Journal of the Korean
Data & Information Science Society
2006, Vol. 17, No. 2, pp. 521 ~ 527

Bayesian Testing for Independence in Bivariate Exponential Model[1]

Jang Sik Cho[2]

Abstract

In this paper, we consider the Bayesian hypotheses testing for independence in bivariate exponential model. In Bayesian testing problem, we use the noninformative priors for parameters which are improper and are defined only up to arbitrary constants. And we use the recently proposed hypotheses testing criterion called the fractional Bayes factor. Also we give some numerical results to illustrate our results.

Keywords : Bayesian test, Bivariate exponential model, Fractional bayes factor, Independence, Posterior

1. Introduction

In many cases of life testing, two components are assumed to have independent life time distributions. However it is more realistic to assume some form of positive dependence among components, in many life testing situations. Let's consider a system which functions only as long as at least one of two identical or very similar components functions. Initially let the two components be independently on test with life distributions that are exponential with parameters λ. Failure of one changes the life distribution of the other to exponential with parameter $\lambda\zeta > 0$, where $\zeta = 1$ implies the independence of the two components lives. For $\zeta > 1$ the workload of the remaining component is increased, thereby decreasing the mean life. We call ζ as dependence parameter. In this case, Weier(1981) obtained Bayes estimators of parameters and reliability using conjugate prior.

In this paper, we consider the Bayesian testing for independence in above

---

1) 이 연구는 2005학년도 경성대학교 연구년 과제로 수행된 것입니다.
2) 608-736 부산광역시 남구 대연동 110-1번지, 경성대학교 정보통계학과 부교수,
E-mail : jscho@ks.ac.kr.

bivariate exponential model by using Bayes factor. But the Bayes factor depend rather strongly on the prior distributions. So subjective elicitation can easily result in poor prior distribution and statistical analysis is often required to appear objective. So, the literature on noninformative priors has grown enormously over recent years. There have been several excellent books or review articles that have been concerned with discussing or comparing different approaches to developing noninformative priors(See Ghosh and Mukerjee, 1992).

However, noninformative priors are typically improper so that such priors are defined only up to arbitrary constants which affects the values of Bayes factors. So, San Martini and Spezzaferri (1984) have made efforts to compensate for that arbitrariness.

Berger and Pericchi (1996b) introduced a new model selection and hypotheses testing criterion, called the Intrinsic Bayes Factors (IBF's) using a data-splitting idea, which would eliminate the arbitrariness of improper priors. This approach has shown to be quite useful (Berger and Pericchi (1996a) and Lingham and Sivaganesan (1997)). O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so called the fraction b.

In this paper, we consider a Bayesian approach to test independence in bivariate exponential model. For dealing Bayesian testing for independence, we use noninformative priors as improper priors. Also we derive fractional Bayes factor to solve our problem and give some numerical results to illustrate our results.

2. Preliminaries

The random variables (X, Y) are said to follow bivariate exponential model with parameters (λ, ζ) if the joint probability density function is given as

$$f(x, y : \lambda, \zeta) = 2\zeta\lambda^2 \exp(-2\lambda x - \lambda\zeta y), \quad x, y > 0, \ \lambda, \zeta > 0 \tag{2.1}$$

Suppose that there are n two components units under study and ith pair of the components have lifetime (x_i, y_i). For a random sample of size n, the likelihood function is

$$f(\mathbf{x}, \mathbf{y} \mid \lambda, \zeta) = 2^n \zeta^n \lambda^{2n} \exp(-2\lambda \sum_{i=1}^n x_i - \lambda\zeta \sum_{i=1}^n y_i). \tag{2.2}$$

The MLE's for (λ, ζ) are given as

$$\hat{\lambda} = \frac{n}{2\sum_{i=1}^n x_i} \quad \text{and} \quad \hat{\zeta} = \frac{2\sum_{i=1}^n x_i}{\sum_{i=1}^n y_i}. \tag{2.3}$$

Now, we introduce Bayes factor in the general hypotheses testing. As a matter of convenience, we introduce the following notations.

$(X, Y) = ((X_1, Y_1), \cdots, (X_n, Y_n))$: observation with density $f(x, y \mid \theta)$, where $\theta \in \Theta$ is
a finite dimensional parameter and Θ is parameter space.

Θ_i : parameter space under ith hypothesis H_i , $i = 1, 2, \cdots, q$.

$f_i(x, y \mid \theta_i)$: the density under H_i , $i = 1, 2, \cdots, q$.

$\pi_i(\theta_i)$: the prior distribution under H_i , $i = 1, 2, \cdots, q$.

$m_i(x, y)$: the marginal density of (X, Y) under H_i when use $\pi_i(\theta_i)$, $i = 1, 2, \cdots, q$.

p_i : the prior probability of H_i being true, $i = 1, 2, \cdots, q$.

$\pi_i^N(\theta_i)$: the improper prior distribution under H_i , $i = 1, 2, \cdots, q$.

$m_i^N(x, y)$: the marginal density of (X, Y) under H_i when use $\pi_i^N(\theta_i)$, $i = 1, 2, \cdots, q$.

Then $\pi_i^N(\theta_i)$ is usually written as $\pi_i^N(\theta_i) \propto h_i(\theta_i)$, where h_i is a function whose integral over the Θ_i-space diverges. Formally, we can write $\pi_i^N(\theta_i) = c_i\, h_i(\theta_i)$, although the normalizing constant c_i does not exist, but treating it as an unspecified constant.

The posterior probability that H_i is true is given as

$$P(H_i \mid x, y) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} B_{ji} \right)^{-1}, \tag{2.4}$$

where B_{ji}, the Bayes factor of H_j to H_i, is defined by

$$B_{ji} = \frac{m_j(x, y)}{m_i(x, y)} = \frac{\int_{\Theta_j} f(x, y \mid \theta_j)\pi_j(\theta_j)d\theta_j}{\int_{\Theta_i} f(x, y \mid \theta_i)\pi_i(\theta_i)d\theta_i}. \tag{2.5}$$

The posterior probabilities in (2.4) are then used to select the most plausible hypothesis.

If one were to use some noninformative priors, then (2.5) becomes

$$B_{ji}^N = \frac{m_j^N(x, y)}{m_i^N(x, y)} = \frac{\int_{\Theta_j} f(x, y \mid \theta_j)\pi_j^N(\theta_j)d\theta_j}{\int_{\Theta_i} f(x, y \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}. \tag{2.6}$$

Hence, the corresponding Bayes factor, B_{ji}^N, is indeterminate. To solve this problem, O'Hagan(1995) proposed the fractional Bayes factor for Bayesian testing and model selection problem as follow. The fractional Bayes factor(FBF) of model H_j to model H_i is

$$B^F_{ji} = \frac{q_j(b, x, y)}{q_i(b, x, y)}, \tag{2.7}$$

where $q_i(b, x, y) = \frac{\int_{\Theta_i} f_i(x, y \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}{\int_{\Theta_i} f_i^b(x, y \mid \theta_i)\pi_i^N(\theta_i)d\theta_i}$ and $f_i(x, y \mid \theta_i)$ is the likelihood function and b specifies a fraction of likelihood which is to be used as a prior density. One frequently suggested choice is b = m/n, where m is the size of the minimal training sample, assuming this is well defined.

3. Bayesian Hypothesis Test

In bivariate exponential model, we want to test the hypotheses of independence test. That is, the hypotheses of independence is $H_1 : \zeta = 1$, v.s. $H_2 : \zeta \neq 1$. Consider samples of sizes n from above bivariate exponential model with parameters $\theta = (\lambda, \zeta)$.

To test the hypothesis of independence based on the M.L.E.'s, we obtained the test statistics as follows;

$$\text{if} \quad \phi_{MLE} = \frac{n(2 - \hat{\zeta}^2 \hat{\lambda}^2 \bar{y}^2)(\hat{\zeta} - 1)^2}{2\hat{\zeta}^2} \geq \chi^2_{(1, 1-\gamma)}, \tag{3.1}$$

then reject H_1 with significance level γ.

To test the hypothesis of independence based on fractional Bayes factor, we need to compute (2.7). The noninformative priors for $H_1 : \zeta = 1$, v.s. $H_2 : \zeta \neq 1$ are respectively given by

$$\pi_1^N(\theta_1) = \frac{1}{D_\lambda} \cdot I(0 < \lambda < \infty) \tag{3.2}$$

and

$$\pi_2^N(\theta_2) = \frac{1}{D_\lambda D_\zeta} \cdot I(0 < \lambda < \infty) \cdot I(0 < \zeta < \infty), \tag{3.3}$$

where D_λ and D_ζ are fixed constants.

The likelihood function under $H_1 : \zeta = 1$ is

$$f_1(\mathbf{x},\mathbf{y}\mid\lambda,1)=2^n\lambda^{2n}\exp(-2\lambda\sum_{i=1}^{n}x_i-\lambda\sum_{i=1}^{n}y_i). \tag{3.4}$$

Then $q_1(b,\mathbf{x},\mathbf{y})$ under $H_1:\ \zeta=1$ is given by

$$\begin{aligned}q_1(b,\mathbf{x},\mathbf{y})&=\frac{\int_{\Theta_1}f_1(\mathbf{x},\mathbf{y}\mid\theta_1)\pi_i^N(\theta_1)d\theta_1}{\int_{\Theta_1}f_1^b(\mathbf{x},\mathbf{y}\mid\theta_1)\pi_i^N(\theta_1)d\theta_1}\\&=\frac{\int 2^n\lambda^{2n}\exp(-\lambda(2\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}y_i))d\lambda}{\int 2^{nb}\lambda^{2nb}\exp(-\lambda b(2\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}y_i)d\lambda}\\&=\frac{2^n\Gamma(2n+1)(2\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}y_i)^{-2n-1}}{2^{nb}\Gamma(2nb+1)(b(2\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}y_i))^{-2nb-1}}.\end{aligned} \tag{3.5}$$

The likelihood function under $H_2:\ \zeta\neq 1$ is

$$f_2(\mathbf{x},\mathbf{y}\mid\lambda,\zeta)=2^n\zeta^n\lambda^{2n}\exp(-2\lambda\sum_{i=1}^{n}x_i-\lambda\zeta\sum_{i=1}^{n}y_i). \tag{3.6}$$

Then $q_2(b,\mathbf{x},\mathbf{y})$ under $H_2:\ \zeta\neq 1$ is given by

$$\begin{aligned}q_2(b,\mathbf{x},\mathbf{y})&=\frac{\int_{\Theta_2}f_2(\mathbf{x},\mathbf{y}\mid\theta_2)\pi_i^N(\theta_2)d\theta_2}{\int_{\Theta_2}f_2^b(\mathbf{x},\mathbf{y}\mid\theta_2)\pi_2^N(\theta_2)d\theta_2}\\&=\frac{\int 2^n\zeta^n\lambda^{2n}\exp(-\lambda(2\sum_{i=1}^{n}x_i+\zeta\sum_{i=1}^{n}y_i))d\lambda}{\int 2^{nb}\zeta^{nb}\lambda^{2nb}\exp(-\lambda b(2\sum_{i=1}^{n}x_i+\zeta\sum_{i=1}^{n}y_i)d\lambda}\\&=\frac{\Gamma(n)\Gamma(n+1)\left(\sum_{i=1}^{n}x_i\right)^{-n}\cdot\left(\sum_{i=1}^{n}y_i\right)^{-n-1}}{\Gamma(nb)\Gamma(nb+1)\left(b\sum_{i=1}^{n}x_i\right)^{-nb}\cdot\left(b\sum_{i=1}^{n}y_i\right)^{-nb-1}}.\end{aligned} \tag{3.7}$$

Therefore, the FBF of H_2 to H_1 is given by

$$B_{21}^F=\frac{\left\{\Gamma(n)\Gamma(n+1)\left(\sum_{i=1}^{n}x_i\right)^{-n}\left(\sum_{i=1}^{n}y_i\right)^{-n-1}\right\}/\left\{\Gamma(nb)\Gamma(nb+1)\left(b\sum_{i=1}^{n}x_i\right)^{-nb}\left(b\sum_{i=1}^{n}y_i\right)^{-nb-1}\right\}}{\left\{2^n\Gamma(2n+1)\left(2\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}y_i\right)^{-2n-1}\right\}/\left\{2^{nb}\Gamma(2nb+1)\left(b\left(2\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}y_i\right)\right)^{-2nb-1}\right\}}. \tag{3.8}$$

4. Simulation Study

In this section, we consider a simulated data to investigate Bayesian test procedure for $H_1 : \zeta=1$, v.s. $H_2 : \zeta\neq1$. We take the prior probability of H_i being true, $p_i=0.5,\ i=1,2.$.

The data are simulated from bivariate exponential model of size 10 for the parameters $\underline{\theta}=(\lambda,\ \zeta)=(1.5,\ 2.0)$. We note that the true hypothesis may be $H_2 : \zeta\neq1$. Table 1 indicates the generated bivariate exponential data.

<Table 1> Simulated data

i	x_i	y_i
1	1.0964	0.1730
2	0.1429	0.4231
3	0.6336	0.0271
4	0.0658	0.0924
5	0.0050	0.5360
6	0.3027	0.0559
7	1.4528	0.1982
8	0.7614	0.9624
9	0.0760	0.0633
10	0.0750	0.6128

Table 2 indicates fractional Bayes factors, the posterior probabilities $P(H_2 \mid x, y)$ and p-value based on M.L.E.'s for testing $H_1 : \zeta=1$, v.s. $H_2 : \zeta\neq1$.

<Table 2> Test statistics, p-value, fractional Bayes factors and the posterior probability for $H_1 : \zeta=1$, v.s. $H_2 : \zeta\neq1$

Tests	ϕ_{MLE}	p-value	B^F_{21}	$P(H_2 \mid x, y)$
H_1 $v.s.$ H_2	1.4738	0.0703	3.8896	0.7955

First, from the Table 2, the Bayes factors B^F_{21} = 3.8896 and the posterior probability $P(H_2 \mid x, y)$ = 0.7955. Also the p-value based on M.L.E.'s is 0.0703. That is, there is evidence for H_2 in terms of the posterior probabilities based on fractional Bayes factors. But there is no evidence for H_2 in terms of the p-value based on M.L.E.'s under significance level 0.05.

In conclusion, FBF is completely automatic Bayes factors, in that they are based only on the data and noninformative priors. FBF methodology can be easily applied to nonnested as well as to irregular problems. They can also be applied in general when the samples come from any distribution.

References

1. Berger, J. O. and Pericchi, L. R.(1996a). The Intrinsic Bayes Factor for Linear Models(with discussion), in *Bayesian Statistics V*, eds. J. M. Bernardo et al., London: Oxford University Press, 25-44.
2. Berger, J. O. and Pericchi, L. R.(1996b). The Intrinsic Bayes Factor for Model Selection and Prediction, *Journal of the American Statistical Association*, 91, 109-122.
3. Berger, J. O. and Pericchi, L. R.(1998). Accurate and Stable Bayesian Model Selection : the Median Intrinsic Bayes Factor, *Sankhya*, 60, 1-18.
4. Ghosh, J.K. and Mukerjee, R.(1992). Non-informative Priors(with Discussion). *In Bayesian Statistics 4*, 195-210, Oxford University Press.
5. Lingham, R. T. and Sivaganesan, S.(1997). Testing Hypotheses ablut the Power Law Process under Failure Truncation using the Intrinsic Bayes Factors, *Annals of the Institute of Statistical Mathematics*, 49, 693-710.
6. O′ Hagan, A.(1995). Fractional Bayes Factors for Model Comparision(with discussion), *Journal of Royal Statistical Society*, 56, 99-118.
7. San Martini, A. and Spezzaferri, F.(1984). A Predictive Model Selection Criterion, *Journal of Royal Statistical Society*, 46, 296-303.
8. Weier, D.R.(1981). Bayes Estimation for a Bivariate Survival Model Based on Exponential Distributions, Communication in statistics-Theory and Methods, 10, 1415-1427.

[received date : Mar. 2006, accepted date : Apr. 2006]