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The UMVUE of $[P(Y>X)]^k$ in a Two Parameter Exponential Distribution

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Abstract

We shall consider the UMVUE of $[P(Y>X)]^k$ in a two parameter exponential distribution.

Keywords : Exponential distribution, UMVUE

1. Introduction

A two parameter exponetial distribution is given by

$$f(x\,;\mu,\sigma)=\frac{1}{\sigma}e^{-(x-\mu)/\sigma},\ \ x>\mu,\ \ where\ \sigma>0,\ \ \mu\in R^1\ .$$

The problem of estimating of the probability that a random variable X is less than an independent random variable Y, arises in a reliability. When X represents the random value of a stress that a device will be subjected to in service and Y represents the strength that varies from item to item in the population of devices, then the reliability R, i.e. the probability that a randomly selected device functions successfully, is equal to P(Y>X). The same problem also arises in the context of statistical tolerancing where X represents the diameter of a draft and Y the diameter of a bearing that is to be mounted on the shaft. The probability that the bearing fits without interference is then P(Y>X). In biometry X represents a patient's remaining years when treated with drug B. If the choice of drug is left to the patient, person's deliberations will center on whether P(Y>X) is less than or greater than 1/2.

Woo and Lee(2001) studied the MLE and the UMVUE of the right-tail

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perobability inn a levy distribution and Kim, et al(2003) studied an inference on P(Y<X) in an exponential distribution.

Here we shall find the UMVUE of $[P(Y>X)]^k$ in a two parameter exponential distribution when the scale parameter is known.

2. The UMVUE of $[P(Y>X)]^k$

Let X and Y be independently random variables with p.d.f.'s:

$$f_X(x\,;\mu_x,\sigma_x)=\frac{1}{\sigma_x}e^{-(x-\mu_x)/\sigma_x},\quad x>\mu_x,\ .$$

and
$$f_Y(y\,;\mu,\sigma)=\frac{1}{\sigma_y}e^{-(y-\mu_y)/\sigma_y},\quad y>\mu_y\ .$$

From th result of Kim, et al.(2003), the reliability is given as the following:

$$R\equiv P(Y>X)=\begin{cases}1-\dfrac{\rho}{\rho+1}e^{-\delta/\sigma_x}, & \text{if } \delta\geq 0\\ \dfrac{1}{\rho+1}e^{\delta/\sigma_y}, & \text{if } \delta<0\end{cases},$$

where, $\rho=\sigma_x/\sigma_y$, and $\delta=\mu_y-\mu_x$.

Assume independent random samples $X_1,X_2,\ldots,X_m$ and $Y_1,Y_2,\ldots,Y_n$ are drawn from $f_X(x)$ and $f_Y(y)$, respectively.

Let $X_{(1)},X_{(2)},\ldots,X_{(m)}$ and $Y_{(1)},Y_{(2)},\ldots,Y_{(n)}$ be the corresponding ordered statistics. Then, from Johnson, et al.(1994), $X_{(1)}$ and $Y_{(1)}$ are complete sufficient statistics for μ_x and μ_y, respectively when σ_x and σ_y are known.

An unbiased estimator of R=P(Y>X) is given by:

$$Z\equiv\begin{cases}1, & \text{if } X_1<Y_1\\ 0, & elsewhere\end{cases}$$

And using Lehmann-Scheffe Theorem in Rohatgi(1976), the UMVUE of R=P(Y>X) is

$$\begin{aligned}E(Z\mid X_{(1)},Y_{(1)})&=P(X_1<Y_1\mid X_{(1)},Y_{(1)})\\&=P(X_1-X_{(1)}<Y_1-Y_{(1)}+D\mid X_{(1)},Y_{(1)})\\&=P(X_1-X_{(1)}<Y_1-Y_{(1)}+D\mid D),\quad D\equiv Y_{(1)}-X_{(1)},\end{aligned}$$

which is a function of D, σ_x, and σ_y , since the distributions of $X_1-X_{(1)}$

and $Y_1 - Y_{(1)}$ don't involve μ_x and μ_y, respectively. Hence an unbiased estimator of R=P(Y>X) based on D will be its UMVUE when σ_x and σ_y are known.

Theorem 1. Let $U \equiv \begin{cases} \dfrac{(n-1)(\sigma_x + m\sigma_y)}{mn(\sigma_x+\sigma_y)} e^{D/\sigma_y}, & \text{if } D < 0 \\ 1 - \dfrac{(m-1)(n\sigma_x+\sigma_y)}{mn(\sigma_x+\sigma_y)} e^{-D/\sigma_x}, & \text{if } D > 0 \end{cases}$.

Then U is the UMVUE of R=P(Y>X) only when σ_x and σ_y are known.

Proof. From the result of Ali, et al.(2004), the pdf of D is given by:

$$f_D(d) = \begin{cases} \dfrac{mn}{n\sigma_x + m\sigma_y} e^{-m(\delta-d)/\sigma_x}, & \text{if } d < \delta \\ \dfrac{mn}{n\sigma_x + m\sigma_y} e^{-n(d-\delta)/\sigma_y}, & \text{if } d \geq \delta \end{cases}.$$

From the pdf of D and Lehmann-Scheffe Theorem, it's sufficient for us to show that an statistics U is an unbiased estimator.

For $\delta \geq 0$,

$$\begin{aligned} E(U) = & \frac{(n-1)(\sigma_x + m\sigma_y)}{mn(\sigma_x+\sigma_y)} \int_{-\infty}^{0} e^{d/\sigma_y} \frac{mn}{n\sigma_x + m\sigma_y} e^{-m\delta/\sigma_x + md/\sigma_x} dd + \\ & \int_0^{\delta} \frac{mn}{n\sigma_x + m\sigma_y} e^{-m\delta/\sigma_x + md/\sigma_x} dd + \int_{\delta}^{\infty} \frac{mn}{n\sigma_x + m\sigma_y} e^{-nd/\sigma_y + n\delta/\sigma_y} dd - \\ & \frac{(m-1)(n\sigma_x + \sigma_y)}{mn(\sigma_x+\sigma_y)} \Big[\frac{mn}{n\sigma_x + m\sigma_y} \int_0^{\delta} e^{-d/\sigma_x - m\delta/\sigma_x + md/\sigma_x} dd + \\ & \frac{mn}{n\sigma_x + m\sigma_y} \int_{\delta}^{\infty} e^{-d/\sigma_x - nd/\sigma_y + n\delta/\sigma_y} dd \Big] \end{aligned}$$

By exponential integrals, we can obtain the expectation of U:

$$E(U) = 1 - \frac{\sigma_x}{\sigma_x + \sigma_y} e^{-\delta/\sigma_x}, \quad \delta \geq 0,$$

Similarly for $\delta < 0$, $E(U) = \dfrac{\sigma_y}{\sigma_x + \sigma_y} e^{\delta/\sigma_y}$, and hence, the statistics D is an unbiased estimator of R=P(Y>X). And hence the statistics U is the UMVUE of R=P(Y>X).

Let $R^k \equiv [P(Y>X)]^k = \begin{cases} \dfrac{\sigma_y^k}{(\sigma_x+\sigma_y)^k} \cdot e^{k\delta/\sigma_y}, & \text{if } \delta<0 \\ \sum_{i=0}^{k}(-1)^i \binom{k}{i} (\dfrac{\sigma_x}{\sigma_x+\sigma_y})^i \cdot e^{-i\delta/\sigma_x}, & \text{if } \delta \geq 0 \end{cases}$,

where k is less a positive integer than both sample sizes m and n.

Now when σ_x and σ_y are known, we shall consider the UMVUE of R^k for $k<\min(m, n)$.

An unbiased estimator of R^k is given by

$$Z_k \equiv \begin{cases} 1, \text{ if } X_1<Y_1, X_2<Y_2, \ldots, X_k<Y_k \\ 0, \quad else \end{cases} .$$

From Lehmann-Scheffe Theorem in Rohatgi(1976), the UMVUE of R^k is

$$\begin{aligned} E(Z_k \mid X_{(1)}, Y_{(1)}) &= P(X_1<Y_1, X_2<Y_2, \ldots, X_k<Y_k \mid X_{(1)}, Y_{(1)}) \\ &= P(X_1-X_{(1)}<Y_1-Y_{(1)}+D, X_2-X_{(1)}<Y_2-Y_{(1)}+D, \ldots, \\ &\quad X_k-X_{(1)}<Y_k-Y_{(1)}+D \mid D) , \end{aligned}$$

which will be a function of D, σ_x, and σ_y , since the distributions of $X_i-X_{(1)}$ and $Y_i-Y_{(1)}$ don't involve μ_x and μ_y, respectively.

Therefore, by Lehmann-Scheffe Theorem in Rohatgi(1976), an unbiased estimator of $R^k=[P(Y>X)]^k$ which is a function of D will be its UMVUE of R^k when σ_x and σ_y are known.

Theorem 2. Let

$$U_k \equiv \begin{cases} \dfrac{(n-k)(k\sigma_x+m\sigma_y)\sigma_y^{k-1}}{mn(\sigma_x+\sigma_y)^k} \cdot e^{kD/\sigma_y}, & \text{if } D<0 \\ \sum_{i=0}^{k}(-1)^i \binom{k}{i} \dfrac{(m-i)(n\sigma_x+i\sigma_y)\sigma_x^{i-1}}{mn(\sigma_x+\sigma_y)^i} \cdot e^{-iD/\sigma_x}, & \text{if } D \geq 0 \end{cases} .$$

If k is less a positive integer than both sample sizes m and n, and σ_x and σ_y are known, then U_k is an unbiased estimator and hence UMVUE of $R^k=[P(Y>X)]^k$.

Proof. For $\delta<0$,

$$E(U_k)=\frac{(n-k)(k\sigma_x+m\sigma_y)\sigma_y^{k-1}}{mn(\sigma_x+\sigma_y)^k}\Big[\int_{-\infty}^{\delta}\frac{mn}{n\sigma_x+m\sigma_y}e^{-m\delta/\sigma_x+d(k/\sigma_y+m/\sigma_x)}dd+$$
$$\int_{\delta}^{0}\frac{mn}{n\sigma_x+m\sigma_y}e^{n\delta/\sigma_y-d(n/\sigma_y-k/\sigma_y)}dd\Big]+\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(m-i)(n\sigma_x+i\sigma_y)\sigma_x^{i-1}}{mn(\sigma_x+\sigma_y)^i}\cdot$$
$$\int_{0}^{\infty}\frac{mn}{n\sigma_x+m\sigma_y}e^{n\delta/\sigma_y-(n/\sigma_y+i/\sigma_x)d}dd\ .$$

By exponential integrals,

$$E(U_k)=[P(Y>X)]^k+B\cdot\frac{e^{n\delta/\sigma_y}}{n\sigma_x+m\sigma_y}\sigma_y\ ,$$

where

$$B\equiv\frac{1}{(\sigma_x+\sigma_y)^k}\Big[\sum_{i=0}^{k}(-1)^i\binom{k}{i}(m-i)\sigma_x^i(\sigma_x+\sigma_y)^{k-i}-\sigma_y^{k-1}(k\sigma_x+m\sigma_y)\Big]\ .$$

From definition of B, coefficient of σ_y^k is zero, "coefficient of $\sigma_x\sigma_y^{k-1}$" is

$$\binom{k}{0}m\binom{k}{k-1}-\binom{k}{1}(m-1)\binom{k-1}{k-1}-k=0\ ,$$

and coefficient of $\sigma_x^{k-t}\cdot\sigma_y^t$, $(t<k-1)$ is given by

$$\sum_{i=0}^{k-t}(-1)^i\binom{k}{i}\binom{k-i}{t}(m-i)=-\frac{k!}{t!(k-t-1)!}\sum_{j=0}^{k-t-1}(-1)^j\binom{k-t-1}{j}=0\ .$$

Therefore, $E(U_k)=[P(Y>X)]^k$ when $\delta<0$.
For $\delta>0$,

$$E(U_k)=\frac{(n-k)(k\sigma_x+m\sigma_y)\sigma_y^{k-1}}{mn(\sigma_x+\sigma_y)^k}\int_{-\infty}^{0}\frac{mn}{n\sigma_x+m\sigma_y}e^{-m\delta/\sigma_x+(k/\sigma_y+m/\sigma_x)d}dd+$$
$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(m-i)(n\sigma_x+i\sigma_y)\sigma_x^{i-1}}{mn(\sigma_x+\sigma_y)^i}\Big[\int_{0}^{\delta}\frac{mn}{n\sigma_x+m\sigma_y}e^{-m\delta/\sigma_x+(m-i)d/\sigma_x}dd+$$
$$\frac{mn}{n\sigma_x+m\sigma_y}\int_{\delta}^{\infty}e^{n\delta/\sigma_y-(n/\sigma_y+i/\sigma_x)d}dd\Big],$$

By exponential integrals,

$$E(U_k)=[P(Y>X)]^k+B\cdot\frac{e^{-m\delta/\sigma_x}\cdot\sigma_x}{(n\sigma_x+m\sigma_y)(\sigma_x+\sigma_y)^k}\ ,$$

where $B \equiv (n-k)\sigma_y^k - \sum_{i=0}^{k}(-1)^i \binom{k}{i}(n\sigma_x + i\sigma_y)(\sigma_x+\sigma_y)^{k-i}\sigma_x^{i-1}$.

From the definition of B, similarly B=0 can be shown, and hence

$$E(U_k) = [P(Y > X)]^k \quad \text{when} \quad \delta \geq 0 \ .$$

Therefore , U_k is an unbiased estimator of $R^k = [P(Y > X)]^k$ and hence it's the UMVUE of $R^k = [P(Y > X)]^k$. Q.E.D.

To get the UMVUE of $Var(U) = E(U^2) - E^2(U)$,

$$\widehat{Var(U)} = U^2 - \widehat{R^2} \text{ , and}$$

from Theorem 2, the UMVUE of R^2 is given by

$$\widehat{R^2} = U_2 = \frac{(n-2)(2\sigma_x + m\sigma_y)\sigma_y}{mn(\sigma_x+\sigma_y)^2} \cdot e^{2D/\sigma_y}, \quad \text{if } D < 0$$

$$1 - \frac{2(m-1)(n\sigma_x+\sigma_y)}{mn(\sigma_x+\sigma_y)} e^{-D/\sigma_x} + \frac{(m-2)(n\sigma_x+2\sigma_y)\sigma_x}{mn(\sigma_x+\sigma_y)^2} e^{-2D/\sigma_x}, \quad \text{if } D \geq 0$$

Clearly, $0 < U_2 < 1$, for m, n>2.

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