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A Note on Maximal Entropy OWA Operator Weights

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Abstract

In this note, we give an elementary simple proof of the main result of Fullér and Majlender [Fuzzy Sets and systems 124(2001) 53-57] concerning obtaining maximal entropy OWA operator weights.

Keywords : Degree of orness, Dispersion, OWA operator

1. Introduction

Yager(1988) introduced a new aggregation technique based on the ordered weighted averaging(OWA) operators. An OWA operator of dimension n is a mapping $F: R^n \to R$ that has an associated weighting vector $W = (w_1, \cdots, w_n)^T$ of having the properties $w_1 + \cdots + w_n = 1,\ 0 \le w_i \le 1,\ i = 1, \cdots, n,$ and such that

$$F(a_1, \cdots, a_n) = \sum_{i=1}^{n} w_i b_i,$$

where b_j is the jth largest element of the collection of the aggregated objects $\{a_1, \cdots, a_n\}$.

Yager(1988) introduced a measure of "orness" associated with the weighting vector W of an OWA operator, defined as

$$orness(W) = \sum_{i=1}^{n} \frac{n-1}{n-i} w_i,$$

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and it characterizes the degree to which the aggregation is like an *or* operation. One important issue in the theory of OWA operators is the determination of the associated weights. He also introduced the measure of "dispersion" of the aggregation, defined as

$$disp(W) = -\sum_{i=1}^{n} w_i \ln w_i,$$

and it measure the degree to which W takes into account all information in the aggregation.

A number of approaches have been suggested for obtaining the associated weights, i.e., quantifier guided aggregation(1988, 1993), exponential smoothing(1988) and learning(1999). Another approaches, suggested by O'Hagan(1988), determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of orness. This approach is based on solving the following mathematical program problem:

$$\begin{aligned} &\text{maximize}: \ disp(W) = -\frac{1}{n}\sum_{i=1}^{n} w_i \ln w_i. \\ &\text{subject to}: orness(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \ 0 \le \alpha \le 1, \\ &\qquad w_1 + \cdots + w_n = 1, 0 \le w_i, i = 1, \cdots, n, \end{aligned} \tag{1}$$

Recently Fullér and Majlender(2003) solved problem (1) analytically and derived the exact maximal entropy OWA weights for any level of orness using the method of Lagrange multipliers. In this note, we give an elementary simple new method deriving the exact maximal entropy OWA weights for the problem (1).

2. New method obtaining maximal entropy OWA weights

We first consider the following weight $ln\ w_i^* = ai + b$ (equivalently $w_i^* = (e^a)^i e^b$) for some a and b satisfying (1). We first observe that

$$\sum w_i^* = e^b(e^a + (e^a)^2 + \cdots (e^a)^n) = e^b e^a \frac{(1-e^{na})}{1-e^a} = 1, \ 0 \le w_i^*, \ i = 1, \cdots, n. \tag{2}$$

From (2), we have that

$$\sum i w_i^* = e^b(e^a + 2(e^a)^2 + \cdots n(e^a)^{n)} = e^b\left(\frac{e^a(1-(e^a)^2)}{(1-e^a)^2} - \frac{n(e^a)^{n+1}}{1-e^a}\right)$$
$$= \frac{1}{1-e^a} - \frac{\neq^{na}}{1-e^{na}} = n-(n-1)\alpha \;\; (\leftrightarrow \sum_{i=1}^{n} \frac{n-i}{n-1} w_i^* = \alpha). \tag{3}$$

If $a = 0(\leftrightarrow e^a = 1)$, then clearly $w_i^* = 1/n$ for $i = 1 \cdots n$. Suppose that $e^a \neq 1$. From (3) , we have

$$(n-1)\alpha e^{(n+1)a} - [(n-1)\alpha + 1]e^{na} + [n-(n-1)\alpha]e^a + (1-n) + (n-1)\alpha = 0$$

We put $e^a = x > 0$ and

$$f(x) = (n-1)\alpha x^{n+1} - [(n-1)\alpha + 1]x^n + [n-(n-1)\alpha]x + (1-n) + (n-1)\alpha. \tag{4}$$

Then we have

$$f'(x) = (n+1)(n-1)\alpha x^n - n[(n-1)\alpha + 1]x^{n-1} + [n-(n-1)\alpha]$$

and

$$f''(x) = n(n-1)x^{n-2}[(n+1)\alpha x - ((n-1)\alpha + 1)].$$

And hence we have that

$$f(0) = (1-n) - (1-n)\alpha < 0,\; f(1) = 0, \tag{5}$$

$$f'(x) \geq 0 \text{ for all } x \geq 1 \text{ and } \alpha > 1/2,\; f'(1) = 0, \tag{6}$$

$$f''(x) = 0 \text{ only for } x^* = \frac{(n-1)\alpha + 1}{(n+1)\alpha}$$
$$f''(x) < 0 \text{ for } x \in (0, x^*) \text{ and } f''(x) > 0 \text{ for } x \in (x^*, 1). \tag{7}$$

Without loss of generality, we can assume that $\alpha \geq 0.5$, because if a weighting vector W is optimal for problem (1) under given degree of orness $\alpha \geq 0.5$, then its reverse, denoted by W^R, and defined as $w_i^R = w_{n-i+1}$ is also optimal for problem (1) under degree of orness $(1-\alpha)$. We note in (7) that if $\alpha > 0.5$, then $x^* < 1$ and if $\alpha = 0.5$, then $x^* = 1$. Then we can see the shape of the graph of f from (5), (6) and (7). There exist unique solution of $f(x) = 0$ between 0 and 1. Then from (2), we can obtain unique $w_i^* = (e^a)^i e^b$, $i = 1, \cdots, n,$ for each $\alpha \geq 0.5$.

We now show that W^* is optimal.

Theorem. The optimal weight for the constrained optimization problem (1) should satisfy the equations $\ln w_i^* = ai + b$ (equivalently $w_i^* = (e^a)^i e^b$), $i = 1 \cdots n$ for some a and b.

Proof. Let W^* and W satisfy (1). We first note that $\sum w_i^* \ln w_i^* = \sum w_i^*$

$(ai+b) = a\sum i w_i^* + b\sum w_i^* = a\sum i w_i + b\sum w_i = \sum w_i(ai+b) = \sum w_i \ln w_i^*$.

Then we have,

$$\sum w_i^* \ln w_i^* - \sum w_i \ln w_i$$

$$= \sum w_i \ln w_i^* - \sum w_i \ln w_i$$

$$= \sum w_i (ln \frac{w_i^*}{w_i})$$

$$\leq \sum w_{i(} \frac{w_i^*}{w_i} - 1)\ (\text{since } \ln x \leq x - 1)$$

$$= \sum w_i^* - \sum w_i$$

$$= 1 - 1 = 0,$$

which completes the proof.

3. Numerical example

Let us suppose that $n = 5$ and $\alpha = 0.6$. Then from the equation (4), we have that $x = e^a = 1.2258$, $y = e^b = 0.1042$. And hence we find that $w_1^* = xy = 0.1278$, $w_2^* = x^2 y = 0.1566$, $w_3^* = x^3 y = 0.1920$, $w_4^* = x^4 y = 0.2353$, $w_5^* = x^5 y = 0.2884$ and , $disp(W^*) = 1.5692$.

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