Journal of the Korean
Data & Information Science Society
2006, Vol. 17, No. 2, pp. 507 ~ 513

Inference on Reliability in an Exponentiated Uniform Distribution

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Abstract

We shall consider an inference of the reliability and an estimation of the right-tail probability in an exponentiated uniform distribution. And we shall compare numerically efficiencies for proposed estimators of the scale parameter and right-tail probability in the small sample sizes.

Keywords : Exponentiated distribution, Reliability, Right tail probability

1. Introduction

Let $F(x)$ be a cdf of a random variable. Then $G(x) \equiv F^{\alpha}(x)$ is also a cdf of a random variable for some positive real number α , and hence the distribution $G(x)$ is called an exponentiated distribution of a given cdf $F(x)$ in Gupta(2001). Gupta(2001) considered an exponentiated exponential family. Kim et al(2003) studied an inference on the reliability in an exponential distribution. A uniform distribution has been studied by many authors in Johnson et al(1995).

We shall define an exponentiated uniform distribution, and hence consider an estimation of the scale parameter in the exponentiated uniform distribution. And we shall consider an inference on the reliability and estimation of the right tail probability in the exponentiated uniform distribution.

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2. Estimation of the scale parameter

First we shall define an exponentiated uniform distribution, and consider moment and estimation of the scale parameter in the exponentiated uniform random variable.

From a cdf of a uniform distribution over $(0, \theta)$, an exponentiated uniform distribution is defined by :

$$G(x) = \left(\frac{x}{\theta}\right)^{\alpha}, \quad 0 < x < \theta. \quad \alpha > 0, \tag{2.1}$$

which has been called a power function distribution in Johnson et al(1995)

From the formula 2.381(1) in Gradshteyn and Ryzhik(1965), the mgf of the exponentiated uniform random variable X can be obtained :

$$M_X(t) = \begin{cases} 1, & \text{if } t = 0 \\ \dfrac{\alpha}{(-t\theta)^{\alpha}} \gamma(\alpha, -\theta t), & \text{if } t < 0, \end{cases}$$

where $\gamma(a, x)$ is an incomplete gamma function.

From existence of its mgf, the k-th moment of the exponentiated uniform random variable X is well-known as :

$$E(X^k) = \frac{\alpha}{\alpha + k} \theta^k, \quad k = 1, 2, 3, \cdots. \tag{2.2}$$

And hence the expectation and variance of X are also well-known as

$$\frac{\alpha}{\alpha+1}\theta \quad \text{and} \quad \frac{\alpha}{(\alpha+1)^2(\alpha+2)}\theta^2, \text{ respectively.}$$

Assume $X_1, \cdots, X_n$ be independent random variables each having the exponentiated uniform distribution (2.1), and $X_{(1)}, \cdots, X_{(n)}$ be the corresponding order statistics. Then, we can obtain the following expectations of the MLE $\hat{\theta} = X_{(n)}$ of the scale parameter θ :

$$\begin{aligned} E(X_{(n)}^k) &= \frac{n\alpha}{n\alpha + k} \theta^k, \\ E\left(\frac{1}{X_{(n)}^k}\right) &= \frac{n\alpha}{n\alpha - k} \frac{1}{\theta^k}, \quad \text{if } n\alpha > k. \end{aligned} \tag{2.3}$$

Define $\tilde{\theta} \equiv \frac{n\alpha + 1}{n\alpha} X_{(n)}$. Then $\tilde{\theta}$ is an unbiased estimator of the scale parameter θ .

From the expectation (2.3), we can obtain the followings:

$$E(\tilde{\theta}^k)=\frac{(an+1)^k}{(an)^{k-1}(an+k)}\theta^k,$$
$$E(\frac{1}{\tilde{\theta}^k})=\frac{(an)^{k+1}}{(an+1)^k(an-k)}\frac{1}{\theta^k}, \quad \text{if } an>k. \qquad (2,4)$$

From the results (2.3) and (2.4),

Fact 1. The unbiased estimator $\tilde{\theta}$ is more efficient in a sense of MSE than the MLE $\hat{\theta}=X_{(n)}$.

3. The reliability

Let X and Y be independent random variables each having the exponentiated uniform cdf (2.1) with parameter θ_1 and θ_2, respectively. Then we can obtain the following :

$$P(Y<X)=\begin{cases} \frac{1}{2}\rho^{\alpha}, & \text{if } \rho<1 \\ 1-\frac{1}{2\rho^{\alpha}}, & \text{if } \rho>1 \end{cases},$$

where $\rho\equiv\theta_1/\theta_2$.

When the parameter α in the density (2.1) is known, $P(Y<X)$ is a monotone function of ρ , and hence, inference on $P(Y<X)$ is equivalent to inference on ρ (see McCool(1991)).

Here we shall consider an inference on $\rho\equiv\theta_1/\theta_2$ in stead of inferencing on the reliability $P(Y<X)$ itself in the exponentiated uniform distribution when the parameter α in the density (2.1) is known.

Assume $X_1,\cdots,X_m$ and $Y_1,\cdots,Y_n$ be two independent samples from an exponentiated uniform distribution with parameters θ_1 and θ_2 , respectively.

Then, from the estimators $\hat{\theta}$ and $\tilde{\theta}$ in the preceding section, we can define the following estimators of ρ :

$$\hat{\rho}=\frac{X_{(m)}}{Y_{(n)}} \quad \text{and} \quad \tilde{\rho}=\frac{n(am+1)}{m(an+1)}\frac{X_{(m)}}{Y_{(n)}} .$$

From the results (2.3) and (2.4), we can obtain the 1st and 2nd moments of $\hat{\rho}$ and $\tilde{\rho}$ to find expectations and variance of $\hat{\rho}$ and $\tilde{\rho}$.

$$E(\hat{\rho}) = \frac{\alpha m}{(\alpha m+1)(\alpha n-1)}\,\rho,$$
$$E(\hat{\rho}^2) = \frac{\alpha^2 mn}{(\alpha m+2)(\alpha n-2)}\,\rho^2,$$
$$E(\tilde{\rho}) = \frac{n^2\alpha^2}{n^2\alpha^2-1}\,\rho^2, \quad \text{and}$$
$$E(\tilde{\rho}^2) = \left(\frac{\alpha m+1}{\alpha n+1}\right)^2 \frac{n^3\alpha^2}{m(\alpha m+2)(\alpha n-2)}\,\rho^2 .$$

From the 1st moment of $\hat{\rho}$, we can define an unbiased estimator of ρ as following :

$$\bar{\rho} = \frac{(\alpha m+1)(\alpha n-1)}{\alpha^2 mn}\,\frac{X_{(m)}}{Y_{(n)}} ,$$

and hence,

$$E(\bar{\rho}^2) = \frac{(\alpha m+1)^2(\alpha n-1)^2}{\alpha^2 mn(\alpha m+2)(\alpha n-2)}\,\rho^2 .$$

From the 1st and 2nd moments of $\hat{\rho}$, $\tilde{\rho}$ and $\bar{\rho}$. Table 1 shows the numerical values of biases and mean squared errors(MSE) of $\hat{\rho}$, $\tilde{\rho}$ and $\bar{\rho}$.

<Table 1> Biases and mean squared errors of three estimators $\hat{\rho}$, $\tilde{\rho}$ and $\bar{\rho}$.
(units: bias and MSE are ρ and ρ^2, respectively)

α	m	n	$\hat{\rho}$		$\tilde{\rho}$		$\bar{\rho}$
			Bias	MSE	Bias	MSE	MSE
0.5	10	15	0.0385	0.0510	0.0181	0.0559	0.0515
0.5	15	10	0.1029	0.1099	0.0417	0.0902	0.0816
0.5	20	25	0.0119	0.0159	0.0064	0.0163	0.0160
0.5	25	20	0.0288	0.0200	0.0101	0.0185	0.0181
0.5	30	35	0.0057	0.0076	0.0033	0.0075	0.0074
0.5	35	30	0.0135	0.0085	0.0045	0.0081	0.0080
2.0	10	15	0.0148	0.0036	0.0011	0.0022	0.0021
2.0	15	10	0.0187	0.0043	0.0025	0.0038	0.00378
2.0	20	25	0.0045	0.0011	0.0004	0.00104	0.00102
2.0	25	20	0.0055	0.0012	0.0006	0.00110	0.00104
2.0	30	35	0.0021	0.00048	0.0002	0.00039	0.00047
2.0	35	30	0.0026	0,00049	0.0003	0.00045	0.00048

From Table 1, we can obtain the followings :

Fact 2. (a) The $\hat{\rho}$ has less MSE than those of $\tilde{\rho}$ and $\bar{\rho}$ when α =0.5, m=10 < n=15, and α =0.5, m=20 <n=25.
(b) The $\bar{\rho}$ has less MSE than those of $\hat{\rho}$ and $\hat{\rho}$ for other cases except α =2, m=30(35) n=35(30).
(c) The $\tilde{\rho}$ has less MSE than those of $\hat{\rho}$ and $\bar{\rho}$, when α =2, m=30(35) n=35(30), (d) Three estimators of ρ can't dominate each other for all cases.

Next we shall consider an interval estimation of ρ when the parameter α in the density (2.1) is known.

From the quotient pdf of Theorem in Rohatgi(1976), the quantity $\hat{\rho}=\frac{1}{\rho}\frac{X_{(m)}}{Y_{(n)}}$ is a pivot quantity, since the distribution of $\hat{\rho}$ doesn't involve ρ as following :

$$f_{\hat{\rho}}(x)=\begin{cases}\frac{\alpha mn}{m+n}x^{\alpha m-1} & \text{, if } 0<x<1\\ \frac{\alpha mn}{m+n}x^{-\alpha n-1} & \text{, if } x\geq 1.\end{cases} \quad (3.1)$$

From the pdf (3.1) of $\hat{\rho}$, we can obtain a $(1-\gamma)100\%$ confidence interval of ρ :

$$\left(\frac{X_{(m)}}{Y_{(n)}}\left(\frac{m+n}{m}\frac{\gamma}{2}\right)^{\frac{1}{\alpha n}},\ \frac{X_{(m)}}{Y_{(n)}}\left(\frac{m+n}{n}\frac{\gamma}{2}\right)^{-\frac{1}{\alpha m}}\right).$$

And we shall consider the following statistical hypothesis test as following :

$$H_o: \rho=1 \quad \text{against} \quad H_1: \rho\neq 1 .$$

As applying the likelihood ratio test in Rohatgi(1976), we can obtain the critical region :

$$\hat{\rho}\leq c_1 \quad \text{or} \quad \hat{\rho}\geq c_2 .$$

From the pdf (3.1) of $\hat{\rho}$, for a given significance level $0<\gamma<1$, if H_0 is true, then we can obtain the critical region of the test as following :

$$\hat{\rho}\leq\left(\frac{m+n}{n}\frac{\gamma}{2}\right)^{\frac{1}{\alpha m}} \quad \text{or} \quad \hat{\rho}\geq\left(\frac{m+n}{m}\frac{\gamma}{2}\right)^{-\frac{1}{\alpha n}} .$$

4. The right-tail probability

Let X be an exponentiated uniform random variable with the pdf (2.1). Then the right-tail probability is

$$R(t) = 1 - \frac{t^{\alpha}}{\theta^{\alpha}}, \quad 0 < t < \theta.$$

Estimators of the right tail probability can be defined as

$$\widehat{R}(t) = 1 - \frac{t^{\alpha}}{\widehat{\theta}^{\alpha}} \quad \text{and} \quad \widetilde{R}(t) = 1 - \frac{t^{\alpha}}{\widetilde{\theta}^{\alpha}},$$

where $\widehat{\theta} = X_{(n)}$ and $\widetilde{\theta} \equiv \frac{n\alpha + 1}{n\alpha} X_{(n)}$.

From the result (2.3), we can obtain expectations and variances of two right-tail probability estimators of R(t) :

$$E(\widehat{R}(t)) = R(t) - \left(\frac{t}{\theta}\right)^{\alpha} \frac{1}{n-1},$$

$$Var(\widehat{R}(t)) = \left(\frac{t}{\theta}\right)^{2\alpha} \frac{n}{(n-1)^2 (n-2)},$$

$$E(\widetilde{R}(t)) = R(t) - \left(\frac{t}{\theta}\right)^{\alpha} \left(\frac{1}{n-1} - \frac{n}{(n-1)(\alpha n + 1)}\right),$$

and

$$Var(\widetilde{R}(t)) = \left(\frac{t}{\theta}\right)^{2\alpha} \frac{n}{(n-1)^2 (n-2)} \frac{n^2 \alpha^2}{(\alpha n + 1)^2}.$$

And hence, we can get the followings :

$$Bias(\widehat{R}(t)) < Bias(\widetilde{R}(t)) \quad \text{and} \quad Var(\widehat{R}(t)) < Var(\widetilde{R}(t)).$$

Therefore, we can obtain the following:

Fact 3. The MLE $\widehat{R}(t)$ has more efficient in a sense of MSE than the estimator $\widetilde{R}(t)$.

References

1. Gupta, R. D. (2001). Exponentiated Exponential Family, An Alternative to Gamma and Weibull Distribution, *Biometrical J*. 43-1 . 117-130.
2. Gradshteyn, I. S. and Ryzhik, I. M. (1965). *Tables of Integrals, Series, and Products,* Academic Press, New York.
3. Johnson, N.L., Kotz, S. and Balakrishnan, N. (1995). *Continue Univariate Distribution-2,* Houghton Mifflin Com., Boston.
4. Kim, J.D,, Moon, Y,G. and Kang J.H. (2003). Inference on P(X<Y) in an Exponential Distribution, *Journal of Korean Data & Information Sciences Society,* 14-4, 989-995.
5. McCool, J.I, (1991). Inference on P(X<Y) in the Weibull case, *Communications. in Statististics, Simulation,* 20(1), 129-148.
6. Rohatgi, V. K. (1976). *An Introduction To Probability Theory and Mathematical Statistics,* John Wiley & Sons, New York.

[received date : Mar. 2006, accepted date : Apr. 2006]