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A Note on the Minimal Variability OWA Operator Weights[1)]

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Abstract

In this note, we give an elementary simple new proof of the main result of *Fullér* and Majlender [Fuzzy Sets and systems 136 (2003) 203-215] concerning obtaining minimal variability OWA operator weights.

Keywords : Fuzzy sets, OWA operator

1. Introduction

Yager(1988) introduced a new aggregation technique based on the ordered weighted averaging(OWA) operators. An OWA operator of dimension n is a mapping $F: R^n \to R$ that has an associated weighting vector $W = (w_1, \cdots, w_n)^T$ of having the properties $w_1 + \cdots + w_n = 1$, $0 \le w_i \le 1$, $i = 1, \cdots, n$, and such that

$$F(a_1, \cdots, a_n) = \sum_{i=1}^{n} w_i b_i,$$

where b_j is the jth largest element of the collection of the aggregated objects $a_1, \cdots, a_n$.

Yager(1988) introduced a measure of "orness" associated with the weighting vector W of an OWA operator, defined as

$$orness(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i,$$

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and it characterizes the degree to which the aggregation is like an *or* operation. One important issue in the theory of OWA operators is the determination of the associated weights. A number of approaches have been suggested for obtaining the associated weights, i.e., quantifier guided aggregation(Yager(1988, 1993)), exponential smoothing(Filev and Yager(1988)) and learning(Yager and Filev(1999)). Another approaches, suggested by O'Hagan(1988), determines a special class of OWA operators having maximal entropy of the OWA weights for a given level of orness; algorithmically it is based on the solution of a constrained optimization problem. Recently, to obtain minimal variability OWA weights under given level of orness, *Fullér* and Majlender(2003) considered the following constrained mathematical programming problem

$$\text{minimize} \quad D^2(W) = \sum_{i=1}^{n} \frac{1}{n}(w_i - E(W))^2 = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2}.$$

$$\text{subject to} \quad orness(W) = \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha, \quad 0 \le \alpha \le 1, \tag{1}$$

$$w_1 + \cdots + w_n = 1, \quad 0 \le w_i, \quad i = 1, \cdots, n,$$

where $E(W) = (w_1 + \cdots + w_n)/n$ stands for the arithmetic mean of weights. And they solved problem (1) analytically and derived the exact minimal variability OWA weights for any level of orness using the Kuhn-Tucker second-order sufficiency conditions for optimality. In this note, we give a simple new method deriving the exact minimal variability OWA weights for the problem (1).

2. New method obtaining minimal variability OWA weights

We first consider the following disjunctive partition of unit interval (0, 1) presented in (*Fullér* and Majlender(2003)):

$$(0,1) = \bigcup_{r=2}^{n-2} J_{r,n} \cup J_{1,n} \cup \bigcup_{s=2}^{n-2} J_{1,s}, \tag{2}$$

where

$$J_{r,n} = \left(1 - \frac{1}{3}\frac{2n+r-2}{n-1}, 1 - \frac{1}{3}\frac{2n+r-3}{n-1}\right], \quad r = 2, \cdots, \quad n-1$$

$$J_{1,n} = \left(1 - \frac{1}{3}\frac{2n-2}{n-1}, 1 - \frac{1}{3}\frac{2n-2}{n-1}\right),$$

$$J_{1,s} = \left[1 - \frac{1}{3}\frac{s-1}{n-1}, 1 - \frac{1}{3}\frac{s-2}{n-1}\right), \quad s = 2, \cdots, n-1$$

Let us consider the constrained optimization problem (1) and suppose that $\alpha \in J_{r,s}$ for some r and s from partition (2). Such r and s always exist for any $\alpha \in (0,1)$, furthermore, $r=1$ or $s=n$ should hold.
Fullér and Majlender(2003) proved that the weighting vector

$$W^* = (0, \cdots, 0, w_r^*, \cdots, w_s^*, 0, \cdots, 0)$$

where

$$w_j^* = 0 \quad \text{if} \quad j \notin I_{\{r,s\}} = \{r, \cdots, s\}, \tag{3}$$

$$w_r^* = \frac{2(2s+r-2)-6(n-1)(1-\alpha)}{(s-r+1)(s-r+2)} \tag{4}$$

$$w_s^* = \frac{6(n-1)(1-\alpha)-2(s+2r-4)}{(s-r+1)(s-r+2)} \tag{5}$$

$$w_j^* = \frac{s-j}{s-r} w_r^* + \frac{j-r}{s-r} w_s^* \quad if \;\; j \in I_{\{r+1,s-1\}} \tag{6}$$

are the optimal solutions for problem (1). According to (6) of convex linear combination property, we have that $w_i^* = a^* i + b^*$ for $i \in I_{\{r,s\}}$ for some a^*, b^*. Using formulas (4) and (5) we find

$$w_r^*, \;\; w_s^* \in [0,1] \;\rightarrow\; \alpha \in \left[1 - \frac{1}{3}\,\frac{2s+r-2}{n-1}\,, 1 - \frac{1}{3}\,\frac{s+2r-4}{n-1}\right].$$

Then, it is also easy to check that $a^* i + b^* \le 0$ for $i \notin I_{\{s,t\}}$. Hence, it is considered that *Fullér* and Majlender(2003) proved the following result.

Theorem (*Fullér* **and Majlender(2003)**) The optimal weight for the constrained optimization problem (1) should satisfy the equations

$$w_i^* = \begin{cases} a^* i + b^*, & \text{if} \quad i \in I_{\{r,s\}} = \{r, \cdots, s\}, \\ 0, & \text{elsewhere}. \end{cases}$$

for some a^*, b^* satisfying $a^* i + b^* \le 0$ for $i \notin I_{\{s,t\}}$ and $r=1$ or $s=n$.

We now give a new simple proof of this theorem.

Proof. Let $w_i^* = a^* i + b^*$, $i \in I_{\{r,s\}} = \{r, \cdots, s\}$ and 0, otherwise satisfying

$$\sum i w_i^* = n - (n-1)\alpha \quad (\Leftrightarrow \sum_{i=1}^{n} \frac{n-i}{n-1} w_i = \alpha) \tag{7}$$

$$\sum w_i^* = 1,\ 0 \le w_i^*,\ i = 1, \cdots, n \tag{8}$$

and let $w_i, i = 1, \cdots, n$ satisfy

$$\sum_{i=1}^{n} i w_i = n - (n-1)\alpha, \tag{9}$$

$$\sum_{i=1}^{n} w_{i=1},\ 0 \le w_i,\ i = 1, \cdots, n. \tag{10}$$

We put $w_i = w_i^* + \beta_i$, $i = 1, \cdots, n$. Then, noting that $w_i = \beta_i$, $i \notin I_{\{r,s\}}$, we have, from (8) and (10),

$$\sum_{i \notin I_{\{r,s\}}} w_i + \sum_{i \in I_{\{r,s\}}} \beta_i = \sum_{i=1}^{n} \beta_i = 0, \tag{11}$$

since $1 = \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} w_i^* + \sum_{i=1}^{n} \beta_i = 1 + \sum_{i=1}^{n} \beta_i$. We also have, from (7) and (9)

$$\sum_{i \notin I_{\{r,s\}}} i w_i + \sum_{i \in I_{\{r,s\}}} i\beta_i = \sum_{i=1}^{n} i\beta_i = 0, \tag{12}$$

since $\sum_{i=1}^{n} i w_i = \sum_{i=1}^{n} i(w_i^* + \beta_i) = \sum_{i=1}^{n} i w_i^* + \sum_{i=1}^{n} i\beta_i$. We now show that

$$\sum_{i=1}^{n} w_i^2 \ge \sum_{i=1}^{n} w_i^{*2}.$$

It is because from (11) and (12)

$$\begin{aligned}
&\sum_{i=1}^{n} w_i^2 - \sum_{i=1}^{n} w_i^{2*} \\
&= \sum_{i=1}^{n}(w_i^* + \beta_i)^2 - \sum_{i=1}^{n} w_i^{2*} \\
&= 2\sum_{i=1}^{n}\beta_i w_i^* + \sum_{i=1}^{n}\beta_i^2 \\
&= 2\sum_{i \in I_{\{r,s\}}} \beta_i (a^* i + b^*) + \sum_{i=1}^{n}\beta_i^2 \\
&= 2a^* \sum_{i \in I_{\{r,s\}}} i\beta_i + 2b^* \sum_{i \in I_{\{r,s\}}} \beta_i + \sum_{i=1}^{n}\beta_i^2 \\
&= 2a^*(-\sum_{i \notin I_{\{r,s\}}} i w_i) + 2b^*(-\sum_{i \notin I_{\{r,s\}}} w_i) + \sum_{i=1}^{n}\beta_i^2 \\
&= -2\sum_{i \notin I_{\{r,s\}}} w_i (a^* i + b^*) + \sum_{i=1}^{n}\beta_i^2 \\
&\geq \sum_{i=1}^{n}\beta_i^2 \geq 0,
\end{aligned}$$

where the fifth equality comes from (11) and (12), and the first inequality comes from the fact that $a^* i + b^* \leq 0$ for $i \notin I_{\{r,s\}}$, and hence we have

$$D^2(W) = \frac{1}{n}\sum_{i=1}^{n} w_i^2 - \frac{1}{n^2} \geq \frac{1}{n}\sum_{i=1}^{n} w_i^{*2} - \frac{1}{n^2} = D^2(W^*)$$

and the equality holds if and only if $W = W^*$ which completes the proof.

We now consider how to find a^* and b^* in the Theorem. We consider the following weight

$$w_i^* = \begin{cases} a^* i + b^*, & \text{if} \quad i \in I_{\{r,s\}} = \{r, \cdots, s\}, \\ 0, & \text{elsewhere}, \end{cases}$$

for some a^* and b^* satisfying

$$\sum i w_i^* = n - (n-1)\alpha \quad (\Leftrightarrow \sum_{i=1}^{n} \frac{n-i}{n-1} w_i^* = \alpha) \tag{13}$$

$$\sum w_i^* = 1,\ 0 \le w_i^*,\ i = 1, \cdots, n. \tag{14}$$

From (13) and (14) , we have

$$a^* \sum_{i=r}^{s} i + (s-r+1)\, b^* = 1,$$

$$\sum_{i=r}^{s} i\,(a^* i + b^*) = n - (n-1)\,\alpha.$$

Since $\sum_{i=r}^{s} i = \frac{(r+s)(s-r+1)}{2}$, $\sum_{i=r}^{s} i^2 = \frac{s(s+1)(2s+1)-(r-1)r(2r-1)}{6}$,
we have

$$\frac{(r+s)(s-r+1)}{2}\, a^* + (s-r+1)\, b^* = 1$$

$$\frac{s(s+1)(2s+1)-(r-1)r(2r-1)}{6}\, a^* + \frac{(r+s)(s-r+1)}{2}\, b^* = n-(n-1)\alpha.$$

Hence , we have by Cramer's rule

$$a^* = \frac{\begin{vmatrix} 1 & s-r+1 \\ n-(n-1)\alpha & \frac{(r+s)(s-r+1)}{2} \end{vmatrix}}{\begin{vmatrix} \frac{(r+s)(s-r+1)}{2} & s-r+1 \\ \frac{s(s+1)(2s+1)-(r-1)r(2r-1)}{6} & \frac{(r+s)(s-r+1)}{2} \end{vmatrix}}$$

$$= \frac{6(r+s-2n+2n\alpha-2\alpha)}{(s-r+1)(r-s)(s-r+2)},$$

$$b^* = \frac{\begin{vmatrix} \frac{(r+s)(s-r+1)}{2} & 1 \\ \frac{s(s+1)(2s+1)-(r-1)r(2r-1)}{6} & n-(n-1)\alpha \end{vmatrix}}{\begin{vmatrix} \frac{(r+s)(s-r+1)}{2} & s-r+1 \\ \frac{s(s+1)(2s+1)-(r-1)r(2r-1)}{6} & \frac{(r+s)(s-r+1)}{2} \end{vmatrix}}$$

$$= \frac{2[3(r+s)(n-n\alpha+\alpha)-(2s^2+2r^2+2rs+s-r)]}{(s-r+1)(r-s)(s-r+2)}.$$

Then, from the equality $w_i^* = a^* i + b^*,\ i = r, \cdots, s$, we can get

$$w_j^* = 0 \quad \text{if} \quad j \notin I_{\{r,s\}},$$

$$w_r^* = a^* r + b^* = \frac{2(2s+r-2) - 6(n-1)(1-\alpha)}{(s-r+1)(s-r+2)}$$

$$w_s^* = a^* s + b^* = \frac{6(n-1)(1-\alpha) - 2(s+2r-4)}{(s-r+1)(s-r+2)}$$

$$w_j^* = a^* j + b^* = \frac{s-j}{s-r} w_r^* + \frac{j-r}{s-r} w_s^* \quad \text{if} \quad j \in I_{\{r+1, s-1\}}.$$

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