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Reliability and Ratio of Two Independent Exponential Distributions

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Abstract

We shall consider an estimation of the reliability $P(Y<X)$, and derive moments of the ratio $X/(X+Y)$ in two independent exponential random variables.

Keywords : Ratio, Reliability

1. Introduction

Let $X_1, X_2, \cdots, X_n$ be independently identical random variables with the pdf

$$f(x;\theta)=\frac{1}{\theta}e^{-\frac{x-\theta}{\theta}}, \quad x>\theta>0. \tag{1.1}$$

In reliability applications the exponential distribution has been considered by many authors in Johnson et al(1994)). If the reliability $R=P(Y<X)$ depends on a parameter ρ only and R is a monotone function of ρ, then inference on ρ is equivalent to inference on R in McCool(1991).

The reliability has often applied to engineering, biological phenomenon and physics. In a recent, Kim, et al(2003) studied a UMVUE of the reliability in an exponential distribution. The distribution of the ratio of two independent random variables arises in a model of ionic current fluctuation in biological membranes. The distribution of the ratio of independent gamma variate was studied by Bowman and Shenton(1998).

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Here we shall consider an estimation of the reliability $P(Y<X)$, and derive a distribution of the ratio $X/(X+Y)$ when X and Y are two independent exponential random variables each having the exponential distribution (1.1) with the different parameters.

2. Reliability

We shall consider an estimation of the reliability in the exponential distribution (1.1). Assume X and Y be two independent random variables each having the following exponential densities:

$$f_X(x)=\frac{1}{\theta_1}\exp(-\frac{x-\theta_1}{\theta_1}),\quad x>\theta_1>0$$
$$f_Y(y)=\frac{1}{\theta_2}\exp(-\frac{y-\theta_2}{\theta_2}),\quad y>\theta_2>0\ . \tag{2.1}$$

Then the reliability can be obtained:

$$R\equiv P(X<Y)=\begin{cases}1-\frac{1}{1+\rho}e^{1-\rho}, & \text{if}\quad \rho\geq 1\\ \frac{\rho}{1+\rho}e^{1-\frac{1}{\rho}}, & \text{if}\quad 0<\rho<1\end{cases} \tag{2.2}$$

where $\rho\equiv\frac{\theta_2}{\theta_1}$. (see Kim et al(2003))

From the result (2.2), the reliability $R=P(Y<X)$ depends on ρ only and is a monotone function of ρ. Because R is a monotone function of ρ, inference on ρ is equivalent to inference on R in McCool(1991). And hence it's sufficient for us to consider an estimation of ρ in stead of estimating the reliability R itself.

Assume $X_1, X_2, \cdots, X_m$ and $Y_1, Y_2, \cdots, Y_n$ be two independent samples from the preceding random variables X and Y each having the density (2.1).

Based on the MLE of parameters θ_1 and θ_2, two estimators of ρ can be defined as:

$$\widehat{\rho}=\frac{Y_{(1)}}{X_{(1)}}\ ,$$

and
$$\tilde{\rho} = \frac{m \sum_{i=1}^{n} Y_i}{n \sum_{i=1}^{m} X_i} .$$

From the densities of the first order statistics $X_{(1)}$ and $Y_{(1)}$ in Johnson et al(2004), and the formulas 2.9 and 2.14 in Oberhettinger & Badii(1973), we can obtain the expectation and variance of $\hat{\rho}$

$$E(\hat{\rho}) = (1+\frac{1}{n}) m\, e^{m} \, [-E_i(-m)] \rho \,,$$
$$Var(\hat{\rho}) = [\frac{1}{n^2} + (1+\frac{1}{n})^2][m + m^2 e^{m} E_i(-m)] \rho^2 - E^2(\hat{\rho}) \tag{2.3}$$

where $-E_i(-x) = \int_x^{\infty} \frac{e^{-t}}{t} dt, \quad x > 0$ is an exponential integral.

From the densities of two sum statistics $\sum_{i=1}^{m} X_i$ and $\sum_{i=1}^{n} Y_i$ in Kim et al(2003), and the formula 3.8 in Oberhettinger and Badii(1973) and the formula 3.5 in Oberhettinger(1974), we can obtain the expectation and variance of $\tilde{\rho}$:

$$E(\tilde{\rho}) = 2 m^{m} e^{m} \, \Gamma(-(m-1), m) \rho \,,$$
$$Var(\tilde{\rho}) = \frac{4n+1}{n} \; m^{\frac{m+1}{2}} \; e^{m/2} \; W_{-\frac{m}{2}-\frac{1}{2}, \, \frac{m}{2}-1}(m) \; \rho^2 - E^2(\tilde{\rho}) \tag{2.4}$$

where $\Gamma(-n, x) = \frac{(-1)^n}{n!} [-E_i(-x) - e^{-x} \sum_{i=0}^{n-1} (-1)^i \frac{i!}{x^{i+1}}]$
and $W_{a,b}(x)$ is the Whittaker function .

Remark 1. The Whittaker function can be represented by an integral form in Gradshteyn et al(1965) to evaluate variance of $\tilde{\rho}$ numerically as the following:

$$W_{-\frac{1}{2}-\frac{m}{2}, \, \frac{m}{2}-1}(m) = \frac{m^{\frac{m}{2}-\frac{1}{2}}}{\Gamma(m)} \; e^{-m/2} \int_0^{\infty} e^{-mt} \, t^{m-1} \, (1+t)^{-2} dt \, .$$

From the results (2.3) and (2.4) of expectations and variances of two estimators $\hat{\rho}$ and $\tilde{\rho}$, and an integral form of the Whittaker function in Remark 1, Table 1 shows the numerical values of mean squared errors of two estimators $\hat{\rho}$ and $\tilde{\rho}$.

<Table 1> shows that the MLE $\hat{\rho}$ is more efficient in a sense of MSE than the estimator $\tilde{\rho}$ when n=10, 20 and m=5(5)20.

<Table 1> Mean squared errors of two estimators $\hat{\rho}$ and $\tilde{\rho}$ (unit: ρ^2)

n	m	$\hat{\rho}$	$\tilde{\rho}$
10	5	0.05383	0.10871
10	10	0.05307	0.09160
10	15	0.04620	0.09087
10	20	0.04272	0.08924
20	5	0.04833	0.09701
20	10	0.04401	0.08715
20	15	0.04068	0.08051
20	20	0.03721	0.07357

3. Distribution of the ratio $\frac{X}{X+Y}$

Let X and Y be independent random variables each having two parameter exponential densities (2.1). To find a distribution of the ratio $V=X/(X+Y)$ when X and Y are two independent exponential random variables each having density (2.1), we consider the joint density of the following random variables W and V.

Let $W\equiv X+Y$ and $V\equiv\frac{X}{X+Y}$. Then, from the joint pdf of W and V, and the formulas 3.381 (2) & (3) in Gradshteyn & Ryzhik(1965), we can obtain the density (3.1) of the ratio V:

$$f_V(x)=\begin{cases} \frac{re^2}{x^2}e^{-(1+\frac{1-x}{x}r)}(1+\frac{1-x}{x}r)^{-2}+\frac{re^2}{x^2}e^{-(1+\frac{1-x}{x}r)}(1+\frac{1-x}{x}r)^{-1}, & \text{if } 0<x<\frac{\theta_1}{\theta_1+\theta_2}, \\ \frac{e^2}{r(1-x)^2}e^{-(1+\frac{x}{r(1-x)})}(1+\frac{x}{r(1-x)})^{-2}+\frac{e^2}{r(1-x)^2}e^{-(1+\frac{x}{r(1-x)})}\cdot(1+\frac{x}{r(1-x)})^{-1}, & \text{if } \frac{\theta_1}{\theta_1+\theta_2}\le x<1 \end{cases}$$

where $r\equiv\frac{\theta_1}{\theta_2}$. (3.1)

We shall introduce the following integrals to find expectation and variance of the ratio V. From the formulas 2.7 and 2.14 in Oberhettinger and Badii(1973), we can obtain the following Fact 1:

Fact 1. (1) $\int_2^\infty \frac{e^{-x}}{x(x+a)}dx=\frac{1}{a}e^a\cdot E_i(-a-2)-\frac{1}{a}E_i(-2)$.

(2) $\int_2^\infty \frac{e^{-x}}{x^2(x+a)}\,dx = \frac{a+1}{a^2} E_i(-2) - \frac{1}{a^2} e^a E_i(-a-2) + \frac{1}{2a} e^{-2}$.

(3) $\int_2^\infty \frac{e^{-x}}{x(x+a)^2}\,dx = \frac{a+1}{a^2} e^a E_i(-a-2) - \frac{1}{a^2} E_i(-2) - \frac{1}{a(a+2)} e^{-2}$.

(4) $\int_2^\infty \frac{e^{-x}}{x^2(x+a)^2}\,dx = \frac{a+4}{2a^2(a+2)} e^{-2} + \frac{a+2}{a^3} E_i(-2) + \frac{a-2}{a^3} e^a E_i(-a-2)$

where $-E_i(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt, \quad x>0$.

From the density of the ratio (3.1) and Fact 1 (1) & (2), we can obtain the expectation of V .

$$E(V) = \begin{cases} r\left[\frac{r-2}{(r-1)^2} e^{r+1} E_i(-r-1) - \frac{\frac{1}{r}-2}{(r-1)^2} e^{\frac{1}{r}+1} E_i\left(-\frac{1}{r}-1\right)\right], & \text{if } r \neq 1, \\ \frac{1}{2}, & \text{if } r \equiv = 1 \end{cases}$$

Next, to find the variance of the ratio, we can obtain 2nd moment of the ratio from the density (3.1) and Fact 1 (1), (2), (3) and (4),

$$\begin{aligned} E(V^2) &= \frac{3r-1}{(1-r)^2} + \frac{1-2r+3r^2-4r^3}{r(1-r)^3} e^{\frac{1}{r}+1} E_i\left(-\frac{1}{r}-1\right) + \frac{r^2(r^2-3)}{(r-1)^3} \\ &\quad \cdot e^{r+1} E_i(-r-1). \qquad \text{if } r \neq 1 \\ &= -\frac{1}{3} - \frac{5}{3} e^2 E_i(-2) \doteqdot 0.26887, \qquad \text{if } r=1 . \end{aligned}$$

Therefore, especially if r=1 (i.e. $\theta_1=\theta_2$), then $Var(V) \doteqdot 0.01887$.

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