

Renewal Reward Processes with Fuzzy Rewards and Fuzzy Inter-arrival Times¹⁾

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Abstract

In this paper, we consider a renewal process in which both the inter-arrival times and rewards are fuzzy random variables. We prove the uniform levelwise convergence of fuzzy renewal and fuzzy renewal rewards. These results improve the result of Popova and Wu[European J. Oper. Research 117(1999), 606-617] and the main result of Hwang [Fuzzy Sets and Systems 116 (2000), 237-244].

Keywords : A fuzzy renewal process, Fuzzy random variable

1. Introduction

The theory of fuzzy sets introduced by Zadeh(1965, 1975) has been extensively studied and applied in statistics and probability areas in recent years. Since Kwakernaak(1978, 1979) and Puri and Ralescu(1986). Kruse(1982) introduced the concept of fuzzy random variable, there has been increasing interests for fuzzy random variable. Among others, strong law of large numbers for independent fuzzy random variables have been studied by several researchers. But there are only few papers investigating the renewal process in fuzzy environments. Popova and Wu(1999) consider a renewal rewards process with random inter-arrival times and fuzzy random rewards. Hwang(2000) considered a renewal process having

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- 1) This work was supported by Grant No. R05-2004-000-10304-0(2005) from the Korea Science and Engineering Foundation.
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inter-arrival times which are fuzzy random variables and proved a theorem for the rate of a renewal process having inter-arrival times which are fuzzy random variables. In this paper, we consider a renewal process in which both the inter-arrival times and rewards are fuzzy random variables. A uniform levelwise convergence of fuzzy renewal and a uniform levelwise convergence of fuzzy renewal rewards are provided using recent results of Molchanov(1999). These results improve both the result of Popova and Wu(1999) and the result of Hwang (2000). Section 2 is devoted to describe some basic concepts of fuzzy random variables. Main results are given in Section 3.

2. Preliminaries

We have the following definition by Zimmermann(2001).

Definition 2.1. A fuzzy number \tilde{M} is a convex normalized fuzzy set \tilde{M} of the real line R such that

(a) It exists exactly one $x_0 \in R$ with the membership function $\tilde{M}(x_0) = 1$ (x_0 is called the mean value of \tilde{M}).

(b) The membership function $\tilde{M}(x)$ is piecewise continuous.

Definition 2.2. The crisp set of elements that belong to the fuzzy set \tilde{A} at least to the degree α is called the α -level-set:

$$A_\alpha = \{x : \tilde{A}(x) \geq \alpha\}, \quad 0 \leq \alpha \leq 1.$$

The α -level-set A_α is a useful tool for the treatment of fuzzy numbers.

Definition 2.3. If the α -level-set A_α is a closed and bounded set for all $\alpha \in (0, 1]$, then the fuzzy set \tilde{A} is said to be a closed and bounded set.

A fuzzy number \tilde{M} may be decomposed into its α -level-sets by

$$\tilde{M} = \sup_{\alpha \in (0, 1] \cap Q} \{ \alpha \cdot I_{M_\alpha} \}$$

or

$$\tilde{M}(x) = \sup_{\alpha \in (0, 1] \cap Q} \{ \alpha \cdot I_{M_\alpha}(x) \},$$

where Q is the set of the rational number and I_{M_α} is the indicator function of the set M_α .

Let \tilde{U} and \tilde{V} be two fuzzy numbers in R . By the extension principle, four operations on fuzzy numbers \tilde{U} and \tilde{V} are defined as follows

$$\begin{aligned}(\tilde{U} + \tilde{V})(t) &= \sup_{x+y=t} \{\tilde{U}(x) \wedge \tilde{V}(y)\}, \\(\tilde{U} - \tilde{V})(t) &= \sup_{x-y=t} \{\tilde{U}(x) \wedge \tilde{V}(y)\}, \\(\tilde{U} * \tilde{V})(t) &= \sup_{x \cdot y=t} \{\tilde{U}(x) \wedge \tilde{V}(y)\}, \\ \left(\frac{\tilde{U}}{\tilde{V}}\right)(t) &= \sup_{x=t \cdot y} \{\tilde{U}(x) \wedge \tilde{V}(y)\},\end{aligned}$$

where \wedge is denoted by "min".

Theorem 2.1. Let \tilde{U} and \tilde{V} be two closed and bounded fuzzy numbers, then

$$(\tilde{U} + \tilde{V})_\alpha = \tilde{U}_\alpha + \tilde{V}_\alpha.$$

Let $\tilde{X} = [x^-, x^+] = \{x \mid x^- \leq x \leq x^+\}$ and $\tilde{Y} = [y^-, y^+] = \{y \mid y^- \leq y \leq y^+\}$ for $x^-, x^+, y^-, y^+ \in R$.

Four operations (+, -, *, and /) on \tilde{X} and \tilde{Y} are defined as follows:

$$\tilde{X} + \tilde{Y} = [x^-, x^+] + [y^-, y^+] = [x^- + y^-, x^+ + y^+],$$

$$\tilde{X} - \tilde{Y} = [x^-, x^+] - [y^-, y^+] = [x^- - y^+, x^+ - y^-],$$

$$\tilde{X} * \tilde{Y} = [x^-, x^+] * [y^-, y^+],$$

$$= [\min(x^-y^-, x^+y^+, x^-y^+, x^+y^-), \max(x^-y^-, x^+y^+, x^-y^+, x^+y^-)],$$

$$\frac{\tilde{X}}{\tilde{Y}} = [x^-, x^+] * \left[\frac{1}{y^+}, \frac{1}{y^-} \right], \quad 0 \notin [y^-, y^+].$$

We define the metric d_∞ on the set of closed and bounded fuzzy numbers by

$$d_{\infty}(\tilde{U}, \tilde{V}) = \sup_{0 \leq \alpha \leq 1} \max(|\tilde{U}_{\alpha}^* - \tilde{V}_{\alpha}^*|, |\tilde{U}_{\alpha}^{**} - \tilde{V}_{\alpha}^{**}|) \quad (1)$$

where $\tilde{U}_{\alpha} = [\tilde{U}_{\alpha}^*, \tilde{U}_{\alpha}^{**}]$ and $\tilde{V}_{\alpha} = [\tilde{V}_{\alpha}^*, \tilde{V}_{\alpha}^{**}]$.

We have the following definition by Kruse(1982),

Definition 2.4. Let (Ω, Γ, P) be a probability space. A fuzzy random variable $\tilde{X}: \Omega \rightarrow F(R)$, where $F(R)$ is the set of fuzzy numbers in R (i.e. for $w \in \Omega$, $\tilde{X}(w) \in F(R)$), that satisfies the following properties:

For $\alpha \in (0, 1]$ and $w \in \Omega$, both X_{α}^* and X_{α}^{**} defined by

$$\begin{aligned} X_{\alpha}^*(w) &= \inf \tilde{X}_{\alpha}(w), \\ X_{\alpha}^{**}(w) &= \sup \tilde{X}_{\alpha}(w) \end{aligned}$$

are finite real valued random variables on (Ω, Γ, P) such that the mathematical expectations EX_{α}^* and EX_{α}^{**} exist.

For $\alpha \in (0, 1]$ and $w \in \Omega$, $X_{\alpha}^*(w) \in \tilde{X}_{\alpha}(w)$ and $X_{\alpha}^{**}(w) \in \tilde{X}_{\alpha}(w)$. In this case, the expectation of \tilde{X}

is the fuzzy number $E\tilde{X}$ defined by

$$(E\tilde{X})_{\alpha} = [EX_{\alpha}^*, EX_{\alpha}^{**}], \quad 0 \leq \alpha \leq 1.$$

Let (Ω, Γ, P) be a probability space and a fuzzy random variable $\tilde{X}_t: \Omega \rightarrow F(R)$, where $t \in T$ be an index set and $F(R)$ be the set of fuzzy numbers in R (i.e. for, $w \in \Omega$, $\tilde{X}_t(w) \in F(R)$). We called $\{\tilde{X}_t\}_{t \in T}$ as a fuzzy stochastic process.

Let \tilde{X}_i , $i = 1, 2, \dots$ be a sequence of fuzzy random variables on (Ω, Γ, P) .

For $\alpha \in (0, 1]$, $w \in \Omega$, $X_{i,\alpha}^*$ and $X_{i,\alpha}^{**}$ defined by

$$\begin{aligned} X_{i,\alpha}^*(w) &= \inf \tilde{X}_{i,\alpha}(w), \\ X_{i,\alpha}^{**}(w) &= \sup \tilde{X}_{i,\alpha}(w) \end{aligned}$$

are sequences of independent and identically distributed crisp random variables. Then the $\tilde{X}_i, i = 1, 2, \dots$ is called a sequence of independent and identically distributed fuzzy random variables.

For a probability space (Ω, Γ, P) let the time between the $(n-1)$ th and the n th event of a process in fuzzy sense be a fuzzy random variable \tilde{X}_n such that $\tilde{X}_n: \Omega \rightarrow F(R^+)$, where $F(R^+)$ is the set of fuzzy numbers in R^+ (i.e. for $w \in \Omega, \tilde{X}_n(w) \in F(R^+)$ and $R^+ = (0, \infty)$) and $n \geq 1$.

For each $\alpha \in (0, 1]$, both $X_{i,\alpha}^*$ and $X_{i,\alpha}^{**}$ defined by

$$X_{i,\alpha}^*(w) = \inf \{t \mid \tilde{X}_i(w)(t) \geq \alpha\},$$

$$X_{i,\alpha}^{**}(w) = \sup \{t \mid \tilde{X}_i(w)(t) \geq \alpha, \text{ for } i = 1, 2, \dots\}$$

are sequences of independent and identically distributed random variables.

We have the α -level-set $\tilde{X}_{i,\alpha}(w) = [X_{i,\alpha}^*(w), X_{i,\alpha}^{**}(w)]$, $i = 1, 2, \dots$.

By the strong law of large numbers in the crisp sense, we have

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_{i,\alpha}^* = EX_{1,\alpha}^*,$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n X_{i,\alpha}^{**} = EX_{1,\alpha}^{**}.$$

For a sequence of independent and identically distributed fuzzy random variables $\{\tilde{X}_1, \tilde{X}_2, \dots\}$, let $\tilde{S}_n = \sum_{i=1}^n \tilde{X}_i$. That is \tilde{S}_n represents the occurred time of n th renewal in the fuzzy sense.

For each $\alpha \in (0, 1]$, $w \in \Omega$, $n \in N$, $S_{n,\alpha}^*(w)$, and $S_{n,\alpha}^{**}(w)$ are defined by

$$S_{n,\alpha}^*(w) = \inf \{t \mid \tilde{S}_n(w)(t) \geq \alpha\},$$

$$S_{n,\alpha}^{**}(w) = \sup \{t \mid \tilde{S}_n(w)(t) \geq \alpha\}.$$

Hence, the α -level-set $(\tilde{S}_n(w))_\alpha = [S_{n,\alpha}^*(w), S_{n,\alpha}^{**}(w)]$.

We also let

$$N_\alpha^*(t) = \sup \{n \mid S_{n,\alpha}^*(w) \leq t\}. \tag{2}$$

$$N_\alpha^{**}(t) = \sup \{n \mid S_{n,\alpha}^{**}(w) \leq t\}. \tag{3}$$

It is clear that $N_\alpha^*(t) \geq N_\alpha^{**}(t)$.

Definition 2.5 [2]. Let the time between the $(n-1)$ th and the n th event of a process in fuzzy sense be a fuzzy random variable, \tilde{X}_n , $n \geq 1$. If $\{\tilde{X}_1, \tilde{X}_2, \dots\}$ is a sequence of independent and identically distributed fuzzy random variables, then the fuzzy stochastic process $\{\tilde{N}(t), t \geq 0\}$ is said to be a fuzzy renewal process, where $\tilde{N}(t)$ is defined by

$$\tilde{N}(t) = \sup_{\alpha \in (0,1] \cap Q} \{\alpha \cdot I_{[N_\alpha^{**}(t), N_\alpha^*(t)]}\},$$

Q is the set of the rational number.

3. Main result

Recently, Molchanov(1999) and Hong(2003) proved the following result.

Theorem 3.1. Let \tilde{X}_n be a sequence of independent and identically distributed fuzzy random variables.

Then we have

$$d_\infty \left(\frac{1}{n} \sum_{i=1}^n \tilde{X}_i, E\tilde{X}_1 \right) \rightarrow 0 \text{ a.s.}$$

Using this result, we prove the following theorem which is a generalized result of Theorem 3.1 of Hwang(2000).

Theorem 3.2 (Fuzzy renewal theorem). With probability one,

$$d_\infty \left(\frac{\tilde{N}(t)}{t}, \frac{1}{E\tilde{X}_1} \right) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Proof. By (3), we have

$$S_{N_\alpha^{**}(t), \alpha}^{**}(w) \leq t \leq S_{N_\alpha^{**}(t)+1, \alpha}^{**}(w)$$

and

$$\frac{S_{N_\alpha^{**}(t), \alpha}^{**}(w)}{N_\alpha^{**}(t)} \leq \frac{t}{N_\alpha^{**}(t)} \leq \frac{S_{N_\alpha^{**}(t)+1, \alpha}^{**}(w)}{N_\alpha^{**}(t)}.$$

Since

$$\frac{S_{N_{\alpha}^{**}(t),\alpha}^{**}}{N_{\alpha}^{**}(t)} = \frac{\sum_{i=1}^{N_{\alpha}^{**}(t)} X_{i,\alpha}^{**}}{N_{\alpha}^{**}(t)}$$

is the average of independent and identically distributed random variables, $X_{i,\alpha}^{**}$, $i = 1, \dots, N_{\alpha}^{**}(t)$, it follows by Theorem 3.1 that

$$\sup_{0 \leq \alpha \leq 1} \left| \frac{S_{N_{\alpha}^{**}(t),\alpha}^{**}(w)}{N_{\alpha}^{**}(t)} - EX_{1,\alpha}^{**} \right| \rightarrow 0 \text{ a.s.}$$

and

$$\sup_{0 \leq \alpha \leq 1} \left| \frac{S_{N_{\alpha}^{**}(t)+1,\alpha}^{**}(w)}{N_{\alpha}^{**}(t)} - EX_{1,\alpha}^{**} \right| \rightarrow 0 \text{ a.s. as } t \rightarrow \infty.$$

Hence

$$\sup_{0 \leq \alpha \leq 1} \left| \frac{t}{N_{\alpha}^{**}(t)} - EX_{1,\alpha}^{**} \right| \rightarrow 0 \text{ a.s. as } t \rightarrow \infty.$$

Since $EX_{1,\alpha}^{**} \geq EX_{1,0}^* > 0$, we have that

$$\sup_{0 \leq \alpha \leq 1} \left| \frac{N_{\alpha}^{**}(t)}{t} \frac{-1}{EX_{1,\alpha}^{**}} \right| \rightarrow 0 \text{ a.s. as } t \rightarrow \infty.$$

Similarly we have

$$\sup_{0 \leq \alpha \leq 1} \left| \frac{N_{\alpha}^*(t)}{t} \frac{-1}{EX_{1,\alpha}^*} \right| \rightarrow \infty \text{ a.s. as } t \rightarrow \infty,$$

and hence,

$$d_{\infty} \left(\tilde{N} \frac{(t)}{t}, \frac{1}{E} \tilde{X}_1 \right) = \sup_{0 \leq \alpha \leq 1} \max \left(\left| \frac{N_{\alpha}^{**}(t)}{t} - \frac{1}{EX_{1,\alpha}^{**}} \right|, \left| \frac{N_{\alpha}^*(t)}{t} - \frac{1}{EX_{1,\alpha}^*} \right| \right) \rightarrow 0$$

a.s. as $t \rightarrow \infty$, which completes the proof.

We denote by \tilde{R}_n the fuzzy rewards earned at the time of the n -th renewal, where \tilde{R}_n is a fuzzy random variable. We shall assume that the \tilde{R}_n for $n \geq 1$ are independent and identically distributed. Let $\tilde{R}(t)$ denote the total fuzzy reward

earned by time t . And we define

$$\tilde{R}(t) = \sum_{i=1}^{\tilde{N}(t)} \tilde{R}_i$$

where $\tilde{N}(t)$ is a fuzzy renewal process and the α -level set

$$\tilde{R}_\alpha(t)(w) = [R_\alpha^*(t)(w), R_\alpha^{**}(t)(w)] = \left[\sum_{i=1}^{N_\alpha^{**}(t)} R_{i,\alpha}^*, \alpha(w), \sum_{i=1}^{N_\alpha^*(t)} R_{i,\alpha}^{**}, \alpha(w) \right]. \quad (4)$$

Theorem 3.3 (Fuzzy renewal reward theorem). With probability one,

$$d_\infty \left(\frac{\tilde{R}(t)}{t}, \frac{E\tilde{R}_1}{E\tilde{X}_1} \right) \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Proof. We first note that from (1) and (4),

$$d_\infty \left(\frac{\tilde{R}(t)}{t}, \frac{E\tilde{R}_1}{E\tilde{X}_1} \right) = \text{“sup}_{0 \leq \alpha \leq 1} \max \left(\left| \frac{\sum_{i=1}^{N_\alpha^{**}(t)} R_{i,\alpha}^*}{t} - \frac{ER_{1,\alpha}^*}{EX_{1,\alpha}^{**}} \right|, \left| \frac{\sum_{i=1}^{N_\alpha^*(t)} R_{i,\alpha}^{**}}{t} - \frac{ER_{1,\alpha}^{**}}{EX_{1,\alpha}^*} \right| \right).$$

Since

$$\begin{aligned} \left| \frac{\sum_{i=1}^{N_\alpha^{**}(t)} R_{i,\alpha}^*}{t} - \frac{ER_{1,\alpha}^*}{EX_{1,\alpha}^{**}} \right| &\leq \left| \frac{\sum_{i=1}^{N_\alpha^{**}(t)} R_{i,\alpha}^*}{N_\alpha^{**}(t)} \frac{N_\alpha^{**}(t)}{t} - \frac{N_\alpha^{**}(t)}{t} ER_{1,\alpha}^* \right| + \\ &\quad \left| \frac{N_\alpha^{**}(t)}{t} ER_{1,\alpha}^* - \frac{ER_{1,\alpha}^*}{EX_{1,\alpha}^{**}} \right| \\ &\leq \left| \frac{N_\alpha^{**}(t)}{t} \right| \left| \frac{\sum_{i=1}^{N_\alpha^{**}(t)} R_{i,\alpha}^*}{N_\alpha^{**}(t)} - ER_{1,\alpha}^* \right| + \\ &\quad \left| ER_{1,\alpha}^* \right| \left| \frac{N_\alpha^{**}(t)}{t} - \frac{1}{EX_{1,\alpha}^{**}} \right|, \end{aligned}$$

by Theorem 3.2, for large n depending on w with probability 1,

$$\begin{aligned} \sup_{0 \leq \alpha \leq 1} \left| \frac{\sum_{i=1}^{N_{\alpha}^{**}(t)} R_{i,\alpha}^*}{t} - \frac{ER_{1,\alpha}^*}{EX_{1,\alpha}^{**}} \right| &\leq \frac{2}{EX_{1,0}^*} d_{\infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{R}_i, E\tilde{R}_1 \right) \\ &+ (|ER_{1,0}^*| + |ER_{1,0}^{**}|) d_{\infty} \left(\frac{\tilde{N}(t)}{t}, \frac{1}{E\tilde{X}_1} \right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \sup_{0 \leq \alpha \leq 1} \left| \frac{\sum_{i=1}^{N_{\alpha}^{**}(t)} R_{i,\alpha}^{**}}{t} - \frac{ER_{1,\alpha}^{**}}{EX_{1,\alpha}^*} \right| &\leq \frac{2}{EX_{1,0}^*} d_{\infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{R}_i, E\tilde{R}_1 \right) \\ &+ (|ER_{1,0}^*| + |ER_{1,0}^{**}|) d_{\infty} \left(\frac{\tilde{N}(t)}{t}, \frac{1}{E\tilde{X}_1} \right), \end{aligned}$$

and hence we have

$$\begin{aligned} d_{\infty} \left(\frac{\tilde{R}(t)}{t}, \frac{E\tilde{R}_1}{E} \tilde{X}_1 \right) &\leq \frac{2}{EX_{1,0}^*} d_{\infty} \left(\frac{1}{n} \sum_{i=1}^n \tilde{R}_i, E\tilde{R}_1 \right) \\ &+ (|ER_{1,0}^*| + |ER_{1,0}^{**}|) d_{\infty} \left(\frac{\tilde{N}(t)}{t}, \frac{1}{E\tilde{X}_1} \right). \end{aligned}$$

Now, letting $t \rightarrow 0$ and $n \rightarrow 0$, then the result follows from Theorem 3.1 and 3.2.

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[received date : Oct. 2005, accepted date : Jan. 2006]