cdot \pi^N_r\left(\theta^*_1, \cdots, \theta^*_r\right) d\theta^*_1 \cdots d\theta^*_r}. \tag{5}$$

Thus if a model M_i is classified r_i distinct groups and a model M_j is classified r_j distinct groups then the FBF of M_j versus M_i is given by

$$B^F_{ji} = \frac{q_j(b, \boldsymbol{x}, \boldsymbol{y})}{q_i(b, \boldsymbol{x}, \boldsymbol{y})},$$

where $q_i(b, \boldsymbol{x}, \boldsymbol{y}) = \dfrac{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} L(\theta^*_1, \cdots, \theta^*_{r_i} \mid \boldsymbol{x}, \boldsymbol{y}) \cdot \pi^N_{r_i}\left(\theta^*_1, \cdots, \theta^*_{r_i}\right) d\theta^*_1 \cdots d\theta^*_{r_i}}{\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} L^b(\theta^*_1, \cdots, \theta^*_{r_i} \mid \boldsymbol{x}, \boldsymbol{y}) \cdot \pi^N_{r_i}\left(\theta^*_1, \cdots, \theta^*_{r_i}\right) d\theta^*_1 \cdots d\theta^*_{r_i}}$.

Hence the FBF for all comparisons can be computed by equation (2). Using these FBF, we can calculate the posterior probability for model M_i, $i = 1, \cdots, N$. Thus, we can select the hypothesis with highest posterior probability in Bayesian multiple comparisons based on FBF.

3. Bayesian Multiple Comparisons for the Ratio of the Failure Rates

Let $X_i = (X_{i1}, \cdots, X_{in_{i1}})$ and $Y_i = (Y_{i1}, \cdots, Y_{in_{i2}})$ be random sample from ith two components system that the lifetimes of the components have independent exponential models with the failure rates λ_i and η_i, respectively. Then the likelihood function for parameter vector $\Theta = (\theta_1, \cdots, \theta_K)$ given $\boldsymbol{x}_i = (x_{i1}, \cdots, x_{in_{i1}})$ and $\boldsymbol{y}_i = (y_{i1}, \cdots, y_{in_{i2}})$ is given by

$$L(\Theta \mid \boldsymbol{x}, \boldsymbol{y}) = \prod_{i=1}^{K} f_i(\boldsymbol{x}_i, \boldsymbol{y}_i \mid \theta_i)$$
$$= \prod_{i=1}^{K} \left[\lambda_i^{\sum_{i=1}^{K} n_{i1}} \cdot \eta_i^{\sum_{i=1}^{K} n_{i2}} \cdot \exp\left(-\lambda_i \sum_{i=1}^{K} \sum_{j=1}^{n_{i1}} x_{ij} - \eta_i \sum_{i=1}^{K} \sum_{j=1}^{n_{i2}} y_{ij} \right) \right]. \tag{6}$$

In this paper, we focus on Bayesian multiple comparisons for the ratio $\zeta_i = \eta_i / \lambda_i$ of the ith populations. Cho and Baek(2002) obtained the reference prior as noninformative prior for the ratio $\zeta = \eta / \lambda$ as follows.

$$\pi^N(\theta) = \frac{1}{\lambda \zeta}, \quad \lambda, \ \zeta > 0, \tag{7}$$

where $\theta = (\lambda, \zeta)$.
Suppose that a model M_i is classified r distinct groups and that we use the noninformative prior for $(\theta_1^*, \cdots, \theta_r^*)$ as follows.

$$\pi_r^N(\theta_1^*, \cdots, \theta_r^*) = \frac{1}{\lambda_1^* \zeta_1^* \cdots \lambda_r^* \zeta_r^*}, \quad 0 < \lambda_i^*, \ \zeta_i^* < \infty, \tag{8}$$

where $\theta_i^* = (\lambda_i^*, \zeta_i^*)$, $i = 1, \cdots, r$. The likelihood function is

$$L(\theta_1^*, \cdots, \theta_r^* \mid \boldsymbol{x}, \boldsymbol{y}).$$
$$= \prod_{t=1}^{r} (\lambda_t^*)^{\sum_{i \in K_t} (n_{i1} + n_{i2})} (\zeta_t^*)^{\sum_{i \in K_t} n_{i2}} \cdot \exp\left(-\lambda_t^* \sum_{i \in K_t} \sum_{j=1}^{n_{i1}} x_{ij} - \lambda_t^* \zeta_t^* \sum_{i \in K_t} \sum_{j=1}^{n_{i2}} y_{ij} \right). \tag{9}$$

Then the element of the FBF is computed as follows;

$$\int_0^\infty \cdots \int_0^\infty L(\theta_1^*, \cdots, \theta_r^* \mid x, y)\pi_r^N(\theta_1^*, \cdots, \theta_r^*)d\theta_1^* \cdots d\theta_r^*$$

$$= \prod_{t=1}^{r} \Gamma\Big(\sum_{i \in K_t} n_{i1}\Big) \cdot \Gamma\Big(\sum_{i \in K_t} n_{i2}\Big) \cdot \Big(\sum_{i \in K_t}\sum_{j=1}^{n_{i1}} x_{ij}\Big)^{-\sum_{i \in K_t} n_{i1}} \cdot \Big(\sum_{i \in K_t}\sum_{j=1}^{n_{i2}} y_{ij}\Big)^{-\sum_{i \in K_t} n_{i2}}$$

and

$$\int_0^\infty \cdots \int_0^\infty L^b(\theta_1^*, \cdots, \theta_r^* \mid x, y)\pi_r^N(\theta_1^*, \cdots, \theta_r^*)d\theta_1^* \cdots d\theta_r^*$$

$$= \prod_{t=1}^{r} \Gamma\Big(b_1 \sum_{i \in K_t} n_{i1}\Big) \cdot \Gamma\Big(b_2 \sum_{i \in K_t} n_{i2}\Big) \cdot \Big(b_1 \sum_{i \in K_t}\sum_{j=1}^{n_{i1}} x_{ij}\Big)^{-b_1\sum_{i \in K_t} n_{i1}} \cdot \Big(b_2 \sum_{i \in K_t}\sum_{j=1}^{n_{i2}} y_{ij}\Big)^{-b_2\sum_{i \in K_t} n_{i2}}.$$

Hence, $q(b, x, y)$ is given by

$$q(b, x, y) = \prod_{t=1}^{r} \frac{\Gamma\Big(\sum_{i \in K_t} n_{i1}\Big) \cdot \Gamma\Big(\sum_{i \in K_t} n_{i2}\Big) \cdot \Big(\sum_{i \in K_t}\sum_{j=1}^{n_{i1}} x_{ij}\Big)^{-\sum_{i \in K_t} n_{i1}} \cdot \Big(\sum_{i \in K_t}\sum_{j=1}^{n_{i2}} y_{ij}\Big)^{-\sum_{i \in K_t} n_{i2}}}{\Gamma\Big(b_1 \sum_{i \in K_t} n_{i1}\Big) \cdot \Gamma\Big(b_2 \sum_{i \in K_t} n_{i2}\Big) \cdot \Big(b_1 \sum_{i \in K_t}\sum_{j=1}^{n_{i1}} x_{ij}\Big)^{-b_1\sum_{i \in K_t} n_{i1}} \cdot \Big(b_2 \sum_{i \in K_t}\sum_{j=1}^{n_{i2}} y_{ij}\Big)^{-b_2\sum_{i \in K_t} n_{i2}}} \tag{10}$$

.

Then the FBF of M_j versus M_i is given by

$$B_{ji}^F(x) = \prod_{\{M_j^* t=1\}}^{r_j} \frac{\Gamma\Big(\sum_{j \in K_t} n_{j1}\Big) \cdot \Gamma\Big(\sum_{j \in K_t} n_{j2}\Big) \cdot \Big(\sum_{j \in K_t}\sum_{l=1}^{n_{j1}} x_{jl}\Big)^{-\sum_{j \in K_t} n_{j1}} \cdot \Big(\sum_{j \in K_t}\sum_{l=1}^{n_{j2}} y_{jl}\Big)^{-\sum_{j \in K_t} n_{j2}}}{\Gamma\Big(b_1 \sum_{j \in K_t} n_{j1}\Big) \cdot \Gamma\Big(b_2 \sum_{j \in K_t} n_{j2}\Big) \cdot \Big(b_1 \sum_{j \in K_t}\sum_{l=1}^{n_{j1}} x_{jl}\Big)^{-b_1\sum_{j \in K_t} n_{j1}} \cdot \Big(b_2 \sum_{j \in K_t}\sum_{l=1}^{n_{j2}} y_{jl}\Big)^{-b_2\sum_{j \in K_t} n_{j2}}}$$

$$\times \prod_{\{M_i^* t=1\}}^{r_i} \frac{\Gamma\Big(b_1 \sum_{i \in K_t} n_{i1}\Big) \cdot \Gamma\Big(b_2 \sum_{i \in K_t} n_{i2}\Big) \cdot \Big(b_1 \sum_{i \in K_t}\sum_{l=1}^{n_{i1}} x_{jl}\Big)^{-b_1\sum_{i \in K_t} n_{i1}} \cdot \Big(b_2 \sum_{i \in K_t}\sum_{l=1}^{n_{i2}} y_{il}\Big)^{-b_2\sum_{i \in K_t} n_{i2}}}{\Gamma\Big(\sum_{i \in K_t} n_{i1}\Big) \cdot \Gamma\Big(\sum_{i \in K_t} n_{i2}\Big) \cdot \Big(\sum_{i \in K_t}\sum_{l=1}^{n_{i1}} x_{il}\Big)^{-\sum_{j \in K_t} n_{i1}} \cdot \Big(\sum_{i \in K_t}\sum_{l=1}^{n_{i2}} y_{il}\Big)^{-\sum_{i \in K_t} n_{i2}}}. \tag{11}$$

4. Numerical Examples

In this section, we use numerical data to illustrate the Bayesian multiple comparisons procedure based on FBF for the ratios of the failure rates in two components system that the lifetimes of the components are independent exponential distributions. Here, we consider K=4 two components systems and sample size of $n_{i1}=n_{i2}=20,\ i=1,\cdots,4$. So the numbers of possible hypotheses are 15. Table 1 denotes the observations for 4 two components systems.

<Table 1> Observations for each system

system	*components*	*observations*
1	*X*	.5631, .3664, 1.5945, .5757, 1.6993, .2503, 1.1338, 1.0221, .1062, .2402, 1.5693, .3287, .3828, .8950, .2513, 1.2370, .9889, .1265, .8551, .3839
	Y	.9588, .4806, .0302, 1.6232, .4264, .5107, .1201, .6587, .0457, 1.6441, 1.2235, 1.6592, 1.2744, .1251, .8506, .9119, .0843, .5889, 1.8410, 1.4418
2	*X*	1.9597, .0631, 1.3502, .1755, .3364, .4560, .3058, 1.9204, .0671, .3195, .4763, .1840, .9519, .1060, .1265, .6916, .7622, .2727, .1900, .4437
	Y	1.3774, .0028, .4075, .7519, .4017, 2.4950, .9412, .0453, 2.2321, .1901, .8292, .4083, .2285, 1.7112, .4496, 1.2518, .4324, .6550, .7987, 3.6654
3	*X*	1.0013.1025, .8579, .4016, 1.0294, .1709, .0537, .5077, .5693, 2.0985, .1287, .1371, 1.2791, .2629, .0825, 1.6988, .7313, 2.1151, .3065, .0877,
	Y	.8342, .1458, .1468, .0145, .1285, .1606, .7284, .7408, .0928, .3788, .5478, .2561, .0075, .5336, .3238, .2451, .4256, .2567, .0068, .0874,
4	*X*	.8718, .2705, .0405, .3782, .0188, .5570, .0245, .0720, .6205, .3022, .1505, .3898, .0051, .2128, .8795, .5008, .4344, .4312, .2421, .2547
	Y	.0124, .0827, .5150, .2314, .1597, .2098, .2821, .1780, .0924, .1607, .0982, .1269, .3604, .0056, .4643, .1229, .0996, .2376, .0941, .2445

From table 1, we obtained the maximum likelihood estimators of the ratios for 4 systems are $\hat{\zeta}_1=0.8831$, $\hat{\zeta}_2=0.5789$, $\hat{\zeta}_2=2.2474$ and $\hat{\zeta}_4=1.7619$, respectively. So we note that the true hypothesis may be $H:\ \zeta_1=\zeta_2\neq\zeta_3\neq\zeta_4$ or $H:\ \zeta_1=\zeta_2\neq\zeta_3=\zeta_4$. Table 2 gives the calculated posterior probabilities for all possible hypotheses based on FBF.

<Table 2> Calculated posterior probabilities based on FBF

Hypothesis	Posterior probabilities
$\zeta_1=\zeta_2=\zeta_3=\zeta_4$	0.0000
$\zeta_1=\zeta_2=\zeta_3\neq\zeta_4$	0.0198
$\zeta_1=\zeta_2=\zeta_4\neq\zeta_3$	0.0000
$\zeta_1=\zeta_2\neq\zeta_3=\zeta_4$	0.4002
$\zeta_1=\zeta_2\neq\zeta_3\neq\zeta_4$	0.4948
$\zeta_1=\zeta_3=\zeta_4\neq\zeta_2$	0.0198
$\zeta_1=\zeta_3\neq\zeta_2=\zeta_4$	0.0000
$\zeta_1=\zeta_3\neq\zeta_2\neq\zeta_4$	0.0078
$\zeta_1=\zeta_4\neq\zeta_2=\zeta_3$	0.0000
$\zeta_1=\zeta_4\neq\zeta_2\neq\zeta_3$	0.0000
$\zeta_1\neq\zeta_2=\zeta_3=\zeta_4$	0.0000
$\zeta_1=\zeta_2\neq\zeta_3=\zeta_4$	0.0015
$\zeta_1\neq\zeta_2=\zeta_4\neq\zeta_3$	0.0000
$\zeta_1\neq\zeta_2\neq\zeta_3=\zeta_4$	0.0250
$\zeta_1\neq\zeta_2\neq\zeta_3\neq\zeta_4$	0.0309

From tabel 2, it is evident that the hypothesis for $\zeta_1=\zeta_2\neq\zeta_3\neq\zeta_4$ has the most large posterior probabilities 0.4948. And the hypothesis for $\zeta_1=\zeta_2\neq\zeta_3=\zeta_4$ has second large posterior probability 0.4002. This suggests that the data lend greatest support to equalities for $\zeta_1=\zeta_2$ or $\zeta_1=\zeta_2$ and $\zeta_3=\zeta_4$ being different from the others.

Up to this point, we have considered the problem of developing Bayesian multiple comparisons of the failure rates for K two components systems. Extension of the above approach to the Bayesian multiple comparison problems for the another population is straightforward. The research topics pertaining to the extension of the method and the examination of its performance are worthy to study and are left as a future subject of research.

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