Analytic Method on Fuzzy Goal Programming Problem

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Abstract

We propose a simple new analytic method for solving a fuzzy goal programming (FGP) problem with general membership functions of fuzzy goals and re-examine a previously defined method for dealing with fuzzy weights for each of the goals. Several illustrative examples are given.

Keywords : Decision analysis, Fuzzy goal programming

1. Introduction

In most of the real world situations the articulation of the goals and objectives of the decision maker are fuzzy in nature. The application of fuzzy set theory to goal programming has been made by Narasimhan(1980), Hannan(1981) and Tiwari et al.(1986). Narasimhan first considered a FGP problem with multiple goals having equal weights and unequal fuzzy weights associated with them. Hannan(1981) illustrated how the goal programing problem with fuzzy goals having linear membership functions may be formulated as a single goal programming problem and re-examined a previously defined method for dealing with fuzzy weights for each of the goal. Tiwari et al.(1986) introduced priority structure in FGP which utilize the lexicographic order of goal program and yields an efficient computational algorithm for solving FGP. Chen(1994) proposed a new algorithm for solving a FGP problem with symmetrically triangular membership functions of fuzzy goals and priority structure. In this paper, we give a new simple method solving FGP with analytic approach. Some illustrative examples are given.

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2. Fuzzy goal programming (FGP)

Consider the following fuzzy goals where the symbol $\cong$ is a "fuzzifier" representing the imprecision in the stated goals and $(AX)_i$ represents the ith equation of AX, and B_i is the ith component of the right-hand-side column vector b. Let μ_i be the membership function of $(AX)_i$ satisfying $\mu_i(b_i) = 1$ and $0 \leq \mu_i(AX) \leq 1$. We define the membership function of the decision set D, $\mu_D(x)$, as

$$\mu_D(x) = \min\{\mu_1(AX), \mu_2(AX), \cdots, \mu_m(AX)\} \tag{1}$$
$$= \text{Min }_i \mu_i(AX)$$

The FGP problem is to find $x \in X$ that maximize $\mu_D(x)$, i.e., the maximizing decision is given by

$$\text{Max }_x \mu_D(x) = \text{Max }_x \text{ Min }_i \mu_i(AX) \tag{2}$$

3. Analytic solution approach

The FGP formulation represented equation (1) may be rewritten in general as follows:

$$
\begin{array}{lll}
G_1 & : & x_1 \cong b_1 \\
G_2 & : & x_2 \cong b_2 \\
\vdots & & \\
G_n & : & x_n \cong b_n \\
G_{n+1} & : & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \cong b_{n+1} \\
G_{n+2} & : & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \cong b_{n+2} \\
\vdots & & \\
G_{n+k} & : & a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \cong b_{n+k} \\
& & x_j \geq 0, \quad j = 1, 2, \cdots, n, \quad n+k=m
\end{array}
\tag{3}
$$

We construct the membership functions as follows :

$$
\mu_i(AX) = \begin{cases}
1 & \text{if } (AX)_i = b_i \\
0 & \text{if } (AX)_i \leq b_i - \triangle_i^L = b_{iL} \\
\mu_{iL}((AX)_i) & \text{if } b_{iL} \leq (AX)_i \leq b_i \\
\mu_{iR}((AX)_i) & \text{if } b_i \leq (AX)_i \leq b_i + \triangle_i^R = b_{iG} \\
0 & \text{if } b_{iG} \leq (AX)_i
\end{cases}
$$

where $\mu_{iL}: [b_{iL}, b_i] \to [0,1]$ is continuous and strictly increasing satisfying $\mu_{iL}(b_{iL}) = 0$, $\mu_{iL}(b_{iL}) = 1$ and $\mu_{iR}: [b_i, b_{iR}] \to [0,1]$ is continuous and strictly decreasing satisfying $\mu_{iR}(b_i) = 1$, $\mu_{iR}(b_{iG}) = 0$. When the membership function is linear,

$$\mu_{iL}((AX)_i) = \frac{(AX)_i - b_{iL}}{\triangle_i^L} \text{ and } \mu_{iR}((AX)_i) = \frac{b_{iR} - (AX)_i}{\triangle_i^R}$$

In this case, we denote $\mu_i = (b_i, \triangle_i^L, \triangle_i^R)$. If x_i $i = 1, 2, \cdots, n$ are fuzzy sets, then $a_{l1}x_1 + a_{l2}x_2 + \cdots + a_{ln}x_n$ is a fuzzy set. Let μ_{n+l}^*, $l = 1, 2, \cdots, k$, be the membership function of $a_{l1}x_1 + a_{l2}x_2 + \cdots + a_{ln}x_n$. It is well-known(Zimmeremann(1991)) that if $x_i = (b_i, \triangle_i^L, \triangle_i^R)$, $i = 1, 2, \cdots, n$ then

$$a_{l1}x_1 + a_{l2}x_2 + \cdots + a_{ln}x_n = \left(\sum_{i=1}^n a_{li}b_i, \sum_{i=1}^n a_{li}\triangle_i^L, \sum_{i=1}^n a_{li}\triangle_i^R \right) \tag{4}$$

for $l = 1, 2, \cdots, k$. In general, using α-cut interval arithmetic operation, for $0 \le \alpha \le 1$

$$\{\mu_{n+l}^*(AX) \ge \alpha\} = \sum_{i=1}^n a_{li}\{\mu_i(x_i) \ge \alpha\}, \quad l = 1, 2, \cdots, k \tag{5}$$

where μ_i is the membership function of X_i, $i = 1, 2, \cdots, n$. Now, considering equation (3), it is equivalent to the followings :

$$\text{Max}\lambda$$
$$\text{s.t.} \quad \bigcap_{i=1}^m \{\mu_i(AX) \ge \lambda\} \ne \varnothing \tag{6}$$

And we note that

$$\bigcap_{i=1}^m \{\mu_i(AX) \ge \lambda\} \ne \varnothing$$
$$\Leftrightarrow \left[\bigcap_{i=1}^n \{\mu_i(AX) \ge \lambda\} \right] \cap \left[\bigcap_{l=1}^k [\{\mu_{n+l}(AX) \ge \lambda\} \cap \{\mu_{n+l}^*(AX) \ge \lambda\}] \right] \ne \varnothing$$
$$\Leftrightarrow \bigcap_{l=1}^k [\{\mu_{n+l}(AX) \ge \lambda\} \cap \{\mu_{n+l}^*(AX) \ge \lambda\}] \ne \varnothing$$

since μ_{n+1}^* contains all information about μ_i, $i = 1, 2, \cdots, n$ as in (5) and (6).

We assume that there exists $\lambda > 0$ satisfying (7).

Let λ_{n+l}, $n = 1, 2, \cdots, k$ be the solution of the following equations (see Fig.1)

$$\begin{cases} \mu_{(n+l)R}^{-1}(\lambda_{n+l}) = \mu^{*}_{(n+l)L}{}^{-1}(\lambda_{n+l}) & \text{if } b_{n+1} \leq \sum_{i=1}^{n} a_{li} b_i \\ \mu_{(n+l)L}^{-1}(\lambda_{n+l}) = \mu^{*}_{(n+l)R}{}^{-1}(\lambda_{n+l}) & \text{if } b_{n+1} > \sum_{i=1}^{n} a_{li} b_i \end{cases} \tag{7}$$
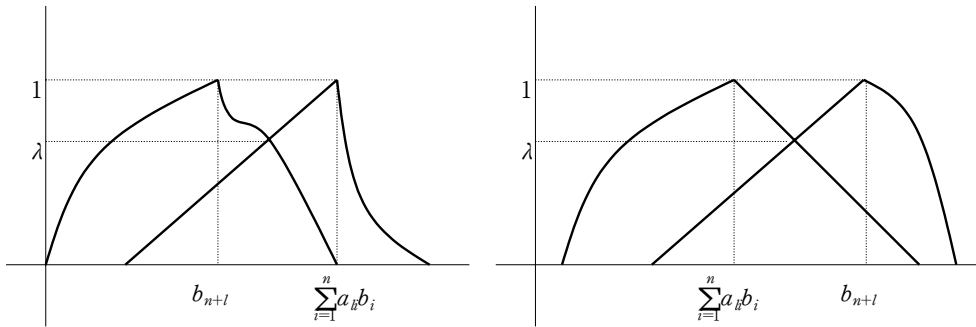
<Figure 1> Solution of the equations (7)

Then, equation (7) is equivalent to

$$\lambda = \text{Min}_{i=1,2,\cdots,k} \lambda_{n+l}. \tag{8}$$

And, in this case, if $\text{Min}_{l=1,2,\cdots,k} \lambda_{n+l} = \lambda_{n+l_0} \equiv \lambda$, then

$$x_i = \begin{cases} \mu_{iL}^{-1}(\lambda) & \text{if } b_{n+l_0} \leq \sum_{i=1}^{n} a_{l_0 i} b_i \\ \mu_{iR}^{-1}(\lambda) & \text{otherwise}. \end{cases} \tag{9}$$
$$i = 1, 2, \cdots, n$$

When μ_i, $l \leq i \leq n$, is linear then (9) is

$$x_i = \begin{cases} b_i - (1-\lambda) \triangle_{iL} & \text{if } b_{n+l_0} \leq \sum_{i=1}^{n} a_{l_0 i} b_i \\ b_i - (1-\lambda) \triangle_{iR} & \text{otherwise}. \end{cases} ,$$

4. Numerical examples

We first consider the same example as given by Narasimhan(1980).

Example 1. Find the solution as close as possible for the following goals:

$$
\begin{aligned}
G_1 &: & x_1 &= (6, 2, 2) = \mu_1 \\
G_2 &: & x_2 &= (4, 2, 2) = \mu_2 \\
G_3 &: & 80x_1 + 40x_2 &= (630, 10, 10) = \mu_3
\end{aligned}
$$

By (4), we have $n = 2$, $k = 1$ and by (5), $\mu_3^* = ((80)(6) + (40)(4), (80)(2) + (40)(2), (80)(2) + (40)(2)) = (640, 240, 240)$. Now $\mu_{3R}^{-1}(\lambda) = 640 - 10\lambda$, $\mu_{3L}^{*-1}(\lambda) = 400 + 240\lambda$, and hence $\lambda = \lambda_3 = \dfrac{24}{25} = 0.96$ by (8, 9). And, Since $\mu_{1L}^{-1}(\lambda) = 4 + 2\lambda$ and $\mu_{2L}^{-1}(\lambda) = 2 + 2\lambda$, $x_1 = \mu_{1L}^{-1}(0.96) = 5.92$, $x_2 = \mu_{2L}^{-1}(0.96) = 3.92$ which is identical to the solution provided in Narasimhan(1980).

Example 2. Find the solution as close as possible for the following goals:

$$
\begin{aligned}
G_1 &: & x_1 &= (6, 2, 2) = \mu_1 \\
G_2 &: & x_2 &= (4, 2, 2) = \mu_2 \\
G_3 &: & 80x_1 + 40x_2 &= (630, 10, 10) = \mu_3 \\
G_4 &: & 50x_1 + 60x_2 &= (600, 20, 10) = \mu_4 \\
G_5 &: & 60x_1 + 60x_2 &= (600, 50, 70) = \mu_5
\end{aligned}
$$

We have $n = 2$, $k = 3$ by (4) and $\mu_3^* = (640, 240, 240)$, $\mu_4^* = (540, 220, 220)$, $\mu_5^* = (600, 240, 240)$ by (5). Now, since

$$
\begin{aligned}
\mu_{3R}^{-1}(\lambda) &= 640 - 10\lambda, & \mu_{3L}^{*-1}(\lambda) &= 400 + 240\lambda \\
\mu_{4R}^{-1}(\lambda) &= 580 + 20\lambda, & \mu_{4L}^{*-1}(\lambda) &= 760 - 220\lambda \\
\mu_{5R}^{-1}(\lambda) &= 670 - 70\lambda, & \mu_{5L}^{*-1}(\lambda) &= 360 + 240\lambda,
\end{aligned}
$$

$\lambda_3 = 0.96$, $\lambda_4 = 0.75$, and $\lambda_5 = 1$ by (8).

Hence $\lambda = \min\{0.96, 0.75, 1\} = 0.75$ and $x_1 = \mu_{1R}^{-1}(0.75) = 6.5$, $x_2 = \mu_{2R}^{-1}(0.75) = 4.5$.

Example 3. Find the solution as close as possible for the following goals:

$$
\begin{array}{lll}
G_1 & : & x_1 \cong 6 = \mu_1 \\
G_2 & : & x_2 \cong 4 = \mu_2 \\
G_3 & : & 80x_1 + 40x_2 \cong 630 = \mu_3
\end{array}
$$

where

$$
\mu_1(x_1) = \begin{cases}
0 & \text{if } x_1 \leq 4, \\
\left(\dfrac{x_1 - 4}{2}\right)^2 & \text{if } 4 \leq x_1 \leq 6, \\
\left(\dfrac{8 - x_1}{2}\right)^2 & \text{if } 6 \leq x_1 \leq 8, \\
0 & \text{otherwise},
\end{cases}
$$

$$
\mu_2(x_2) = \begin{cases}
0 & \text{if } x_2 \leq 2, \\
\left(\dfrac{x_2 - 2}{2}\right)^2 & \text{if } 2 \leq x_2 \leq 4, \\
\left(\dfrac{6 - x_2}{2}\right)^2 & \text{if } 4 \leq x_2 \leq 6, \\
0 & \text{otherwise},
\end{cases}
$$

and

$$
\mu_3(80x_1 + 40x_2) = \begin{cases}
0 & \text{if } 80x_1 + 40x_2 \leq 620, \\
\left(\dfrac{(80x_1 + 40x_2) - 620}{10}\right)^{-\frac{1}{2}} & \text{if } 620 \leq 80x_1 + 40x_2 \leq 630, \\
\left(\dfrac{640 - (80x_1 + 40x_2)}{10}\right)^{-\frac{1}{2}} & \text{if } 630 \leq 80x_1 + 40x_2 \leq 640, \\
0 & \text{if } 80x_1 + 40x_2 \geq 640.
\end{cases}
$$

Since the membership functions of x_1 and x_2 are of same type, we easily have that (see Zimmeremann(1991))

$$\mu_3^*(80x_1 + 40x_2) = \begin{cases} 0 & \text{if } 80x_1 + 40x_2 \leq 400, \\ \left(\dfrac{(80x_1 + 40x_2) - 400}{240}\right)^2 & \text{if } 400 \leq 80x_1 + 40x_2 \leq 640, \\ \left(\dfrac{880 - (80x_1 + 40x_2)}{240}\right)^2 & \text{if } 640 \leq 80x_1 + 40x_2 \leq 880, \\ 0 & \text{if } 80x_1 + 40x_2 \geq 880, \end{cases}$$

and hence $\mu_{3L}^{*-1}(\lambda) = 400 + 240\sqrt{\lambda}$. We also know that $\mu_{3R}^{-1}(\lambda) = 640 - 10\lambda^2$. Hence by (8), $\lambda = 0.9293$ and by (9) $x_1 = \mu_{1L}^{-1}(0.9293) = 5.7272$, $x_2 = \mu_{2L}^{-1}(0.9293) = 3.7272$

5. FGP problems with fuzzy weights

The extension of the FGP problem to a problem containing fuzzy priorities as well as fuzzy goals entails the introduction of linguistic variable such as "important", "not very important", and "very important". Narasimhan defines $\mu_{W_i}(\mu_i(AX))$ as the weighted contribution of the ith goal to the overall objective, where μ_W represents the membership function corresponding to the fuzzy priority associated with the ith goal.

The maximizing decision is given by Narasimhan(1980)

$$\text{Max}_{X \geq 0} \mu_D(x) = \text{Max}_{x \geq 0} \text{Min}_i \mu_i(\mu_i(AX)).$$

Thus, the weighted attributed to a given goal is dependent upon the degree to which the goal has been achieved.

Example 1 is extended by Narasimhan after defining the following membership functions for the fuzzy priority :

$$\mu_{W_1}(\mu_1(AX)) = \begin{cases} \dfrac{\mu_1(AX) - 0.8}{0.2} & \text{if } 0.8 \leq \mu_1(AX) \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{W_2}(\mu_2(AX)) = \begin{cases} \dfrac{\mu_2(AX) - 0.6}{0.2} & \text{if } 0.6 \leq \mu_2(AX) \leq 0.8 \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_{W_3}(\mu_3(AX)) = \begin{cases} \dfrac{\mu_3(AX) - 0.6}{0.2} & \text{if } 0.6 \leq \mu_3(AX) \leq 0.8 \\ 0 & \text{otherwise} \end{cases}$$

Let the following linguistic variables characterize in a fuzzy sense the importance of the goals in Example 1 :

1   G_1 :   "more or less important"

2   G_2 :   "more or less important"

3   G_3 :   "very important".
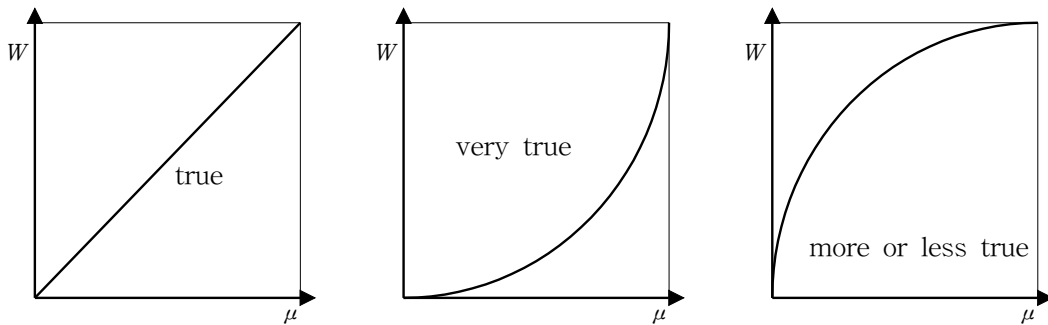
<Figure 2> Linguistic variables

We now consider the following new types of membership function for the fuzzy priorities. (see Fig. 2) :

$$\mu_{W_1}(\mu_1(AX)) = \mu_1(AX)^{\frac{1}{2}},$$

$$\mu_{W_2}(\mu_2(AX)) = \mu_2(AX)^{\frac{1}{2}},$$

$$\mu_{W_3}(\mu_3(AX)) = (\mu_3(AX))^2.$$

Using these membership functions for the weights with the same method as in Example 3, the optimal solutions is :

$$x_1 = 5.9177 \quad x_2 = 3.9177 \quad x_3 = 0.9792$$

If we compare the solutions for the two cases, in the second case where the third goal is more important than the other goals, the optimal total profit is 630.112 whereas in Example 1, where all three goals are equally important, the optimal total profit is 630.40.

The optimal values of $x_1 = 5.92$ and $x_2 = 3.92$ are a little closer to their individual goals in Example 1 as compared to the second case where $x_1 = 5.9177$, $x_2 = 3.9177$.

This is consistent with the fuzzy priorities associated with the individual goals in the second case.

5. Conclusion

We have introduced a new analytic method for solving a FGP and considered a new types of membership functions for the fuzzy priorities. The   main advantages of our method is easy and simple. Solving FGP problem, we just prove a few equations and compare them. We don't need computer works.

References

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