Journal of Korean
Data & Information Science Society
2006, Vol. 17, No. 1, pp. 233 ~ 244

Bayesian Multiple Comparison of Binomial Populations based on Fractional Bayes Factor

Dal Ho Kim[1)]· Sang Gil Kang[2)]· Woo Dong Lee[3)]

Abstract

In this paper, we develop the Bayesian multiple comparisons procedure for the binomial distribution. We suggest the Bayesian procedure based on fractional Bayes factor when noninformative priors are applied for the parameters.

An example is illustrated for the proposed method. For this example, the suggested method is straightforward for specifying distributionally and to implement computationally, with output readily adapted for required comparison. Also, some simulation was performed.

Keywords : Bayesian Multiple Comparison, Fractional Bayes Factor, Noninformative Priors, Posterior Probability

1. Introduction

The primary objective of this paper is to provide a Bayesian multiple comparison procedure (MCP) based on the fractional Bayes factor for binomial populations when noninformative priors are used. It is well known that binomial distribution is widely used parametric distribution in many areas.

Related with testing equality of two independent binomial proportions, classical tests such as exact test or approximate Z-test are widely used. But the test of equality of proportions more than three populations relies on likelihood ratio test statistic which is distributed as approximately χ^2 distribution. And classical tests

1) First Author : Department of Statistics, Kyungpook National University, Taegu, 702-701, Korea
E-mail : dalkim@knu.ac.kr

2) Department of Applied Statistics, Sangji University, Wonju, 220-702, Korea

3) Corresponding Author : Department of Asset Management, Daegu Hanny University, Kyungsan, 712-715, Korea
E-mail : wdlee@오u.ac.kr

only decide whether the null hypothesis, commonly the equality of proportions, will be rejected or not. When the null hypothesis is rejected, there remains a problem which hypothesis is best for describing the equality of parameters. And the researches for multiple comparison more than three binomial populations are rare. So, we want to propose a Bayesian MCP for this problem.

Bayesian MCP selects the model with the highest posterior probability. The posterior probabilities are calculated using the Bayes factors. And we can compute all the posterior probabilities of the hypotheses under consideration.

It is well known that the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constrains often forces the use of noninformative priors. Since noninformative priors such as Jeffrey's priors or reference priors are typically improper so that such priors are only up to arbitrary constants which affects the values of Bayes factors.

Many people have made efforts to compensate for that arbitrariness (O'Hagan 1995, Berger and Pericchi 1996).

Among the many people, Berger and Pericchi (1996) introduced the intrinsic Bayes factor (IBF) using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor (FBF). For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in several statistical areas.

In developing the Bayesian MCP, we will suggest a method based on FBF rather than IBF. In MCP for a reasonable number of populations, the use of IBF encountered some of difficulties as follows. Firstly, there is a difficulty in recognizing which is the more complex model, and some models having the same level of complexity. This fact concerns with the stability problem. Secondly, the IBF does not have multiple model coherence. Finally, it takes much time to compute the IBF because it averages out all possible outcomes of the minimal training sample.

The outline of the remaining sections is as follows. In Section 2, we review the concept of the FBF methodology and develop the Bayesian MCP. In Section 3, we derive expressions of the Bayesian MCP for several binomial populations. And we give some real examples to illustrate our procedure. Finally, we give some numerical examples. From these results, our Bayesian MCP based on FBF very well select the target model.

2. The Bayesian Multiple Comparisons Procedure Using Fractional Bayes Factor

Models (or Hypotheses) M_1, M_2, $\cdots$, M_q are under consideration, with the data

$\boldsymbol{X} = (X_1, X_2, \cdots, X_n)$ having probability density function (pdf) $f_i(\boldsymbol{x} \mid \theta_i)$ under model $M_i, i = 1, 2, \cdots, q$. The parameter vectors θ_i are unknown. Let $\pi_i(\theta_i)$ be the prior distribution of model M_i, and let p_i be the prior probabilities of model M_i, $i = 1, 2, \cdots, q$. Then the posterior probability that the model M_i is true is

$$P(M_i \mid \boldsymbol{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji}\right)^{-1},$$

where B_{ji} is the Bayes factor of model M_j to model M_i defined by

$$B_{ji} = \frac{m_j(\boldsymbol{x})}{m_i(\boldsymbol{x})} = \frac{\int f_j(\boldsymbol{x} \mid \theta_j)\pi_j(\theta_j)\,d\theta_j}{\int f_i(\boldsymbol{x} \mid \theta_i)\pi_i(\theta_i)\,d\theta_i}. \tag{1}$$

The B_{ji} interpreted as the comparative support of the data for the model j to i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Usually, one can use the noninformative prior, often improper, for parameters such as uniform prior, Jeffreys prior, reference prior or probability matching prior. Denote it as π_i^N. The use of improper priors $\pi_i^N(\cdot)$ in (1) causes the B_{ji} to contain arbitrary constants.

To solve this problem, O'Hagan (1995) proposed the fractional Bayes factor for Bayesian testing and model selection problem as follow.

When the $\pi_i^N(\theta_i)$ is noninformative prior under M_i, equation (1) becomes

$$B_{ji}^N(\boldsymbol{x}) = \frac{\int f_j(\boldsymbol{x} \mid \theta_j)\pi_j^N(\theta_j)\,d\theta_j}{\int f_i(\boldsymbol{x} \mid \theta_i)\pi_i^N(\theta_i)\,d\theta_i}.$$

Then the fraction Bayes factor (FBF) of model M_j versus model M_i is

$$B_{ji}^F = \frac{q_j(b, \boldsymbol{x})}{q_i(b, \boldsymbol{x})},$$

where

$$q_i(b, \boldsymbol{x}) = \frac{\int f_i(\boldsymbol{x} \mid \theta_i)\pi_i^N(\theta_i)\,d\theta_i}{\int f_i^b(\boldsymbol{x} \mid \theta_i)\pi_i^N(\theta_i)\,d\theta_i},$$

and $f_i(\boldsymbol{x} \mid \theta_i)$ is the likelihood function and b specifies a fraction of the likelihood which is to be used as a prior density. He proposed three ways for the choice of the fraction b. That is, (a) $b = m/n$, when robustness is no concern, (b) $b = n^{-1}\max\{m, \sqrt{n}\}$ when robustness is a serious concern, and (c) $b=n^{-1}\max\{n, \log n\}$, as an intermediate option. One frequently suggested choice is $b = m/n$, where m is the size of the minimal training sample (MTS), assuming this is well defined. (see O'Hagan, 1995 and the discussion by Berger and Mortera of O'Hagan, 1995).

Consider k populations with parameters $\theta = (\theta_1, \cdots, \theta_k)^T$. Let $\boldsymbol{X}_i = (x_{i1}, \cdots, x_{in_i})^T$ be a $n_i \times 1$ vector of independent observations on θ_i with density $f(x_{ij} \mid \theta_i)$, $i = 1,\cdots,k$, $j=1,\cdots,n_i$. Then the likelihood function for θ given $\boldsymbol{X} = (\boldsymbol{X}_1, \cdots, \boldsymbol{X}_k)$ is

$$L(\theta \mid \boldsymbol{x}) = \prod_{i=1}^{k}\prod_{j=1}^{n_i} f(x_{ij} \mid \theta_i).$$

The MCP of k populations is to make inferences concerning relationships among the θ_i's based on $\boldsymbol{X}$.

Let $\Omega = \{\theta = (\theta_1, \cdots, \theta_k) \mid \theta_i \in R, i = 1, 2, \cdots, k\}$ be the k-dimensional parameter space. Equality and inequality relationships among the θ_i's induce statistical hypotheses that subsets of Ω. Say, $M_0 : \Omega_0 = \{\theta_i \mid \theta_1 = \cdots = \theta_k\}$, $M_1 : \Omega_1 = \{\theta_i \mid \theta_1 \neq \theta_2 = \cdots = \theta_k\}$, and so on up to $M_Q : \Omega_Q = \{\theta_i \mid \theta_1 \neq \cdots \neq \theta_k\}$. The hypotheses $(M_r : \Omega_r;\ r=1,\cdots,Q)$, are disjoint, and $\Omega = \cup_{r=1}^{Q} \Omega_r$.

Each hypothesis can classified $r (r = 1, \cdots, k)$ distinct groups. Let $(\theta_1^*, \cdots, \theta_r^*)$ denote the set of distinct θ_i's, where r is the number of distinct elements in the vector Ω. We define the configuration notation.

Definition 1 (*Configuration*). The configuration $S = \{S_1, \cdots, S_k\}$ determines a classification of θ into r distinct groups. Write I_j for the set of indices of parameters in group j, $I_j = \{i \mid S_i = j\}$. Let $n_{I_j} = \{n_i \mid i \in I_j\}$ be the index set of observations and θ_j^* be the common parameter value for I_j.

There is a one-to-one correspondence between hypotheses and configurations. Therefore the Bayes factor for MCP can easily compute by this configuration. As an illustration, let $k=5$ and $S=\{1,2,1,2,3\}$. Then $r=3$, $I_1=\{1,3\}$, θ_1^*, $n_{I_1}=\{n_1, n_3\}$, $I_2=\{2,4\}$, θ_2^*, $n_{I_2}=\{n_2,n_4\}$, $I_3=\{5\}$, θ_3^* and $n_{I_3}=\{n_5\}$. And the noninformative prior for a model with r distinct groups denoted by $\pi_r^N(\theta_1^*,\cdots,\theta_r^*)$.

Now we will develop Bayesian multiple comparisons procedure using fractional Bayes factor. Suppose that a model classified r distinct groups. Then the likelihood function is given by

$$L(\theta_1^*,\cdots,\theta_r^* \mid \boldsymbol{x}) = \prod_{t=1}^{r}\prod_{\{i:i\in I_t\}}\prod_{j\in n_{I_i}} f(x_{ij}\mid\theta_t).$$

And the noninformative prior for the model is $\pi_r^N(\theta_1^*,\cdots,\theta_r^*)$. Thus the element of the FBF is given by

$$q(b,\boldsymbol{x}) = \frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} L(\theta_1^*,\cdots,\theta_r^* \mid \boldsymbol{x})\pi_r^N(\theta_1^*,\cdots,\theta_r^*)\,d\theta_1^*\cdots d\theta_r^*}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} L^b(\theta_1^*,\cdots,\theta_r^* \mid \boldsymbol{x})\pi_r^N(\theta_1^*,\cdots,\theta_r^*)\,d\theta_1^*\cdots d\theta_r^*}.$$

Thus if a model M_i classified r_i distinct groups and a model M_j classified r_j distinct groups then the FBF of M_j versus M_i is given by

$$B_{ji}^F(\boldsymbol{x}) = \frac{q_j(b,\boldsymbol{x})}{q_i(b,\boldsymbol{x})},$$

where

$$q_i(b,\boldsymbol{x}) = \frac{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} L(\theta_1^*,\cdots,\theta_{r_1}^* \mid \boldsymbol{x})\pi_{r_1}^N(\theta_1^*,\cdots,\theta_{r_1}^*)\,d\theta_1^*\cdots d\theta_{r_1}^*}{\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} L^b(\theta_1^*,\cdots,\theta_{r_1}^* \mid \boldsymbol{x})\pi_{r_1}^N(\theta_1^*,\cdots,\theta_{r_1}^*)\,d\theta_1^*\cdots d\theta_{r_1}^*}. \tag{2}$$

Hence the FBF for all comparisons can computed by equation (2). Using these FBF, we calculate the posterior probability for model M_i, $i=1,\cdots Q$. Thus for MCP, we select the hypothesis with highest posterior probability. Note that as the number k of populations increase, the number of hypotheses increases exponentially. The number of hypotheses is given by the Bell exponential number B_m (see Berge 1971). The sequence B_m can be generated by the recursion

$$B_{m+1} = \sum_{i=0}^{m}\binom{m}{i} B_i,\ m=0,1,\cdots$$

where $B_0=1$. When the number of populations under consideration is k, then

$Q = B_k$ for $k \geq 2$.

For a reasonably moderate number of treatments, such as 8 and 9, the number of hypotheses to be considered is 4,140 and 21,147, respectively. These numbers are very large. But our developed procedure in this section runs quickly and given a correct results. However, for a large number of treatments $k \geq 10$, the proposed procedure needs much time on our computer with Pentium IV processor.

3. The Binomial Sampling

Let $\boldsymbol{X}_i$ be an independent sample from a binomial with parameters n_i and θ_i. Suppose that a model M_i classified r distinct groups. Then the noninformative prior for $\theta_i^*, \cdots, \theta_r^*$ is

$$\pi_r(\theta_1^*, \cdots, \theta_r^*) = \frac{1}{\sqrt{\theta_1^*(1-\theta_1^*) \cdots \theta_r^*(1-\theta_r^*)}}, 0 \leq \theta_1^* \leq 1, \cdots, 0 \leq \theta_r^* \leq 1.$$

The likelihood function is

$$L(\theta_1^*, \cdots, \theta_r^* \mid \boldsymbol{x}) = S_1 \cdots S_k \prod_{t=1}^{r} (\theta_t^*)^{\sum_{i \in I_t} x_i} (1-\theta_t^*)^{\sum_{i \in I_t} (n_i - x_i)},$$

where $S_i = \binom{n_i}{x_i}$, $i = 1, \cdots, k$.

Then the elements of FBF are given by

$$\int_0^1 \cdots \int_0^1 L(\theta_1^*, \cdots, \theta_r^* \mid \boldsymbol{x}) \pi_r(\theta_1^*, \cdots, \theta_r^*) d\theta_1^* \cdots d\theta_r^*$$

$$= S_1 \cdots S_k \prod_{t=1}^{r} \frac{\Gamma(\frac{1}{2} + \sum_{i \in I_t} x_i) \Gamma(\frac{1}{2} + \sum_{i \in I_t} (n_i - x_i))}{\Gamma(1 + \sum_{i \in I_t} n_i)},$$

and

$$\int_0^1 \cdots \int_0^1 L^b(\theta_1^*, \cdots, \theta_r^* \mid \boldsymbol{x}) \pi_r(\theta_1^*, \cdots, \theta_r^*) d\theta_1^* \cdots d\theta_r^*$$

$$= (S_1 \cdots S_k)^b \prod_{t=1}^{r} \frac{\Gamma(\frac{1}{2} + \sum_{i \in I_t} b x_i) \Gamma(\frac{1}{2} + \sum_{i \in I_t} b(n_i - x_i))}{\Gamma(1 + \sum_{i \in I_t} b n_i)}.$$

Thus

$$q(b,\boldsymbol{x}) = \frac{S_1 \cdots S_k}{(S_1 \cdots S_k)^b} \times \prod_{t=1}^{r} \frac{\Gamma(\frac{1}{2} + \sum_{i \in I_t} x_i)\Gamma(\frac{1}{2} + \sum_{i \in I_t} (n_i - x_i))\Gamma(1 + \sum_{i \in I_t} bn_i)}{\Gamma(\frac{1}{2} + \sum_{i \in I_t} bx_i)\Gamma(\frac{1}{2} + \sum_{i \in I_t} b(n_i - x_i))\Gamma(1 + \sum_{i \in I_t} n_i)}.$$

Therefore if a model M_i classified r_i distinct groups and a model M_j classified r_j distinct groups then the FBF of M_j versus M_i is given by

$$B_{ji}^F(\boldsymbol{x}) = \prod_{t=1}^{r_j} \frac{\Gamma(\frac{1}{2} + \sum_{i \in I_t} x_i)\Gamma(\frac{1}{2} + \sum_{i \in I_t} (n_i - x_i))\Gamma(1 + \sum_{i \in I_t} bn_i)}{\Gamma(\frac{1}{2} + \sum_{i \in I_t} bx_i)\Gamma(\frac{1}{2} + \sum_{i \in I_t} b(n_i - x_i))\Gamma(1 + \sum_{i \in I_t} n_i)} \times \prod_{t=1}^{r_i} \frac{\Gamma(\frac{1}{2} + \sum_{i \in I_t} bx_i)\Gamma(\frac{1}{2} + \sum_{i \in I_t} b(n_i - x_i))\Gamma(1 + \sum_{i \in I_t} n_i)}{\Gamma(\frac{1}{2} + \sum_{i \in I_t} x_i)\Gamma(\frac{1}{2} + \sum_{i \in I_t} (n_i - x_i))\Gamma(1 + \sum_{i \in I_t} bn_i)}.$$

Example. To investigate the effect of planting longleaf and slash pine seedlings 1/2 inch too high or too deep in winter on their mortality in following fall, an experiment was conducted involving 200 pine seedlings of both types, 100 of which were planted too high and 100 too deep. Because the number of plants for each combination of Seeding Type (LS=Longleaf Seedling, SS=Slash Seedling) and Depth of Planting (DH=Depth too High, DL=Depth too Low) was fixed by design, we have four binomial experiments each of size $n_i = 100, i = 1, 2, 3, 4$, corresponding to the four combinations (LS, DH), (LS, DL), (SS, DH) and (SS, DL). The data are reported in the Table 1. (For more details on this experiment and analysis of the data, see Fienberg 1980, Consonni and Veronese 1995).

<Table 1> Mortality of Pine Seedlings Data

	LS DH	LS DL	SS DH	SS DL
Experiment	1	2	3	4
x_i	59	89	88	95
$\hat{\theta}_i$	0.59	0.89	0.88	0.95

We assume that the prior probabilities are equal. Then Table 2 gives the posterior probabilities for hypotheses.

<Table 2> Posterior Probabilities for Hypotheses

Hypothesis	Posterior Probability	Hypothesis	Posterior Probability
1 1 1 1	0.0000	1 2 2 1	0.0000
1 1 1 2	0.0000	1 2 2 2	0.4015
1 1 2 1	0.0000	1 2 2 3	0.3639
1 1 2 2	0.0000	1 2 3 1	0.0000
1 1 2 3	0.0000	1 2 3 2	0.1051
1 2 1 1	0.0000	1 2 3 3	0.0731
1 2 1 2	0.0000	1 2 3 4	0.0564
1 2 1 3	0.0000		

The number of possible hypotheses for the MCP is 15. The hypothesis 1 2 2 2 $(\theta_1 \neq \theta_2 = \theta_3 = \theta_4)$ has the largest posterior probability. Next the hypothesis $(\theta_1 \neq \theta_2 = \theta_3 \neq \theta_4)$ has the second largest posterior probability.

For the problem of combining information related to k binomial experiments, Consonni and Veronese (1995) considered a partition of the experiments and take the θ_i's belonging to the same partition subset to be exchangeable (partial exchangeability) and the θ_i's belonging to distinct subsets to be independent. They revealed the hypothesis $(\theta_1 \neq \theta_2 = \theta_3 = \theta_4)$ has the highest posterior probability for the above data. But they analyzed the posterior probability of a selected collection of partitions that are most supported by data.

4. Monte Carlo Simulation Studies

We examine whether the our procedure for MCP work well. Although all configurations for $k \geq 3$ populations are considered, we examine our procedure for the MCP under some population and configuration to save the space. We consider $k=5$ populations. And, for simplicity, we only assume two configurations, $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5$ and $\theta_1 = \theta_2 = \theta_3 \neq \theta_4 = \theta_5$ for 5 binomial populations. We consider that $\boldsymbol{X} = (\boldsymbol{X}_1, \cdots, \boldsymbol{X}_5)$ be a set of independent sample, where $\boldsymbol{X}_i$ is a sample from a binomial distribution with parameter θ_i. Let the first true configuration $\theta_1 = \theta_2 = \theta_3 = \theta_4 = \theta_5 = 0.1$ and the second true configuration $\theta_1 = \theta_2 = \theta_3 = 0.1, \theta_4 = \theta_5 = 0.2$ with the sample sizes $n_1 = \cdots = n_5 = n = 30, 50, 100$. And we assume that the prior probabilities are equal. Under 1,000 replications, Table 3 and 4 give the posterior probabilities for hypotheses.

From the results, our procedure work well for small and moderate sample sizes. From the all simulation results for the considered models, our procedure for MCP always select the true hypothesis from small sample size to moderate sample size. Though we do not report simulation results for the other configurations, the number of populations, and sample sizes, we verified that our proposed procedure worked very well.

5. Conclusions

We have considered the problem of developing a Bayesian MCP for binomial populations. We proposed the Bayesian MCP based on fraction Bayes factor when the noninformative prior is used.

The suggested Bayesian MCP allows for probability calculations of hypotheses of equality and inequality under the moderate number of populations and gives a correct results.

As some application of our procedure, we can apply our method to calculating the posterior probabilities related with the Bayesian model averaging, variable selection of regression and Bayesian analysis of mixtures with an unknown number of components under parametric models.

<Table 3> Posterior Probabilities for Hypotheses

n_i	Hypothesis	Post. Prob.	Hypothesis	Post. Prob.	Hypothesis	Post. Prob.	Hypothesis	Post. Prob.
30	11111	0.0802	11233	0.0135	12212	0.0306	12321	0.0147
	11112	0.0334	11234	0.0063	12213	0.0132	12322	0.0151
	11121	0.0340	12111	0.0358	12221	0.0331	12323	0.0144
	11122	0.0306	12112	0.0319	12222	0.0353	12324	0.0066
	11123	0.0141	12113	0.0150	12223	0.0152	12331	0.0148
	11211	0.0348	12121	0.0329	12231	0.0144	12332	0.0145
	11212	0.0316	12122	0.0318	12232	0.0146	12333	0.0153
	11213	0.0145	12123	0.0140	12233	0.0140	12334	0.0067
	11221	0.0330	12131	0.0152	12234	0.0064	12341	0.0068
	11222	0.0310	12132	0.0138	12311	0.0153	12342	0.0065
	11223	0.0140	12133	0.0141	12312	0.0138	12343	0.0065
	11231	0.0152	12134	0.0064	12313	0.0139	12344	0.0065
	11232	0.0135	12211	0.0317	12314	0.0064	12345	0.0030
50	11111	0.1117	11233	0.0116	12212	0.0332	12321	0.0125
	11112	0.0361	11234	0.0042	12213	0.0112	12322	0.0130
	11121	0.0372	12111	0.0394	12221	0.0364	12323	0.0121
	11122	0.0337	12112	0.0344	12222	0.0389	12324	0.0044
	11123	0.0121	12113	0.0127	12223	0.0130	12331	0.0129
	11211	0.0378	12121	0.0352	12231	0.0122	12332	0.0124
	11212	0.0339	12122	0.0348	12232	0.0126	12333	0.0134
	11213	0.0123	12123	0.0118	12233	0.0122	12334	0.0045
	11221	0.0360	12131	0.0129	12234	0.0043	12341	0.0045
	11222	0.0339	12132	0.0117	12311	0.0132	12342	0.0043
	11223	0.0118	12133	0.0122	12312	0.0116	12343	0.0043
	11231	0.0129	12134	0.0043	12313	0.0117	12344	0.0045
	11232	0.0115	12211	0.0347	12314	0.0042	12345	0.0016
100	11111	0.1642	11233	0.0090	12212	0.0360	12321	0.0096
	11112	0.0379	11234	0.0023	12213	0.0087	12322	0.0101
	11121	0.0396	12111	0.0418	12221	0.0386	12323	0.0094
	11122	0.0359	12112	0.0371	12222	0.0416	12324	0.0024
	11123	0.0091	12113	0.0097	12223	0.0099	12331	0.0098
	11211	0.0408	12121	0.0376	12231	0.0094	12332	0.0097
	11212	0.0368	12122	0.0369	12232	0.0098	12333	0.0103
	11213	0.0095	12123	0.0089	12233	0.0094	12334	0.0025
	11221	0.0389	12131	0.0099	12234	0.0024	12341	0.0025
	11222	0.0364	12132	0.0090	12311	0.0103	12342	0.0024
	11223	0.0091	12133	0.0093	12312	0.0092	12343	0.0024
	11231	0.0101	12134	0.0023	12313	0.0092	12344	0.0025
	11232	0.0090	12211	0.0375	12314	0.0024	12345	0.0006

<Table 4> Posterior Probabilities for Hypotheses

n_i	Hypothesis	Post. Prob.	Hypothesis	Post. Prob.	Hypothesis	Post. Prob.	Hypothesis	Post. Prob.
	11111	0.0528	11233	0.0267	12212	0.0197	12321	0.0104
	11112	0.0318	11234	0.0108	12213	0.0159	12322	0.0158
	11121	0.0325	12111	0.0311	12221	0.0214	12323	0.0106
	11122	0.0621	12112	0.0202	12222	0.0279	12324	0.0073
	11123	0.0248	12113	0.0157	12223	0.0150	12331	0.0116
	11211	0.0275	12121	0.0201	12231	0.0166	12332	0.0107
30	11212	0.0208	12122	0.0341	12232	0.0151	12333	0.0174
	11213	0.0151	12123	0.0154	12233	0.0270	12334	0.0078
	11221	0.0209	12131	0.0157	12234	0.0108	12341	0.0076
	11222	0.0362	12132	0.0158	12311	0.0173	12342	0.0073
	11223	0.0163	12133	0.0275	12312	0.0101	12343	0.0077
	11231	0.0148	12134	0.0109	12313	0.0112	12344	0.0123
	11232	0.0164	12211	0.0369	12314	0.0076	12345	0.0049
	11111	0.0533	11233	0.0332	12212	0.0158	12321	0.0072
	11112	0.0346	11234	0.0105	12213	0.0158	12322	0.0136
	11121	0.0353	12111	0.0287	12221	0.0173	12323	0.0075
	11122	0.1005	12112	0.0162	12222	0.0256	12324	0.0055
	11123	0.0312	12113	0.0138	12223	0.0131	12331	0.0083
	11211	0.0245	12121	0.0160	12231	0.0159	12332	0.0076
50	11212	0.0168	12122	0.0388	12232	0.0137	12333	0.0154
	11213	0.0136	12123	0.0151	12233	0.0340	12334	0.0058
	11221	0.0169	12131	0.0137	12234	0.0106	12341	0.0055
	11222	0.0410	12132	0.0152	12311	0.0152	12342	0.0053
	11223	0.0160	12133	0.0338	12312	0.0069	12343	0.0059
	11231	0.0124	12134	0.0104	12313	0.0080	12344	0.0119
	11232	0.0159	12211	0.0417	12314	0.0058	12345	0.0037
	11111	0.0358	11233	0.0514	12212	0.0080	12321	0.0034
	11112	0.0308	11234	0.0112	12213	0.0131	12322	0.0107
	11121	0.0313	12111	0.0166	12221	0.0084	12323	0.0033
	11122	0.2118	12112	0.0092	12222	0.0172	12324	0.0035
	11123	0.0465	12113	0.0093	12223	0.0097	12331	0.0034
	11211	0.0171	12121	0.0077	12231	0.0142	12332	0.0037
100	11212	0.0084	12122	0.0417	12232	0.0087	12333	0.0107
	11213	0.0098	12123	0.0139	12233	0.0500	12334	0.0035
	11221	0.0084	12131	0.0099	12234	0.0108	12341	0.0036
	11222	0.0444	12132	0.0134	12311	0.0109	12342	0.0033
	11223	0.0138	12133	0.0497	12312	0.0039	12343	0.0035
	11231	0.0091	12134	0.0110	12313	0.0034	12344	0.0126
	11232	0.0146	12211	0.0436	12314	0.0035	12345	0.0027

References

1. Berge, C. (1971), *Principle of Combinatorics* , New York: Academic Press.
2. Berger, J. O. and Pericchi, L. R. (1996), The Intrinsic Bayes Factor for Model Selection and Prediction, *Journal of the American Statistical Association*, 91, 109-122.
3. Consonni, G. and Veronese, P. (1995), A Bayesian Method for Combining Results from Several Binomial Experiments, *Journal of the American Statistical Association*, 90, 935-944.
4. Fienberg, S. E. (1980). *The Analysis of Cross-Classified Categrical Data*, Cambridge, MA: The MIT Press.
5. O′ Hagan, A.(1995), Fractional Bayes Factors for Model Comparison(with discussion), *Journal of Royal Statistical Society*, 56, 99-118.

[recieved date : Jan. 2006, accepted date : Feb. 2006]