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Bayesian One-Sided Testing for the Ratio of Poisson Means

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Abstract

When X and Y have independent Poisson distributions, we develop a Bayesian one-sided testing procedures for the ratio of two Poisson means. We propose the objective Bayesian one-sided testing procedures for the ratio of two Poisson means based on the fractional Bayes factor and the intrinsic Bayes factor. Some real examples are provided.

Keywords : Fractional Bayes factor, Intrinsic Bayes factor, One-sided testing, Ratio of Poisson means, Reference prior

1. Introduction

The Poisson distribution is applied to model many processes in a broad variety of field such as biology, ecology, epidemiology, medicine, industrial quality control and agriculture. The comparison of Poisson means from two independent samples is of great interest. For instance in the comparison of incidence of breast cancer study two groups of women were compared to determine whether those who had been examined using X-ray fluoroscopy during treatment for tuberculosis has a higher rate of breast cancer than those who had not been examined using X-ray fluoroscopy (Graham, et. al. 2003; Ng and Tang, 2005).

The present paper focuses on Bayesian testing procedure for the ratio of two

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Poisson means. In Bayesian testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' priors or reference priors (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegalhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegalhalter and Smith (1982) used the device of imaginary training samples in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so, there is no guarantee that the Bayes factor of Spiegalhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas.

For the comparison for two Poisson means, the most common method of testing the difference between two means is the conditional method that was first proposed by Przyborowski and Wilenski (1940). The conditional distribution follows binomial distribution whose success probability is a function of the ratio of two means. Therefore hypothesis testing and interval estimation procedures can be readily developed from the exact methods for making inferences about the binomial success probability. In particular, Chapman (1952) proposed a confidence interval for the ratio of two means which is deduced from the exact confidence interval for the binomial success probability due to Clopper and Pearsons (1934). Since then some papers have addressed these inferential procedures based on the conditional distribution (Gail, 1974; Shiue and Bain, 1982; Nelson, 1991). Schwertman and Martinez (1994) give several binomial-normal based approximate methods for constructing confidence interval for difference of two means.

Although the conditional test is exact and simple to use, in the two-sampling binomial case such a conditional test is known to be less powerful than some unconditional tests. For example, see Suissa and Schuster (1985) and Storer and Kim (1990). So Krishnamoorthy and Thomson (2004) proposed an unconditional test for testing about the difference of two Poisson means. In numerical studies, they showed that the unconditional test is very satisfactory in terms of sample size, and is more powerful than the conditional test due to Przyborowski and Wilenski (1940).

Almost all the work mentioned above is the analysis based on the frequentist point of view, there is a little work on this problem from the viewpoint of the objective Bayesian framework. So we feel a strong necessity to develop objective

Bayesian testing procedure for the ratio of two Poisson means. For dealing this problem, we use the fractional Bayes factor (O'Hagan, 1995) and the intrinsic Bayes factor (Berger and Pericchi, 1996).

The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian model selection based on the Bayes factor. In Section 3, using the reference priors, we provide the Bayesian testing procedure based on the fractional Bayes factor and intrinsic Bayes factor for testing the ratio of two Poisson means. In Section 4, some real examples are given.

2. Bayesian Model Selection Methods

Models (or Hypotheses) $M_1, M_2, \cdots, M_q$ are under consideration, with the data $x=(x_1, x_2, \cdots, x_n)$ having probability density function $f_i(x \mid \theta_i)$ under model $H_i, i=1,2,\cdots,q$. The parameter vectors θ_i are unknown. Let $\pi_i(\theta_i)$ be the prior distribution of model M_i, and let p_i be the prior probabilities of model M_i, $i=1,2,\cdots,q$. Then the posterior probability that the model M_i is true is

$$P(M_i \mid x)=\left(\sum_{j=1}^{q}\frac{p_j}{p_i}\cdot B_{ji}\right)^{-1}, \tag{1}$$

where B_{ji} is the Bayes factor of model M_j to model M_i defined by

$$B_{ji}=\frac{\int f_j(x \mid \theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(x \mid \theta_i)\pi_i(\theta_i)d\theta_i}=\frac{m_j(x)}{m_i(x)}. \tag{2}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta)$. Usually, one can use the noninformative prior, often improper, such as uniform prior, Jeffreys prior, reference prior or probability matching prior. Denote it as π_i^N. The use of improper priors $\pi_i^N(\cdot)$ in (2) causes the B_{ji} to contain unspecified constants.

2.1 The Intrinsic Bayes Factor Approach

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $x(l)$ denote the part of the data to be so used and let

$x(-l)$ be the remainder of the data, such that

$$0 < m_i^N(x(l)) < \infty, \quad i = 1, \cdots, q. \tag{3}$$

In view of (3), the posteriors $\pi_i^N(\theta_i \mid x(l))$ are well defined. Now, consider the Bayes factor, $B_{ji}(l)$, for the rest of the data $x(-l)$, using $\pi_i^N(\theta_i \mid x(l))$ as the priors:

$$B_{ji}(l) = \frac{\int_{\Theta_j} f(x(-l) \mid \theta_j, x(l)) \pi_j^N(\theta_j \mid x(l)) d\theta_j}{\int_{\Theta_i} f(x(-l) \mid \theta_i, x(l)) \pi_i^N(\theta_i \mid x(l)) d\theta_i} = B_{ji}^N \cdot B_{ij}^N(x(l)) \tag{4}$$

where

$$B_{ji} = B_{ji}^N(x) = \frac{m_j^N(x)}{m_i^N(x)} \quad \text{and} \quad B_{ij}^N(x(l)) = \frac{m_i^N(x(l))}{m_j^N(x(l))}$$

are the Bayes factors that would be obtained for the full data x and training samples $x(l)$, respectively. The Bayes factor (4) depends on the specific training sample $x(l)$. To avoid the difficulty of choosing $x(l)$, Berger and Perrich (1996) proposed the use of a minimal training sample to compute the Bayes factor (4). Then, an average over all the possible minimal training samples contained in the sample is computed. This gives the arithmetic intrinsic Bayes factor (AIBF) of M_j to M_i as

$$B_{ji}^{AI} = B_{ji}^N \cdot \frac{1}{L} \sum_{l=1}^{L} B_{ij}^N(x(l)). \tag{5}$$

where L is the number of minimal training samples $x(l)$ contained in x. Also Berger and Pericchi (1998) gives the median intrinsic Bayes factor (MIBF) of M_j to M_i as

$$B_{ji}^{MI} = B_{ji}^N \cdot ME[B_{ij}^N(x(l))], \tag{6}$$

where ME indicates the median, here to be taken over Bayes factor with respect to all the training samples. The MIBF is the most robust and widely applicable intrinsic Bayes factor. We can also calculate the posterior probability of M_i using (1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (5) and (6).

2.2 The Fractional Bayes Factor Approach

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction, b, of each likelihood function, $L(\theta_i)=f_i(x \mid \theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor of model M_j versus model M_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int f_i^b(x \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}{\int f_j^b(x \mid \theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \cdot F_{ij}^b,$$

and $f_i(x \mid \theta_i)$ is the likelihood function, b specifies a fraction of the likelihood which is to be used as a prior density, and $F_{ij}^b = m_i^b(x)/m_j^b(x)$ is the correction term. Among the choice of fraction b in O'Hagan (1995), we use $b=m/n$, where m is the size of the minimal training sample size, because of the appealing arguments in its favor given by Berger and Mortera (1995) and Kass and Wasserman (1995).

3. Bayesian Test Procedures

Let X be a Poisson distribution with probability mass function

$$f(x \mid \lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x=0,1,2,\cdots \tag{7}$$

where $\lambda > 0$ is the mean parameter. Suppose that $X_1,\cdots,X_{n_1}$ and $Y_1,\cdots,Y_{n_2}$ denote independent random samples from Poisson distributions with the mean λ and the mean $\eta\lambda$, respectively. Then the joint probability mass function of $X_1,\cdots,X_{n_1}$ and $Y_1,\cdots,Y_{n_2}$ is

$$f(x, y \mid \lambda,\eta) = \frac{\lambda^{n_1\bar{x}+n_2\bar{y}}\eta^{n_2\bar{y}}\exp(-n_1\lambda-n_2\eta\lambda)}{\prod_{i=1}^{n_1}x_i!\prod_{i=1}^{n_2}y_i!},$$

where $\lambda > 0$ and $\eta > 0$. The parameter η is the ratio of two Poisson means. Kang, Lee and Kim (2002) derived the reference prior (Berger and Bernardo, 1989, 1992) for η as follows:

$$\pi(\lambda, \eta) = \eta^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} (n_1 + n_2 \eta)^{-\frac{1}{2}}.$$

Also they showed that the propriety of posteriors for a general class of priors which include the reference prior and Jeffreys' prior, and the reference prior meet very well the target coverage probabilities. Note that the marginal posterior distribution of η under reference prior and Jeffreys' prior is the same, but in our Bayesian hypothesis testing problem, the results based on reference prior and Jeffreys' prior are different because of the constant in computation of the marginal density.

We want to test the hypotheses $H_1: \eta \le \eta_0$ vs. $H_2: \eta > \eta_0$. Our interest is to develop a Bayesian one-sided test based on the fractional and intrinsic Bayes factors for H_1 vs. H_2 under the reference priors.

3.1 Bayesian Test Procedure based on the Fractional Bayes Factor

Under the hypothesis H_1, the reference prior for λ and η is

$$\pi_1(\lambda, \eta) = \eta^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} (n_1 + n_2 \eta)^{-\frac{1}{2}} I[\eta \le \eta_0].$$

and under H_2, the reference prior for λ and η is

$$\pi_2(\lambda, \eta) = \eta^{-\frac{1}{2}} \lambda^{-\frac{1}{2}} (n_1 + n_2 \eta)^{-\frac{1}{2}} I[\eta > \eta_0].$$

where $I[\cdot]$ is the indicator function. Then the elements of B_{21}^N in fractional Bayes factor are given by

$$\begin{aligned} m_1^N(x, y) &= \int_0^{\eta_0} \int_0^{\infty} L(\eta, \lambda \mid x, y) \pi_1(\eta, \lambda) d\lambda d\eta \\ &= \frac{\Gamma(n_1 \bar{x} + n_2 \bar{y} + 0.5)}{\prod_{i=1}^{n_1} x_i! \prod_{i=1}^{n_2} y_i!} \int_0^{\eta_0} \eta^{n_2 \bar{y} - 0.5} (n_1 + n_2 \eta)^{-(n_1 \bar{x} + n_2 \bar{y} + 1)} d\eta \\ &= \frac{\Gamma(n_1 \bar{x} + n_2 \bar{y} + 0.5)}{\prod_{i=1}^{n_1} x_i! \prod_{i=1}^{n_2} y_i!} S_1(x, y) \end{aligned}$$

and

$$
\begin{aligned}
m_2^N(x, y) &= \int_{\eta_0}^{\infty}\int_0^{\infty} L(\eta,\lambda \mid x, y)\pi_2(\eta,\lambda)d\lambda d\eta \\
&= \frac{\Gamma(n_1\bar{x}+n_2\bar{y}+0.5)}{\prod_{i=1}^{n_1} x_i!\prod_{i=1}^{n_2} y_i!}\int_{\eta_0}^{\infty}\eta^{\,n_2\bar{y}-0.5}(n_1+n_2\eta)^{-(n_1\bar{x}+n_2\bar{y}+1)}d\eta. \\
&= \frac{\Gamma(n_1\bar{x}+n_2\bar{y}+0.5)}{\prod_{i=1}^{n_1} x_i!\prod_{i=1}^{n_2} y_i!} S_2(x, y),
\end{aligned}
$$

where $S_1(x, y)=\int_0^{\eta_0}\eta^{\,n_2\bar{y}-0.5}(n_1+n_2\eta)^{-(n_1\bar{x}+n_2\bar{y}+1)}d\eta$ and $S_2(x, y)=\int_{\eta_0}^{\infty}\eta^{\,n_2\bar{y}-0.5}(n_1+n_2\eta)^{-(n_1\bar{x}+n_2\bar{y}+1)}d\eta.$ Since the training sample for hypotheses H_1 and H_2 consists of two independent and identically distributed random variables, it seems natural in the fractional setting to formulate the correction term F_{12}^b with the fractions $1/n_1$ and $1/n_2$ of each likelihood. Thus the element of correction term F_{12}^b is given as follows.

$$
\begin{aligned}
m_1^b(x, y) &= \int_0^{\eta_0}\int_0^{\infty} L^{\frac{1}{n_1}}(\lambda \mid x)L^{\frac{1}{n_2}}(\eta,\lambda \mid y)\pi_1(\eta,\lambda)d\lambda d\eta \\
&= \frac{\Gamma(n_1\bar{x}+n_2\bar{y}+0.5)}{[\prod_{i=1}^{n_1} x_i!]^{1/n_1}[\prod_{i=1}^{n_2} y_i!]^{1/n_2}}\int_0^{\eta_0}\eta^{\,\bar{y}-0.5}(1+\eta)^{-(\bar{x}+\bar{y}+0.5)}(n_1+n_2\eta)^{-0.5}d\eta \\
&= \frac{\Gamma(n_1\bar{x}+n_2\bar{y}+0.5)}{[\prod_{i=1}^{n_1} x_i!]^{1/n_1}[\prod_{i=1}^{n_2} y_i!]^{1/n_2}} S_1(x, y; b)
\end{aligned}
$$

and

$$
\begin{aligned}
m_2^b(x, y) &= \int_{\eta_0}^{\infty}\int_0^{\infty} L^{\frac{1}{n_1}}(\lambda \mid x)L^{\frac{1}{n_2}}(\eta,\lambda \mid y)\pi_2(\eta,\lambda)d\lambda d\eta \\
&= \frac{\Gamma(n_1\bar{x}+n_2\bar{y}+0.5)}{[\prod_{i=1}^{n_1} x_i!]^{1/n_1}[\prod_{i=1}^{n_2} y_i!]^{1/n_2}}\int_{\eta_0}^{\infty}\eta^{\,\bar{y}-0.5}(1+\eta)^{-(\bar{x}+\bar{y}+0.5)}(n_1+n_2\eta)^{-0.5}d\eta. \\
&= \frac{\Gamma(n_1\bar{x}+n_2\bar{y}+0.5)}{[\prod_{i=1}^{n_1} x_i!]^{1/n_1}[\prod_{i=1}^{n_2} y_i!]^{1/n_2}} S_2(x, y; b),
\end{aligned}
$$

where $S_1(x, y; b)=\int_0^{\eta_0}\eta^{\,\bar{y}-0.5}(1+\eta)^{-(\bar{x}+\bar{y}+0.5)}(n_1+n_2\eta)^{-0.5}d\eta$ and $S_2(x, y; b)$

$= \int_{\eta_0}^{\infty} \eta^{\bar{y}-0.5}(1+\eta)^{-(\bar{x}+\bar{y}+0.5)}(n_1+n_2\eta)^{-0.5}d\eta$. Thus the fractional Bayes factor of H_2 versus H_1 is given by

$$B^F_{21} = \frac{S_2(x, y)}{S_1(x, y)} \cdot \frac{S_1(x, y; b)}{S_2(x, y; b)}.$$

Note that the calculation of fractional Bayes factor of H_2 versus H_1 requires an one dimensional integration.

3.2 Bayesian Test Procedure based on the Intrinsic Bayes Factor

The element B^N_{21} of the intrinsic Bayes factor is computed in the fractional Bayes factor. So using minimal training sample, we only calculate the marginal densities under H_1 and H_2, respectively. The marginal density of (X_i, Y_j) is finite for all $1 \le i \le n_1, 1 \le j \le n_2$ and hypotheses. Thus we conclude that any training sample of size two is a minimal training sample.

The marginal densities $m^N_1(x_i, y_j)$ under H_1 is given by

$$\begin{aligned} m^N_1(x_i, y_j) &= \int_0^{\eta_0}\int_0^{\infty} f(x_i \mid \lambda) f(y_j \mid \eta, \lambda)\pi_1(\eta, \lambda) d\lambda d\eta \\ &= \frac{\Gamma(x_i+y_j+0.5)}{x_i!\, y_j!}\int_0^{\eta_0} \eta^{y_j-0.5}(1+\eta)^{-x_i-y_j-0.5}(n_1+n_2\eta)^{-0.5}d\eta \\ &\equiv \frac{\Gamma(x_i+y_j+0.5)}{x_i!\, y_j!} T_1(x_i, y_j), \end{aligned}$$

and

$$\begin{aligned} m^N_2(x_i, y_j) &= \int_{\eta_0}^{\infty}\int_0^{\infty} f(x_i \mid \lambda) f(y_j \mid \eta, \lambda)\pi_2(\eta, \lambda) d\lambda d\eta \\ &= \frac{\Gamma(x_i+y_j+0.5)}{x_i!\, y_j!}\int_{\eta_0}^{\infty} \eta^{y_j-0.5}(1+\eta)^{-x_i-y_j-0.5}(n_1+n_2\eta)^{-0.5}d\eta \\ &\equiv \frac{\Gamma(x_i+y_j+0.5)}{x_i!\, y_j!} T_2(x_i, y_j), \end{aligned}$$

where $T_1(x_i, y_j) = \int_0^{\eta_0} \eta^{y_j-0.5}(1+\eta)^{-x_i-y_j-0.5}(n_1+n_2\eta)^{-0.5}d\eta$ and $T_2(x_i, y_j) = \int_{\eta_0}^{\infty} \eta^{y_j-0.5}(1+\eta)^{-x_i-y_j-0.5}(n_1+n_2\eta)^{-0.5}d\eta$. Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{S_2(x, y)}{S_1(x, y)} \cdot \left[\frac{1}{L} \sum_{i,j} \frac{T_1(x_i, y_j)}{T_2(x_i, y_j)} \right].$$

where $L = n_1 n_2$. And the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(x, y)}{S_1(x, y)} \cdot ME\left[\frac{T_1(x_i, y_j)}{T_2(x_i, y_j)} \right].$$

Note that the calculation of the AIBF and MIBF of H_2 versus H_1 requires an one dimensional integration.

However AIBF are often not suitable for non-nested situation, especially when one-sided hypotheses as here (see Dmochowski, 1996). An attractive alternative, given by Berger and Pericchi (1996) is to embed the competing models in a larger encompassing model H_0 so that all of the $H_i, i = 1, 2$ are nested within H_0. The encompassing arithmetic intrinsic Bayes factor (EIBF) is then defined as

$$B_{ji}^{EI} = B_{ji}^{N} \cdot \frac{\sum_{l=1}^{L} B_{i0}^{N}(x(l))}{\sum_{l=1}^{L} B_{j0}^{N}(x(l))},$$

where $B_{i0}^{N}(x(l)) = m_i^N(x(l)) / m_0^N(x(l))$. Therefore the marginal densities $m_0^N(x_i, y_j)$ under $H_0 (= H_1 \cup H_2): \eta > 0$ is given by

$$\begin{aligned} m_0^N(x_i, y_j) &= \int_0^\infty \int_0^\infty f(x_i \mid \lambda) f(y_j \mid \eta, \lambda) \pi_0(\eta, \lambda) d\lambda d\eta \\ &= \frac{\Gamma(x_i + y_j - 0.5)}{x_i! y_j!} \int_0^\infty \eta^{y_j - 0.5} (1+\eta)^{-x_i - y_j - 0.5} (n_1 + n_2 \eta)^{-0.5} d\eta \\ &= \frac{\Gamma(x_i + y_j + 0.5)}{x_i! y_j!} T_0(x_i, y_j), \end{aligned}$$

where $T_0(x_i, y_j) = \int_0^\infty \eta^{y_j - 0.5}(1+\eta)^{-x_i - y_j - 0.5}(n_1 + n_2\eta)^{-0.5} d\eta$. Therefore the EIBF of H_2 versus H_1 is given by

$$B_{21}^{EI} = \frac{S_2(x, y)}{S_1(x, y)} \cdot \left(\frac{\sum_{i,j} T_1(x_i, y_j) / T_0(x_i, y_j)}{\sum_{i,j} T_2(x_i, y_j) / T_0(x_i, y_j)} \right).$$

Note that the calculation of MIBF and EIBF of H_2 versus H_1 requires an one dimensional integration.

4. Numerical Studies

In this section, we compare the proposed Bayesian test procedures with two frequentist methods. The frequentist methods are as follow.

The conditional test (C-test) due to Prizborowski and Wilenski (1940) is based on the conditional distribution of X_1 given $X_1+X_2=k$, where $X_1=\sum_{i=1}^{n_1} X_{1i}$ and $X_2=\sum_{i=1}^{n_2} X_{2i}$ are the Poisson distributions with means $n_1\lambda_1$ and $n_2\lambda_2$, respectively. Let k_1 and k_2 be the observed values of X_1 and X_2. Note that the distribution of X_1 conditionally given $X_1+X_2=k$ is binomial with the number of trials k and success probability

$$p(\lambda_1/\lambda_2)=(n_1/n_2)(\lambda_1/\lambda_2)/[1+(n_1/n_2)(\lambda_1/\lambda_2)]. \quad (8)$$

This conditional test rejects $H_0 : \lambda_1/\lambda_2 \le c$, whenever the p-values

$$p(X_1 \ge k_1 \mid k, p(c))=\sum_{i=k_1}^{k}\binom{k}{i}[p(c)]^i[1-p(c)]^{k-i} \le \alpha,$$

where $p(c)$ is the expression in (8) with λ_1/λ_2 replaced by c.

The unconditional test (E-test) by Krishnamoorthy and Thomson (2004) is obtained by suitably modifying the binomial test due to Storer and Kim (1990). Since the unconditional test is essentially based on the estimated p-values of the standardized difference. The test rejects $H_0 : \lambda_1-\lambda_2 \le d$, whenever the p-values

$$\sum_{x_1=0}^{\infty}\sum_{x_2=0}^{\infty}\frac{e^{-n_1(\hat{\lambda}_{2k}+d)}[n_1(\hat{\lambda}_{2k}+d)]^{x_1}}{x_1!}\frac{e^{-n_2\hat{\lambda}_{2k}}(n_2\hat{\lambda}_{2k})^{x_2}}{x_2!}I[T_{x_1,x_2} \ge T_{k_1,k_2}] \le \alpha,$$

where $I[\cdot]$ denotes the indicator function,

$$\hat{\lambda}_{2k}=\frac{k_1+k_2}{n_1+n_2}-\frac{dn_1}{n_1+n_2}, \quad \hat{V}_X=\frac{X_1/n_1}{n_1}+\frac{X_2/n_2}{n_2},$$
$$T_{X_1,X_2}=\frac{X_1/n_1-X_2/n_2-d}{\sqrt{\hat{V}_X}}, \quad T_{k_1,k_2}=\frac{k_1/n_1-k_2/n_2-d}{\sqrt{\hat{V}_k}}.$$

For our purpose, we set $\eta_0=1$ in Bayesian hypotheses testing, then $c=1$ and $d=0$ in frequentist tests, respectively.

Example 1. This example is given in Snedecor and Cochran (1980). An experiment designed to investigate various treatments for the control of cabbage loopers. Table 1 records the number of loopers on 50 cabbage plants per plot after the application of five treatments to each of four plots.

The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. The p-value, value of the Bayes Factor and the posterior probability are given in Table 2. From Table 2, the frequentist tests are very conservative whereas the Bayes factors are not. That is, the frequentist tests are biased to H_1. Also with plot means, the fractional Bayes factor and the intrinsic Bayes factors give fairly reasonable answers.

<Table 1> Number of Loopers on 50 Cabbage Plants in a Plot

Treatment	Loopers per Plot	Plot Mean
1	11, 4, 4, 5	6.00
2	6, 4, 3, 6	4.75
3	8, 6, 4, 11	7.25
4	7, 4, 9, 14	8.50

<Table 2> p-value, Bayes Factor and Posterior Probability

Treatment	p-value :C-Test	p-value :E-Test	B^F_{21}	B^{MI}_{21}	B^{EI}_{21}	$P^F_{H_1}$	$P^{MI}_{H_1}$	$P^{EI}_{H_1}$
2 vs. 1	0.271	0.227	1.836	1.984	2.555	0.353	0.335	0.281
2 vs. 3	0.097	0.075	3.234	3.438	5.365	0.236	0.225	0.157
2 vs. 4	0.027	0.020	5.599	8.671	17.267	0.152	0.103	0.055

Example 2. To show that the frequentist tests are more biased than the Bayesian testing procedures, we examine the cases when $(n_1, n_2)=(5,5)$, $(k_1, k_2)=(20,21),(20,22),(20,30)$ and $(n_1, n_2)=(5,10)$, $(k_1, k_2)=(20,42),(20,44),(20,60)$.

<Table 3> p-value, Fractional Bayes Factor and Posterior Probability

(n_1, n_2)	(k_1, k_2)	p-value :C-Test	p-value :E-Test	B^F_{21}	$P^F_{H_1}$
5, 5	20, 21	0.500	0.444	1.116	0.473
	20, 22	0.439	0.385	1.243	0.446
	20, 30	0.101	0.080	2.775	0.265
5, 10	20, 42	0.488	0.438	1.024	0.494
	20, 44	0.418	0.370	1.142	0.467
	20, 60	0.069	0.055	2.560	0.281

For not so clear data, the C-test and E-test choose the hypothesis H_1 whereas the fractional Bayes factor chooses the hypothesis H_2. Therefore from this result, frequentist tests are biased to H_1. However the fractional Bayes factor give fairly reasonable answers for all cases.

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