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A Test of Multivariate Normality Oriented for Testing Elliptical Symmetry

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Abstract

A chi-squared test of multivariate normality is suggested which is oriented for detecting deviations from elliptical symmetry. We derive the limiting distribution of the test statistic via a central limit theorem on empirical processes. A simulation study is conducted to study the accuracy of the limiting distribution in finite samples. Finally, we compare the power of our method with those of other popular tests of multivariate normality under a non-normal distribution.

Keywords : Central limit theorem, Chi-squared test, Empirical process

1. Introduction

Elliptically symmetric distributions are a much broader class of multivariate distributions than the multivariate normal distributions, and so they can serve as the basis for the development of more robust analyses than the standard normal-theory procedures. There is a large literature on these distributions and their use in statistics (see Fang and Anderson (1990), and Fang, Kotz, and Ng (1990)). In this paper, a chi-squared test is proposed which might be useful for testing elliptical symmetry. This test, however, is tentatively proposed as a test of multivariate normality before it is further developed into a test of elliptical symmetry in near future.

Our method can be briefly summarized as follows. We first transform the original observations into scaled residuals that have sample correlations of zero and have sample means and sample variances equal to zero and one, respectively. We next discretize the scaled residuals based on the signs of each variable of the scaled residuals and on the squared distances of the scaled residuals. In other

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words, we divide the space of the scaled residuals by quadrants and the sample quantiles of the scaled residuals so that each cell is (at least approximately) equiprobable. From these categorical variables, a multiway contingency table is formed and a chi-squared test is employed as a test of independence in the contingency table. The main idea leading to this chi-squared test of independence is that the signs and radius of each scaled residual are approximately mutually independent under multivariate normality.

Our paper is organized as follows. In Chapter 2, we explain our method in more detail, and then provide main results on the asymptotic distribution of the cell counts and on the limiting distribution of the chi-squared test under multivariate normality. In Chapter 3, we provide rigorous proofs of main results in Chapter 2. In Chapter 4, we provide a simulation study to check the accuracy of the limiting distribution of the statistic for finite sample sizes. We then provide a simulation study to compare the power of our test with those of other popular tests of multivariate normality under a non-normal distribution.

2. Main Results

First, some brief remarks on notation. We will use I, e and 0 to denote an identity matrix, a column vector of ones, and a column vector or matrix of zeros respectively. The dimensions will usually be clear from context, but will be specified by subscripts if necessary. Unless otherwise noted vectors will be column vectors, but for convenience they will be written in text as row vectors.

Let $n\times p$ data matrix $Y=(y_{ij})$ denote an original data matrix whose n rows $y_1, y_2, \cdots, y_n$ are a random sample from a p-variate normal distribution. We spherize Y to obtain a transformed matrix Z in which the variables are uncorrelated and are standardized to have zero mean and unit variance. More formally, the $n\times p$ matrix $Z=(z_{ij})$ of transformed data is defined by

$$Z= Q_e Y R(S), \tag{2.1}$$

where $Q_e = I_n - ee^t/n$, e is the n-vector of ones, and $R(S)$ is a $p\times p$ matrix chosen so that $Z^tZ/n = I_p$. We require the matrix $R(S)$ to be a function of the sample covariance matrix S defined by $S = n^{-1}Y^tQ_eY$. By Lemma 3.1 in Huffer and Park (2002), we can choose any R satisfying $R^tSR = I$ to obtain $Z^tZ/n = I$ and the distribution of Z does not depend on the parameter $\theta = (\mu, \Sigma)$ of the multivariate normal distribution.

After obtaining the transformed matrix, we discretize the transformed matrix Z based on the signs of each variable and on the squared distances of each case z_i from 0. In other words, we first use the signs of each variable to obtain 2 groups labeled ± 1. We then use the sample quantiles of the squared distances

$$\hat{d}_i^2 = z_i^t z_i = (y_i - \bar{y})^t S^{-1} (y_i - \bar{y})$$

to obtain c groups (labeled $1, 2, \cdots, c$) of equal size n/c. (If n is not divisible by c, the group size will not be exactly equal.) We now form a contingency table of $K \equiv cg = c2^p$ cells from the discretized data, where $g \equiv 2^p$ is the number of cells generated by signs of each variable. Let $\pi = (\pi_1, \pi_2, \cdots, \pi_{p+1}) = (\pi^*, \pi_{p+1})$ denote a particular cell in our table, and $U_{n\pi}$ to denote the cell count in that cell. The first part π^* correspond to the signs of each variable and π_{p+1} corresponds to the c groups of the squared distances, so that every component π_i of π^* will take ± 1 for $i = 1, 2, \cdots, p$, whereas π_{p+1} will take values $1, 2, \cdots, c$.

For two given vectors $u = (u_1, \cdots, u_p)$ and $v = (v_1, \cdots, v_p)$, we define $u \cdot v = (u_1 v_1, \cdots, u_p v_p)$ and we also use vector inequality in such a way that $u = (u_1, \cdots, u_p) > 0$ if and only if $u_i > 0$ for all $i = 1, 2, \cdots, p$. Let q_{ni} be the (i/c)-th sample quantiles of $\hat{d}_1^2, \hat{d}_2^2, \cdots, \hat{d}_n^2$. We take $q_{n0} = 0$ and $q_{nc} = \infty$. A more precise definition of the cell counts $U_{n\pi}$ is

$$U_{n\pi} = \sum_{i=1}^{n} I\left(\pi^* \cdot z_i > 0, q_{n(\pi_{p+1}-1)} < z_i^t z_i \leq q_{n\pi_{p+1}}\right). \tag{2.2}$$

We have excluded the cases where $\pi^* \cdot z_i = 0$ but they have zero probability.

Our chi-squared statistic X^2 is defined by

$$X^2 = \sum_{\pi} \frac{(U_{n\pi} - np_0)^2}{np_0} \tag{2.3}$$

where $p_0 = 1/K = 1/(cg)$ is the asymptotic probability of each observation belonging to a particular cell. This is just the usual Pearson chi-squared statistic for testing equality of probabilities of belonging to cells in a multi-way contingency table. Note that, in our situation, the 'expected' number of

observations in each cell is taken simply to be $np_0 = n/K$.

Before presenting main results, we first introduce some matrices which we need in the statements of results. Let $U_n = (U_{n\pi})$ be a $K\times 1$ vector of cell counts. For easy presentation of results, we assume that the elements of U_n are arranged in a standard way; i.e. the first coordinate changes -1 to 1 the fastest, the second coordinate changes the second fastest, and so on.

Let q_{0i} be the (i/c)-th quantiles of $\chi^2(p)$ distribution for $i = 0, 1, \cdots, c$. Note that $q_{00} = 0$, $q_{0c} = \infty$. For $i = 0, 1, \cdots, c$, define

$$\begin{aligned} a_i &= \frac{1}{2^{p-1}} \int_{S_{1i}} z_1 \phi(z) dz, \quad b_i = \frac{1}{2^p} \int_{S_{1i}} z_1^2 \phi(z) dz - \frac{p_0}{2}, \\ c_i &= \frac{1}{2^{p-2}} \int_{S_{2i}} z_1 z_2 \phi(z) dz, \end{aligned} \tag{2.4}$$

where $\phi(z) = (2\pi)^{-p/2} \exp(-z^t z/2)$, $S_{1i} = \{z \in R^p : z_1 > 0,\ q_{0(i-1)} < z^t z \le q_{0i}\}$, and $S_{2i} = \{z \in R^p : z_1 > 0, z_2 > 0,\ q_{0(i-1)} < z^t z \le q_{0i}\}$.

Let D_1 be the design matrix of the main effects and D_2 be the design matrix of the first-order interaction effects in $g = 2^p$ factorial design. More precisely, D_1 is a $g \times p$ matrix whose i-th column is 2^{p-i} repetitions of the vector $(-e_{2^{i-1}}, e_{2^{i-1}})$, and D_2 is the $g \times p(p-1)/2$ matrix obtained from D_1 such that the columns of D_2 are all possible products of two distinct columns from D_1. We then define

$$A = \begin{pmatrix} a_1 D_1 \\ a_2 D_1 \\ \vdots \\ a_c D_1 \end{pmatrix}, \quad B = \begin{pmatrix} b_1 e_g e_p^t \\ b_2 e_g e_p^t \\ \vdots \\ b_c e_g e_p^t \end{pmatrix}, \quad C = \begin{pmatrix} c_1 D_2 \\ c_2 D_2 \\ \vdots \\ c_c D_2 \end{pmatrix}.$$

We now present the asymptotic distributions of the vector of cell counts $U_{n\pi}$ and of the chi-squared test statistics.

Theorem 1. If $y_1, y_2, \cdots, y_n$ are a random sample from $N(\mu, \Sigma)$ where Σ is nonsingular, then

$$(U_n - np_0 e)/\sqrt{np_0} \to N(0, \Psi) \quad \text{as } n \to \infty$$

where $\Psi = I - E - AA^t/p_0 - CC^t/p_0$, $E = \text{diag}(e_g e_g^t/g, \cdots, e_g e_g^t/g)$ with $p_0 = 1/K = 1/(cg)$ and $g = 2^p$.

Theorem 2. Under the assumptions of Theorem 1,

$$X^2 \to W_1 + (1-a^*)W_2 + (1-c^*)W_3 \quad \text{as } n \to \infty$$

where W_1, W_2, and W_3 are independent chi-squared variates with degrees of freedom $K - c - p - p(p-1)/2$, p, and $p(p-1)/2$ respectively, and

$$a^* = g\sum_{i=1}^{c} a_i^2/p_0,\ c^* = g\sum_{i=1}^{c} c_i^2/p_0$$

with a_i's and c_i's defined in (2.4) and $g = 2^p$.

After some algebra, simple computing formulas of a^*, c^* can be given as follows: Define

$$f_a(x) = F_{p-1}(x)/\sqrt{2\pi} - (2\pi)^{-p/2}\exp(-x/2)\, V_{p-1}(\sqrt{x})$$
$$f_c(x) = F_{p-2}/(2\pi) - (2\pi)^{-p/2}\exp(-x/2)\, V_{p-2}(\sqrt{x})(1 + x/p)$$

for $x \geq 0$, where F_p is the distribution function of $\chi^2(p)$ distribution and $V_p(r)$ is the volume of the hypersphere with radius r; i.e. $V_p(r) = \int_{z'z<r} dz = 2r^p B_p$ where

$$B_p = \begin{cases} (2\pi)^{(p-2)/2}\pi\left[\prod_{j=0}^{k-1}(p-2j)\right]^{-1}, & \text{if } p = 2k \text{ is even}, \\ (2\pi)^{(p-1)/2}\left[\prod_{j=0}^{k-1}(p-2j)\right]^{-1}, & \text{if } p = 2k+1 \text{ is odd}. \end{cases}$$

We take $f_a(\infty) = 1/\sqrt{2\pi}$, $f_b(\infty) = 1/(2\pi)$. Then

$$a^* = 4c\sum_{i=1}^{c}\{f_a(q_{0i}) - f_a(q_{0(i-1)})\}^2,\quad c^* = 16c\sum_{i=1}^{c}\{f_c(q_{0i}) - f_c(q_{0(i-1)})\}^2,$$

where q_{0i} is defined just prior to (2.4).

3. Proofs

Let $\theta=(\mu,\Sigma)$ denote the parameter of the multivariate normal distribution. Because the transformed matrix Z is ancillary by Lemma 3.1 in Huffer and Park (2002), we may assume without loss of generality that Y is sampled from a population with $\theta_0=(0, I)$. We can also note that the distribution of Z does not depend on the choice of rotation R and we will use Gram-Schmidt method that takes R to be lower triangular with positive elements.

The cell count $U_{n\pi}$ is the number of observations y_i lying in a region $\Lambda_{n\pi}$. We introduce some notation to describe this region. Let $q=(q_1, q_2, \cdots, q_{c-1})$ be a vector satisfying $q_1\le q_2\le\cdots\le q_{c-1}$. We take $q_0=0$ and $q_c=\infty$. For given $\theta=(\mu,\Sigma)$ and q, for a given cell $\pi=(\pi^*, \pi_{p+1})$, we define regions

$$A_{\pi^*}(\theta)=\{y\in R^p: \pi^*\cdot z>0,\ z=R^t(\Sigma)(y-\mu)\},$$
$$B_{\pi_{p+1}}(\theta,q)=\left\{y\in R^p: q_{(\pi_{p+1}-1)}<(y-\mu)^t\Sigma^{-1}(y-\mu)\le q_{\pi_{p+1}}\right\},$$

and

$$\Lambda_\pi(\theta,q)=A_{\pi^*}(\theta)\cap B_{\pi_{p+1}}(\theta,q),$$

where $\pi^*=(\pi_1,\pi_2,\cdots,\pi_p)$ is defined prior to (2.2).

Let $q_n=(q_{n1}, q_{n2}, \cdots, q_{n(c-1)})$ be the vector of sample quantiles of the squared distances $z_i^t z_i$ and let $q_0=(q_{01}, q_{02}, \cdots, q_{0(c-1)})$ be the vector of population quantiles of $\chi^2(p)$ distribution, where q_{ni}'s and q_{0i}'s are defined prior to (2.2) and (2.4), respectively. We can easily show that $q_n\to q_0$ in probability. Let θ_n be the maximum likelihood estimator of $\theta=(\mu,\Sigma)$ based on $y_1, y_2, \cdots, y_n$, that is $\theta_n=(\bar{y}, S)$, so that $\theta_n\to\theta_0$ in probability. We now define the vector of regions $\Lambda_n=(\Lambda_{n\pi})$ by $\Lambda_{n\pi}=\Lambda_\pi(\theta_n, q_n)$ and the limiting vector of regions $\Lambda_0=(\Lambda_{0\pi})$ by $\Lambda_{0\pi}=\Lambda_\pi(\theta_0, q_0)$.

For an arbitrary region $\Gamma\subset R^p$, we define $U_n(\Gamma)=\sum_{i=1}^n I(y_i\in\Gamma)$ and define $P(\Gamma,\theta)$ to be the probability assigned to Γ by the normal distribution with parameter θ. We also let $D(\Gamma)$ be $\partial P(\Gamma,\theta)/\partial\theta$ evaluated at $\theta=\theta_0$. For a vector of regions $\Gamma=(\Gamma_i)$ we use the obvious vector analogs of the above definitions so that $U_n(\Gamma)=(U_n(\Gamma_i))$, $P(\Gamma,\theta)=(P(\Gamma_i,\theta))$ is a vector of

probabilities, and $D(\Gamma)$ is a matrix of partial derivatives. Our vector of cell counts U_n may now be written as $U_n(\Lambda_n)$. Finally, we define the process $V_n(\Gamma) = n^{-1/2}\{U_n(\Gamma) - nP(\Gamma, \theta_0)\}$.

Lemma 1.

$$n^{-1/2}\{U_n(\Lambda_n) - nP(\Lambda_n, \theta_n)\} = V_n(\Lambda_0) - D(\Lambda_0)\sqrt{n}(\theta_n - \theta_0) + o_{p(1)}.$$

Proof: It is well known that the space $\mathcal{T}_1$ of all sets expressible as intersections of a finite number of open or closed half-spaces and the space $\mathcal{T}_2$ of regions generated by differences of hyperellipsoids are Donsker classes for any probability measure on R^p for any p. Thus the class $\mathcal{T}$ of $\Lambda_\pi(\theta, q)$ is a Donsker class since $\Lambda_\pi(\theta, q)$ can be expressed as $A \cap B$ where $A \in \mathcal{T}_1$ and $B \in \mathcal{T}_2$. Now we can apply almost the same arguments as in the proof of Lemma 3.2 of Huffer and Park (2002) to derive the above result. □

Proof of Theorem 1

Suppose the coordinates of $\theta = (\mu, \Sigma)$ are arranged so that $\theta = (\mu, \sigma, \rho)$, where $\sigma = (\sigma_{11}, \sigma_{22}, \cdots, \sigma_{pp})$ and $\rho = (\sigma_{12}, \sigma_{13}, \cdots, \sigma_{(p-1)p})$ are the diagonal and off-diagonal elements of Σ, respectively. Then it is easy to show that $D(\Lambda_0) = (A, B, C)$, where A, B, C are defined prior to Theorem 1. For any vector $x = (x_1, x_2, \cdots, x_p)$, we define the column vectors $s(x) = (x_1^2, x_2^2, \cdots, x_p^2)$, and $r(x) = (x_1x_2, x_1x_3, \cdots, x_{p-1}x_p)$ with dimensions p and $p(p-1)/2$, respectively. Then since $S = Y^tY/n + o_p(n^{-1/2})$, Lemma 1 leads to

$$\begin{aligned} n^{-1/2}\ &\{U_n(\Lambda_n) - nP(\Lambda_n, \theta_n)\} = \\ &V_n(\Lambda_0) - \sqrt{n}\left(A\bar{y} + B\sum_{i=1}^{n}(s(y_i) - e)/n + C\sum_{i=1}^{n} r(y_i)/n\right) + o_{p(1)}. \end{aligned} \tag{3.1}$$

Using E defined in Theorem 1, we define a $K \times K$ projection matrix $\Pi = I - E$ which removes the marginal means of the last component π_{p+1}. We now apply Π to both sides of (3.1). It is easy to show that

$$\Pi U_n(\Lambda_n) = U_n(\Lambda_n) - np_0e + O(1),\ \ \Pi A = A\ ,\ \ \Pi B = 0,$$

and $\Pi C = C$, where $p_0 = 1/K$ defined in (2.3). (Note that the equality $\Pi U_n(\Lambda_n) = U_n(\Lambda_n) - np_0 e$ is exact when n is divisible by c.) Since $P(\Lambda_\pi(\theta, q), \theta)$ does not depend on θ, i.e. $P(\Lambda_\pi(\theta, q), \theta) = P(\Lambda_\pi(\theta_0, q), \theta_0)$, we have

$$P(\Lambda_n, \theta_n) = g^{-1}(\hat{p}_1 e_g, \hat{p}_2 e_g, \cdots, \hat{p}_c e_g),$$

where $g = 2^p$ and $\hat{p}_i = F_p(\hat{q}_{ni}) - F_p(\hat{q}_{n(i-1)})$ for each i with F_p the distribution function of $\chi^2(p)$ distribution and $\hat{q}_{ni}$'s defined prior to (2.2). Thus $\Pi P(\Lambda_n, \theta_n) = 0$. Putting this all together we have

$$n^{-1/2}\{U_n(\Lambda_n) - np_0 e\} = \Pi V_n(\Lambda_0) - n^{1/2}\left(A\bar{y} + C\sum_i r(y_i)/n\right) + o_{p(1)}.$$

By the finite dimensional central limit theorem, $\left(V_n(\Lambda_0), n^{1/2}\bar{y}, n^{-1/2}\sum_i r(y_i)\right)$ converges in distribution to a normal distribution with zero mean vector and covariance matrix given by

$$\begin{pmatrix} p_0(I - p_0 ee^t) & A & C \\ A^t & I & 0 \\ C^t & 0 & I \end{pmatrix}.$$

Since $\Pi A = A$, $\Pi B = 0$, $\Pi C = C$, and $\Pi e = 0$, we can show that

$$n^{-1/2}(U_n(\Lambda_n) - np_0 e) \to N(0, p_0\Pi - AA^t - CC^t) \quad \text{as} \quad n \to \infty.$$

This completes the proof. □

Proof of Theorem 2

We need to calculate the eigenvalues of

$$\Psi = I - E - AA^t/p_0 - CC^t/p_0 = I - E - a^* P_1 - c^* P_2$$

where $P_1 = AA^t/(g\sum_{i=1}^{c} a_i^2)$, $P_2 = CC^t/(g\sum_{i=1}^{c} c_i^2)$, and a^*, c^*, and $g = 2^p$ are defined in this theorem. Since $A^tA = g\sum_{i=1}^{c} a_i^2 I$, $C^tC = g\sum_{i=1}^{c} c_i^2 I$ with

$EA=0, EC=0,$ and $A^tC=0$, it is easy to see that $E, P_1,$ and P_2 are mutually orthogonal, symmetric, and idempotent matrices of ranks $c, p,$ and $p(p-1)/2$, respectively. These implies that non-zero eigenvalues of Ψ are 1 with multiplicity $c(g-1)$, $1-a^*$ with multiplicity p, and $1-c^*$ with multiplicity $p(p-1)/2$. This implies the desired result. □

4. Simulation Study

We first provide a simulation study illustrating the accuracy of the limiting distribution of X^2 for finite sample sizes. Although we have tried many configurations, almost all of them show identical results unless the number K of cells is small. Therefore we present a simulation results corresponding to the case where $p=3$, $c=5$, so that the limiting distribution of the chi-square statistic is

$$\chi^2(29)+0.2613\chi^2(3)+0.3730\chi^2(3).$$

We consider four different sample sizes, $n=40, 80, 200$ and 400, which have an average of 1, 2, 5 and 10 observations per cell, respectively.

For each sample size n, we generate 1000 samples of size n from $N_3(0, I)$ and then calculate the chi-squared statistics for each of them. These 1000 values are plotted against the expected order statistics of a sample of size 1000 from the limiting distribution. The expected order statistics are generated from the limiting distribution as follows: We generate a random sample of size one million from the weighted chi-squared distribution and the expected order statistics are computed by taking every 1000-th order statistic of the sample starting from the 500-th order statistic. The resulting quantile-quantile plots are displayed in Figure 1.

Each plot displays the reference line with slope 1 and intercept 0, which corresponds to the ideal case where empirical and theoretical distributions coincide. Examining the plots, we find that the limiting distribution is a good approximation for the cases where average cell counts np_0 are 2, 5 and 10. However the discreteness of the chi-squared values are apparent in the case where np_0 is 1 but the points do not deviate much from the reference line even in this case. We now present a simulation study comparing the power of X^2 with other popular tests of multivariate normality. In our comparisons, four popular tests of multivariate normality such as the well-known skewness and kurtosis tests of Mardia (1970), the test by Ozturk and Romeu (1992), and the test statistic by Baringhaus and Henze (1988) are included. These tests will be denoted by Skew,

<Figure 1> quantile-quantile plots for four different sample sizes

Kurt, OR, and BH respectively in this simulation study. These tests are chosen since they performed well in previous studies such as Romeu and Ozturk (1993) and Manzotti and Quiroz (2001).

A non-normal distribution considered in this simulation is a spiral distribution with density

$$f(x_1, x_2 | b) = \frac{1}{2\pi} \{1 + \cos[2(\theta - br)]\} \exp(-r^2/2)$$

where r and θ are radius and angle of (x_1, x_2) in polar coordinates. A plot of density shows two symmetric spiral arms, and as b increases, the moments of this distribution converge rapidly to those of the bivariate normal distribution $N_2(0, I)$. In our simulation we take $b = 2$ and, in computing X^2, we use $c = 5$. Empirical powers, out of 1000 replicates, of our test and four other tests of multivariate normality for $\alpha = .01, .05$ and $.1$ are given in Table 1.

<Table 1> Empirical powers, out of 1000 replicates, of five tests

	$n = 100$			$n = 200$		
α	.01	.05	.1	.01	.05	.1
X^2	.391	.634	.738	.866	.948	.984
Skew	.013	.068	.115	.014	.074	.120
Kurt	.007	.027	.075	.011	.037	.077
OR	.048	.115	.172	.033	.136	.223
BH	.016	.101	.198	.053	.179	.274

From the results in Table 1, we can see that our method has great power to detect this spiral pattern whereas OR and BH are the only other tests that have some power. The poor performance of Skew and Kurt is understandable since the moments of the spiral distribution are close to those of the normal distribution.

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