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Topologies induced by two types uniformities

Yong Chan Kim[1] and Young Sun Kim[2]

[1] **Department of Mathematics, Kangnung National University, Gangneung, 201-702, Korea**
[2] **Department of Applied Mathematics, Pai Chai University, Daejeon, 302-735, Korea**

Abstract

In strictly two-sided, commutative biquantale, we show that Hutton $(L,\otimes)$-uniform spaces and $(L,\odot)$-uniform spaces induce enriched $(L,\odot)$-topological spaces and enriched $(L,\odot)$-interior spaces.

Key words : Hutton $(L,\otimes)$-uniform spaces, $(L,\odot)$-uniform spaces, enriched $(L,\odot)$-topological spaces, enriched $(L,\odot)$-interior spaces

1. Introduction

Recently, Gutiérrez García et al.[2] introduced L-valued Hutton unifomity where a quadruple $(L,\leq,\otimes,*)$ is defined by a GL-monoid $(L,*)$ dominated by $\otimes$, a cl-quasi-monoid $(L,\leq,\otimes)$. Kubiak et al.[12] studied the relationships between the categories of $I(L)$-uniform spaces and L-uniform spaces. Kim et al. [9,10], as a somewhat different aspect in [2], introduced the notion of Hutton $(L,\otimes)$-uniformities as a view point of the approach using uniform operators defined by Rodabaugh [15] and $(L,\odot)$-uniformities in a sense Lowen [12] and Höhle [3] based on powersets of the form $L^{X\times X}$. Futhermore, the category **HUnif** of all Hutton $(L,\otimes)$-uniform spaces and H-uniformly continuous maps and the category **HUnif** of all $(L,\odot)$-uniform spaces and uniformly continuous maps are isomorphic.

In this paper, we introduce the notion of enriched $(L,\odot)$-topologies and enriched $(L,\odot)$-interior spaces. We investigate the relations between them. Moreover, we show that Hutton $(L,\otimes)$-uniform spaces and $(L,\odot)$-uniform spaces induce enriched $(L,\odot)$-topologies and enriched $(L,\odot)$-interior spaces.

2. Preliminaries

Definition 2.1 [4-7, 14] A triple $(L,\leq,\odot)$ is called a *strictly two-sided, commutative biquantale* (stsc-biquantale, for short) iff it satisfies the following properties:

(L1) $L=(L,\leq,\vee,\wedge,\top,\bot)$ is a completely distributive lattice where $\top$ is the universal upper bound and $\bot$ denotes the universal lower bound;

(L2) $(L,\odot)$ is a commutative semigroup;

(L3) $a=a\odot\top$, for each $a\in L$;

(L4) $\odot$ is distributive over arbitrary joins, i.e.

$$(\bigvee_{i\in\Gamma} a_i)\odot b=\bigvee_{i\in\Gamma}(a_i\odot b).$$

(L5) $\odot$ is distributive over arbitrary meets, i.e.

$$(\bigwedge_{i\in\Gamma} a_i)\odot b=\bigwedge_{i\in\Gamma}(a_i\odot b).$$

In this paper, we always assume that $(L,\leq,\odot,^*)$ is a stsc-biquantale with strong negation * where $a^*=a\rightarrow 0$ unless otherwise specified.

All algebraic operations on L can be extended pointwisely to the set L^X as follows: for all $x\in X$, $f,g\in L^X$ and $\alpha\in L$,

(1) $f\leq g$ iff $f(x)\leq g(x)$;

(2) $(f\odot g)(x)=f(x)\odot g(x)$;

(3) $1_X(x)=\top,\quad \alpha\odot 1_X(x)=\alpha$ and $1_\emptyset(x)=\bot$;

(4) $(\alpha\rightarrow\lambda)(x)=\alpha\rightarrow\lambda(x)$ and $(\lambda\rightarrow\alpha)(x)=\lambda(x)\rightarrow\alpha$;

(5) $(\alpha\odot\lambda)(x)=\alpha\odot\lambda(x)$.

Definition 2.2 [9,10] Let $\Omega(X)$ be a subset of $(L^X)^{(L^X)}$ such that

(O1) $\lambda\leq\phi(\lambda)$, for each $\lambda\in L^X$,

(O2) $\phi(\bigvee_{i\in\Gamma}\lambda_i)=\bigvee_{i\in\Gamma}\phi(\lambda_i)$, for $\{\lambda_i\}_{i\in\Gamma}\subset L^X$,

(O3) $\alpha\odot\phi(\lambda)=\phi(\alpha\odot\lambda)$, for each $\lambda\in L^X$.

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Lemma 2.3 [9,10] *For $\phi, \phi_1, \phi_2, \phi_3 \in \Omega(X)$, we define, for all $\lambda \in L^X$,*

$$\phi^{-1}(\lambda) = \bigwedge\{\rho \in L^X \mid \phi(\rho^*) \leq \lambda^*\},$$

$$\phi_1 \circ \phi_2(\lambda) = \phi_1(\phi_2(\lambda)),$$

$$\phi_1 \otimes \phi_2(\lambda) = \bigwedge\{\phi_1(\lambda_1) \odot \phi_2(\lambda_2) \mid \lambda = \lambda_1 \odot \lambda_2\}.$$

Then the following properties hold:

(1) *If $\phi(1_{\{x\}}) = \rho_x$ for all $x \in X$, then $\phi(\lambda) = \bigvee_{z\in X} \lambda(z) \odot \rho_z$.*

(2) *If $\phi_1(1_{\{x\}}) = \phi_2(1_{\{x\}})$ for all $x \in X$, then $\phi_1 = \phi_2$.*

(3) $\phi^{-1}, \phi_1 \circ \phi_2, \phi_1 \otimes \phi_2 \in \Omega(X)$.

(4) $\phi_1 \otimes \phi_2 \leq \phi_1$ *and* $\phi_1 \otimes \phi_2 \leq \phi_2$.

(5) $(\phi_1 \otimes \phi_2) \otimes \phi_3 = \phi_1 \otimes (\phi_2 \otimes \phi_3)$,

(6) $(\phi_1 \otimes \phi_2) \circ (\phi_1 \otimes \phi_2) \leq (\phi_1 \circ \phi_1) \otimes (\phi_2 \circ \phi_2)$.

(7) *Define $\phi_\top \in \Omega(X)$ as $\phi_\top(1_{\{x\}}) = 1_X, \forall x \in X$. Then $\phi \leq \phi_\top$ for all $\phi \in \Omega(X)$.*

Definition 2.4 [9,10] A nonempty subset $\mathbf{U}$ of $\Omega(X)$ is called a Hutton $(L, \otimes)$-*uniformity* on X if it satisfies the following conditions:

(U1) If $\phi \leq \psi$ with $\phi \in \mathbf{U}$ and $\psi \in \Omega(X)$, then $\psi \in \mathbf{U}$.

(U2) For each $\phi, \psi \in \mathbf{U}$, $\phi \otimes \psi \in \mathbf{U}$.

(U3) For each $\phi \in \mathbf{U}$, there exists $\psi \in \mathbf{U}$ such that $\psi \circ \psi \leq \phi$.

(U4) For each $\phi \in \mathbf{U}$, there exists $\phi^{-1} \in \mathbf{U}$.

The pair $(X, \mathbf{U})$ is said to be a *Hutton $(L, \otimes)$-uniform space.*

Definition 2.5 [9,10] Let $E(X \times X) = \{u \in L^{X\times X} \mid u(x,x) = 1\}$ be a subset of $L^{X\times X}$. A nonempty subset $\mathbf{D}$ of $E(X \times X)$ is called an $(L, \odot)$-*uniformity* on X if it satisfies the following conditions:

(D1) If $u \leq v$ with $u \in \mathbf{D}$ and $v \in E(X \times X)$, then $v \in \mathbf{D}$.

(D2) For each $u, v \in \mathbf{D}$, $u \odot v \in \mathbf{D}$.

(D3) For each $u \in \mathbf{D}$, there exists $v \in \mathbf{D}$ such that $v \circ v \leq u$ where

$$v \circ v(x,y) = \bigvee_{z\in X} (v(x,z) \odot v(z,y)).$$

(D4) For each $u \in \mathbf{D}$, there exists $u^s \in \mathbf{U}$ where $u^s(x,y) = u(y,x)$.

The pair $(X, \mathbf{D})$ is said to be an *$(L, \odot)$-uniform space.*

Theorem 2.6 [9,10] *We define a mapping $\Gamma : E(X \times X) \to \Omega(X)$ as follows:*

$$\Gamma(u)(\lambda)(y) = \bigvee_{x\in X} \lambda(x) \odot u(x,y).$$

Then we have the following properties:

(1) *For $u \in E(X \times X)$, $\Gamma(u) \in \Omega(X)$.*

(2) *Γ has a right adjoint mapping $\Lambda : \Omega(X) \to E(X \times X)$ as follows:*

$$\Lambda(\phi)(x,y) = \phi(1_{\{x\}})(y).$$

(3) $\Gamma \circ \Lambda = 1_{\Omega(X)}$ *and* $\Lambda \circ \Gamma = E(X \times X)$.

Theorem 2.7 [9,10] *Let $u, u_1, u_2 \in E(X \times X)$. Then we have the following properties:*

(1) *If $u_1 \leq u_2$, $\Gamma(u_1) \leq \Gamma(u_2)$.*

(2) $\Gamma(u_1 \odot u_2) \leq \Gamma(u_1) \otimes \Gamma(u_2)$.

(3) $\Gamma(1_\Delta) = 1_{L^X}$.

(4) $\Gamma(u)^{-1} = \Gamma(u^s)$.

(5) *$\Gamma(u)^{-1}(\lambda \to \bot) = \Gamma(u)^{\leftarrow}(\lambda) \to \bot$, for all $\lambda \in L^X$.*

(6) $\Gamma(u_1 \circ u_2) = \Gamma(u_2) \circ \Gamma(u_1)$.

(7) $\Gamma(\alpha \odot u) = \alpha \odot \Gamma(u)$.

(8) *If u is an $\odot$-equivalence relation on X, then*

$$(\Gamma(u))^{-1} = \Gamma(u^s) = \Gamma(u), \;\; \Gamma(u) \circ \Gamma(u) = \Gamma(u).$$

Theorem 2.8 [9,10] *Let $\phi, \phi_1, \phi_2 \in \Omega(X)$. Then we have the following properties:*

(1) *If $\phi_1 \leq \phi_2$, then $\Lambda(\phi_1) \leq \Lambda(\phi_2)$.*

(2) $\Lambda(\phi_1) \odot \Lambda(\phi_2) = \Lambda(\phi_1 \otimes \phi_2)$.

(3) $\Lambda(1_{L^X}) = 1_\Delta$.

(4) $\Lambda(\phi)^s = \Lambda(\phi^{-1})$.

(5) $\Lambda(\phi_1) \circ \Lambda(\phi_2) = \Lambda(\phi_2 \circ \phi_1)$.

(6) $\Lambda(\alpha \odot \phi) = \alpha \odot \Lambda(\phi)$.

(7) *If $\phi \circ \phi = \phi$ and $\phi = \phi^{-1}$, then $\Lambda(\phi)$ is an $\odot$-equivalence relation.*

Theorem 2.9 [9,10] *Let $\mathbf{D}$ be an $(L, \odot)$-uniform space. We define a subset $\mathbf{U_D}$ of $\Omega(X)$ as follows:*

$$\mathbf{U_D} = \{\phi \in \Omega(X) \mid \exists u \in \mathbf{D}, \Gamma(u) \leq \phi\}.$$

Then $\mathbf{U_D}$ is a Hutton $(L, \otimes)$- uniformity on X.

Theorem 2.10 [9,10] *Let $\mathbf{U}$ be a Hutton $(L, \otimes)$-uniformity on X. We define a subset $\mathbf{D_U}$ of $E(X \times X)$ as follows:*

$$\mathbf{D_U} = \{u \in E(X \times X) \mid \exists \phi \in \mathbf{U}, \Lambda(\phi) \leq u\}.$$

Then:

(1) *$\mathbf{D_U}$ is an $(L, \odot)$-uniformity on X.*

(2) *$\mathbf{D}_{\mathbf{U_D}} = \mathbf{D}$ and $\mathbf{U}_{\mathbf{D_U}} = \mathbf{U}$.*

Lemma 2.11 [10] *Let $f : X \to Y$ be a function. We define the image and preimage operators*

$$f^{\Rightarrow} : (L^X)^{(L^X)} \to (L^Y)^{(L^Y)}, \;\; f^{\Leftarrow} : (L^Y)^{(L^Y)} \to (L^X)^{(L^X)}$$

such that for each $\phi \in (L^X)^{(L^X)}$ and $\psi \in (L^Y)^{(L^Y)}$ for all $\mu, \mu_1, \mu_2 \in L^X, \rho_1, \rho_2 \in L^Y$,

$$f^{\Rightarrow}(\phi)(\rho) = (f^{\rightarrow} \circ \phi \circ f^{\leftarrow})(\rho) = f^{\rightarrow}(\phi(f^{\leftarrow}(\rho))),$$

$$f^{\Leftarrow}(\psi)(\mu) = (f^{\leftarrow} \circ \psi \circ f^{\rightarrow})(\mu) = f^{\leftarrow}(\psi(f^{\rightarrow}(\mu))).$$

For each $\psi, \psi_1, \psi_2 \in \Omega(Y)$ and $\phi_1, \phi_2 \in \Omega(X)$, we have the following properties.

(1) *The pair $(f^{\Rightarrow}, f^{\Leftarrow})$ is a Galois connection; i.e., $f^{\Rightarrow} \dashv f^{\Leftarrow}$.*

(2) $f^{\rightarrow}(\mu_1 \odot \mu_2) \leq f^{\rightarrow}(\mu_1) \odot f^{\rightarrow}(\mu_2)$ *with equality if f is injective and* $f^{\leftarrow}(\rho_1 \odot \rho_2) = f^{\leftarrow}(\rho_1) \odot f^{\leftarrow}(\rho_2)$.

(3) $f^{\Leftarrow}(\psi) \in \Omega_X$.

(4) *If* $\psi_1 \leq \psi_2$, *then* $f^{\Leftarrow}(\psi_1) \leq f^{\Leftarrow}(\psi_2)$.

(5) $f^{\Leftarrow}(\psi_1) \circ f^{\Leftarrow}(\psi_2) \leq f^{\Leftarrow}(\psi_1 \circ \psi_2)$ *with equality if f is onto.*

(6) $(f^{\Leftarrow}(\psi))^{-1} = f^{\Leftarrow}(\psi^{-1}) \in \Omega_X$.

(7) $f^{\Leftarrow}(\psi_1) \otimes f^{\Leftarrow}(\psi_2) = f^{\Leftarrow}(\psi_1 \otimes \psi_2)$ *and* $f^{\Rightarrow}(\phi_1) \otimes f^{\Rightarrow}(\phi_2) \geq f^{\Rightarrow}(\phi_1 \otimes \phi_2)$.

(8) $f^{\rightarrow}((f^{\Leftarrow}(\psi))^{-1}(\mu)) \leq \psi^{-1}(f^{\rightarrow}(\mu))$, *for all* $\mu \in L^X$.

Lemma 2.12 [10] *Let $f : X \to Y$ be a function. For each $v, v_1, v_2 \in E(Y \times Y), \phi \in \Omega(Y)$ and $\lambda \in L^X$, we have:*

(1) $f^{\Leftarrow}(\Gamma(v)) = f^{\leftarrow} \circ \Gamma(v) \circ f^{\rightarrow} = \Gamma((f \times f)^{\leftarrow}(v))$.

(2) $(f \times f)^{\leftarrow}(\Lambda(\phi)) = \Lambda(f^{\Leftarrow}(\phi))$.

(3) $\Gamma((f \times f)^{\leftarrow}(v^s)) = \Gamma(((f \times f)^{\leftarrow}(v))^s) = \Gamma((f \times f)^{\leftarrow}(v))^{-1}$,

(4) $(f \times f)^{\leftarrow}(v_1 \odot v_2) = (f \times f)^{\leftarrow}(v_1) \odot (f \times f)^{\leftarrow}(v_2)$,

(5) $(f \times f)^{\leftarrow}(v) \circ (f \times f)^{\leftarrow}(v) \leq (f \times f)^{\leftarrow}(v \circ v)$.

Definition 2.13 [9,10] (1) Let $(X, \mathbf{U}_1)$ and $(Y, \mathbf{U}_2)$ be Hutton $(L, \otimes)$-uniform spaces. A function $f : (X, \mathbf{U}_1) \to (Y, \mathbf{U}_2)$ is *H-uniformly continuous* if $f^{\Leftarrow}(\psi) \in \mathbf{U}_1$, for every $\psi \in \mathbf{U}_2$.

(2) Let $(X, \mathbf{D}_1)$ and $(Y, \mathbf{D}_2)$ be $(L, \odot)$-uniform spaces. A function $f : (X, \mathbf{D}_1) \to (Y, \mathbf{D}_2)$ is *uniformly continuous* if $(f \times f)^{\leftarrow}(v) \in \mathbf{D}_1$, for every $v \in \mathbf{D}_2$.

3. Topologies induced by two types uniform spaces

Definition 3.1 A subset $\mathbf{T}$ of L^X is called an *$(L, \odot)$-topology* on X if it satisfies the following conditions:

(T1) $1_X, 1_\emptyset \in \mathbf{T}$.

(T2) If $\lambda_1, \lambda_2 \in \mathbf{T}$, then $\lambda_1 \odot \lambda_2 \in \mathbf{T}$.

(T3) If $\lambda_1, \lambda_2 \in \mathbf{T}$, then $\lambda_1 \wedge \lambda_2 \in \mathbf{T}$.

(T4) If $\lambda_i \in \mathbf{T}$ for all $i \in \Gamma$, then $(\bigvee_{i \in \Gamma} \lambda_i) \in \mathbf{T}$

The pair $(X, \mathbf{T})$ is called an *$(L, \odot)$-topological space.*

An $(L, \odot)$-topological space is called *enriched* iff it satisfies:

(E) If $\lambda \in \mathbf{T}$, then $\alpha \odot \lambda \in \mathbf{T}$.

Let $(X, \mathbf{T}_1)$ and $(Y, \mathbf{T}_2)$ be $(L, \odot)$-topological spaces. A function $f : (X, \mathbf{T}_1) \to (Y, \mathbf{T}_2)$ is *L-continuous* if $f^{\leftarrow}(\lambda) \in \mathbf{T}_1$, for every $\lambda \in \mathbf{T}_2$.

Definition 3.2 A function $\mathbf{I} : L^X \to L^X$ is called an *$(L, \odot)$-interior operator* on X iff $\mathbf{I}$ satisfies the following conditions:

(I1) $\mathbf{I}(1_X) = 1_X$.

(I2) $\mathbf{I}(\lambda) \leq \lambda$.

(I3) $\mathbf{I}(\lambda \odot \mu) \geq \mathbf{I}(\lambda) \odot \mathbf{I}(\mu)$.

(I4) $\mathbf{I}(\lambda \wedge \mu) \geq \mathbf{I}(\lambda) \wedge \mathbf{I}(\mu)$.

The pair $(X, \mathbf{I})$ is called an *$(L, \odot)$-interior space.*

An $(L, \odot)$-interior space $(X, \mathbf{I})$ is called *topological* if

(T) $\mathbf{I}(\mathbf{I}(\lambda)) \geq \mathbf{I}(\lambda), \ \forall \lambda \in L^X$.

An $(L, \odot)$-interior space $(X, \mathbf{I})$ is called *enriched* if

(E) $\mathbf{I}(\alpha \odot \lambda) \geq \alpha \odot \mathbf{I}(\lambda), \ \forall \alpha \in L, \lambda \in L^X$.

Theorem 3.3 (1) *Let $(X, \mathbf{T})$ be an enriched $(L, \odot)$-topological space. Define a map $\mathbf{I_T} : L^X \to L^X$ as follows:*

$$\mathbf{I_T}(\lambda) = \bigvee \{\rho \in L^X \mid \rho \leq \lambda, \ \ \rho \in \mathbf{T}\}.$$

Then $\mathbf{I_T}$ is an enriched topological $(L, \odot)$-interior operator on X induced by $\mathbf{T}$.

(2) *Let $(X, \mathbf{I})$ be an enriched topological $(L, \odot)$-interior space. Define a subset $\mathbf{T_I}$ of L^X by*

$$\mathbf{T_I} = \{\lambda \in L^X \mid \lambda \leq \mathbf{I}(\lambda)\}.$$

Then $\mathbf{T_I}$ is an enriched $(L, \odot)$-topology on X induced by $\mathbf{I}$.

(3) $\mathbf{I_{T_I}} = \mathbf{I}$ *and* $\mathbf{T_{I_T}} = \mathbf{T}$.

Proof. (1) For all $\alpha \in L, \lambda_1, \lambda_2, \lambda \in L^X$, we have

$$\begin{aligned}
&\mathbf{I_T}(\lambda_1) \odot \mathbf{I_T}(\lambda_2) \\
&= \bigvee \{\rho_1 \in L^X \mid \rho_1 \leq \lambda_1, \ \ \rho_1 \in \mathbf{T}\} \\
&\ \odot \bigvee \{\rho_2 \in L^X \mid \rho_2 \leq \lambda_2, \ \ \rho_2 \in \mathbf{T}\} \\
&\leq \bigvee \{\rho_1 \odot \rho_2 \mid \rho_1 \odot \rho_2 \leq \lambda_1 \odot \lambda_2, \ \ \rho_1 \odot \rho_2 \in \mathbf{T}\} \\
&\leq \mathbf{I_T}(\lambda_1 \odot \lambda_2).
\end{aligned}$$

$$\begin{aligned}
&\alpha \odot \mathbf{I_T}(\lambda) \\
&= \alpha \odot \bigvee \{\rho \in L^X \mid \rho \leq \lambda, \ \ \rho \in \mathbf{T}\} \\
&= \bigvee \{\alpha \odot \rho \mid \alpha \odot \rho \leq \alpha \odot \lambda, \ \ \alpha \odot \rho \in \mathbf{T}\} \\
&\leq \mathbf{I_T}(\alpha \odot \lambda).
\end{aligned}$$

Other cases and (2) are similarly proved.

(3) Since $\mathbf{I}(\lambda) \in \mathbf{T_I}$ and $\mathbf{I}(\lambda) \leq \lambda$, by the definition of $\mathbf{I_{T_I}}$, $\mathbf{I_{T_I}}(\lambda) \geq \mathbf{I}(\lambda)$.

Suppose there exists $\lambda \in L^X$ such that $\mathbf{I_{T_I}}(\lambda) \not\leq \mathbf{I}(\lambda)$. Then there exists $\rho \in L^X$ with $\rho \in \mathbf{T_I}$ and $\rho \leq \lambda$ such that $\rho \not\leq \mathbf{I}(\lambda)$. On the other hand, since $\rho = \mathbf{I}(\rho) \leq \lambda$, then $\rho = \mathbf{I}(\rho) = \mathbf{I}(\mathbf{I}(\rho)) \leq \mathbf{I}(\lambda)$. It is a contradiction. Hence $\mathbf{I_{T_I}} \leq \mathbf{I}$.

Let $\mu \in \mathbf{T}$. Then $\mathbf{I_T}(\mu) = \mu$. Thus $\mu \in \mathbf{T_{I_T}}$. Let $\mu \in \mathbf{T_{I_T}}$. Then $\mathbf{I_T}(\mu) = \mu \in \mathbf{T}$. Hence $\mathbf{T_{I_T}} = \mathbf{T}$.

Theorem 3.4 *Let* $\mathbf{U}$ *be a Hutton* $(L,\otimes)$*-uniformity on* X. *We define a mapping* $\mathbf{I_U}: L^X \to L^X$ *as follows:*

$$\mathbf{I_U}(\lambda) = \bigvee\{\rho \in L^X \mid \exists \phi \in \mathbf{U}, \phi(\rho) \le \lambda\}.$$

Then:

(1) $\mathbf{I_U}$ *is an enriched topological* $(L,\odot)$*-interior operator on* X.

(2)

$$\begin{aligned}
\mathbf{I}_1(\lambda) &= \bigvee\{\rho \in L^X \mid \exists \phi \in \mathbf{U}, \phi(\phi(\rho)) \le \lambda\}\\
\mathbf{I}_2(\lambda) &= \bigvee\{\phi(\rho) \in L^X \mid \exists \phi \in \mathbf{U}, \phi(\phi(\rho)) \le \lambda\}\\
\mathbf{I}_3(\lambda) &= \bigvee\{\alpha \odot 1_{\{x\}} \mid \exists \phi \in \mathbf{U}, \phi(\alpha \odot 1_{\{x\}}) \le \lambda\}\\
\mathbf{I}_4(\lambda) &= \bigvee\{\alpha \odot 1_{\{x\}} \mid \exists \phi \in \mathbf{U}, \phi(1_{\{x\}}) \le \alpha \to \lambda\}
\end{aligned}$$

Then $\mathbf{I_U}(\lambda) = \mathbf{I}_i(\lambda)$ *for* $i = 1,2,3,4$.

(3) $\mathbf{T_{I_U}}$ *is an enriched* $(L,\odot)$*-topology induced by* $\mathbb{U}$.

Proof. (1) (I1) Since $\phi(1_X) = 1_X$, we have $\mathbf{I_U}(1_X) = 1_X$.

(I2) Since $\rho \le \phi(\rho) \le \lambda$, $\mathbf{I_U}(\lambda) \le \lambda$ for all $\lambda \in L^X$.

(I3) Suppose $\mathbf{I_U}(\lambda \odot \mu) \not\ge \mathbf{I_U}(\lambda) \odot \mathbf{I_U}(\mu)$. By the definition of $\mathbf{I_U}$ and (L4), there exist $\rho, \gamma \in L^X$ and $\phi, \psi \in \mathbb{U}$ with $\phi(\rho) \le \lambda$, $\psi(\gamma) \le \mu$ such that

$$\mathbf{I_U}(\lambda \odot \mu) \not\ge \rho \odot \gamma.$$

Since $\phi \otimes \psi \in \mathbf{U}$,

$$(\phi \otimes \psi)(\rho \odot \gamma) \le \phi(\rho) \odot \psi(\gamma) \le \lambda \odot \mu,$$

Thus, $\mathbf{I_U}(\lambda \odot \mu) \ge \rho \odot \gamma$. It is a contradiction. Thus (I3) holds.

(I4) Suppose $\mathbf{I_U}(\lambda \wedge \mu) \not\ge \mathbf{I_U}(\lambda) \wedge \mathbf{I_U}(\mu)$. By the definition of $\mathbf{I_U}(\lambda)$ and a completely distributive lattice L, there exist $\rho, \gamma \in L^X$ and $\phi, \psi \in \mathbf{U}$ with $\phi(\rho) \le \lambda$, $\psi(\gamma) \le \mu$ such that

$$\mathbf{I_U}(\lambda \wedge \mu) \not\ge \rho \wedge \gamma.$$

Since $\phi \otimes \psi \in \mathbf{U}$, we have $(\phi \otimes \psi)(\rho \wedge \gamma) \le \phi(\rho \wedge \gamma) \odot \psi(\overline{1}) = \phi(\rho \wedge \gamma)$, simiarly $(\phi \otimes \psi)(\rho \wedge \gamma) \le \psi(\rho \wedge \gamma)$. It implis

$$\begin{aligned}
(\phi \otimes \psi)(\rho \wedge \gamma) &\le \phi(\rho \wedge \gamma) \wedge \psi(\rho \wedge \gamma)\\
&\le \phi(\rho) \wedge \psi(\gamma) \le \lambda \wedge \mu.
\end{aligned}$$

Thus, $\mathbf{I_U}(\lambda \wedge \mu) \ge \rho \wedge \gamma$. It is a contradiction. Thus (I4) holds.

(T) Suppose there exists $\lambda \in L^X$ such that $\mathbf{I_U}(\mathbf{I_U}(\lambda)) \not\ge \mathbf{I_U}(\lambda)$. By the definition of $\mathbf{I_U}(\lambda)$, there exist $\rho \in L^X$, $\phi \in \mathbf{U}$ with $\phi(\rho) \le \lambda$ such that $\mathbf{I_U}(\mathbf{I_U}(\lambda)) \not\ge \rho$.

On the other hand, since $\phi \in \mathbf{U}$, there exists $\psi \in \mathbb{U}$ with $\psi \circ \psi \le \phi$. It implies $\psi(\psi(\rho)) \le \phi(\rho) \le \lambda$. By the definition of $\mathbf{I_U}(\lambda)$, we have $\psi(\rho) \le \mathbf{I_U}(\lambda)$. By the definition of $\mathbf{I_U}(\mathbf{I_U}(\lambda))$, it follows that $\mathbf{I_U}(\mathbf{I_U}(\lambda)) \ge \rho$. It is a contradiction. Hence, $\mathbf{I_U}(\mathbf{I_U}(\lambda)) \ge \mathbf{I_U}(\lambda)$.

(E)

$$\begin{aligned}
&\alpha \odot \mathbf{I_U}(\lambda)\\
&= \alpha \odot \bigvee\{\rho \in L^X \mid \exists \phi \in \mathbf{U}, \phi(\rho) \le \lambda\}\\
&\le \bigvee\{(\alpha \odot \rho) \in L^X \mid \exists \phi \in \mathbf{U}, \phi(\alpha \odot \rho) \le \alpha \odot \lambda\}\\
&\le \mathbf{I_U}(\alpha \odot \lambda).
\end{aligned}$$

(2) Let $\rho \in L^X$ and $\phi \in \mathbf{U}$ such that $\phi(\rho) \le \lambda$. Since $\phi \in \mathbf{U}$, there exists $\psi \in \mathbf{U}$ with $\psi \circ \psi \le \phi$ such that $(\psi \circ \psi)(\rho) \le \phi(\rho) \le \lambda$. Thus, $\mathbf{I_U}(\lambda) \le \mathbf{I}_1(\lambda)$. Since $\rho \le \psi(\rho)$, $\mathbf{I}_1 \le \mathbf{I}_2$. Hence $\mathbf{I_U}(\lambda) \le \mathbf{I}_1(\lambda) \le \mathbf{I}_2(\lambda)$. Trivially, $\mathbf{I}_2(\lambda) \le \mathbf{I_U}(\lambda)$. Hence $\mathbf{I_U}(\lambda) = \mathbf{I}_1(\lambda) = \mathbf{I}_2(\lambda)$.

Trivially, $\mathbf{I}_3(\lambda) \le \mathbf{I_U}(\lambda)$. Suppose $\mathbf{I}_3(\lambda) \not\ge \mathbf{I_U}(\lambda)$. By the definition of $\mathbf{I_U}(\lambda)$, there exist $\rho \in L^X$, $\phi \in \mathbf{U}$ with $\phi(\rho) \le \lambda$ such that $\mathbf{I}_3(\lambda) \not\ge \rho$. Since $\rho = \bigvee_{z \in X} \rho(z) \odot 1_{\{z\}}$,

$$\phi(\rho) = \phi(\bigvee_{z \in X} \rho(z) \odot 1_{\{z\}}) = \bigvee_{z \in X} \phi(\rho(z) \odot 1_{\{z\}}) \le \lambda$$

Put $\alpha_z = \rho(z)$. Since $\phi(\alpha_z \odot 1_{\{z\}}) \le \lambda$ for all $z \in X$, $\rho = \bigvee_{z \in X} \rho(z) \odot 1_{\{z\}} \le \mathbf{I}_3(\lambda)$. It is a contradiction. Hence $\mathbf{I}_3(\lambda) = \mathbf{I_U}(\lambda)$.

Since $\phi(\alpha \odot 1_{\{x\}}) = \alpha \odot \phi(1_{\{x\}}) \le \lambda$ iff $\phi(1_{\{x\}}) \le \alpha \to \lambda$, we have $\mathbf{I}_3 = \mathbf{I}_4$.

Theorem 3.5 *Let* $\mathbf{D}$ *be an* $(L,\odot)$*-uniformity on* X. *We define a mapping* $\mathbf{I_D}: L^X \to L^X$ *as follows:*

$$\mathbf{I_D}(\lambda) = \bigvee\{\rho \in L^X \mid \exists u \in \mathbf{D}, \Gamma(u)(\rho) \le \lambda\}.$$

Then:

(1) $\mathbf{I_D}$ *is an enriched topological* $(L,\odot)$*-interior operator on* X.

(2)

$$\mathbf{I_D}(\lambda) = \bigvee\{\alpha \odot 1_{\{y\}} \mid \exists u \in \mathbf{D}, \alpha \odot u(y,-) \le \lambda\}.$$

(3) $\mathbf{T_{I_D}}$ *is an enriched* $(L,\odot)$*-topology induced by* $\mathbf{D}$. *Moreover,* $\mathbf{I_{U_D}} = \mathbf{I_D}$.

(4) *If* $\mathbf{U}$ *is a Hutton* $(L,\otimes)$*-uniformity on* X, *then* $\mathbf{I_{D_U}} = \mathbf{I_U}$.

Proof. (1) (I1) Since $\Gamma(u)(1_X) \le 1_X$, we have $\mathbf{I_D}(1_X) = 1_X$.

(I2) Since $\rho \le \Gamma(u)(\rho) \le \lambda$, $\mathbf{I_D}(\lambda) \le \lambda$ for all $\lambda \in L^X$.

(I3) Suppose $\mathbf{I_D}(\lambda \odot \mu) \not\ge \mathbf{I_D}(\lambda) \odot \mathbf{I_D}(\mu)$. By the definition of $\mathbf{I_D}$ and (L4), there exist $\rho, \gamma \in L^X$ and $u, v \in \mathbf{D}$ with $\Gamma(u)(\rho) \le \lambda$, $\Gamma(v)(\gamma) \le \mu$ such that

$$\mathbf{I_D}(\lambda \odot \mu) \not\ge \rho \odot \gamma.$$

Since $\Gamma(u), \Gamma(v) \in \Omega(x)$ from Theorem 2.6(1) and $u \odot v \in \mathbf{D}$, by Theorem 2.7(2), we have:

$$\begin{aligned}
&\Gamma(u \odot v)(\rho \odot \gamma) \le \Gamma(u) \otimes \Gamma(v)(\rho \odot \gamma)\\
&\le \Gamma(u)(\rho) \odot \Gamma(v)(\gamma) \le \lambda \odot \mu.
\end{aligned}$$

Thus, $\mathbf{I_D}(\lambda \odot \mu) \geq \rho \odot \gamma$. It is a contradiction. Thus (I3) holds.

(I4) Suppose $\mathbf{I_D}(\lambda \wedge \mu) \not\geq \mathbf{I_D}(\lambda) \wedge \mathbf{I_D}(\mu)$. By the definition of $\mathbf{I_D}(\lambda)$ and a completely distributive lattice L, there exist $\rho, \gamma \in L^X$ and $u, v \in \mathbf{D}$ with $\Gamma(u)(\rho) \leq \lambda$, $\Gamma(v)(\gamma) \leq \mu$ such that

$$\mathbf{I_D}(\lambda \wedge \mu) \not\geq \rho \wedge \gamma.$$

Since $u \odot v \in \mathbf{D}$, by Theorem 2.7(2),

$$\begin{aligned}&\Gamma(u \odot v)(\rho \wedge \gamma) \leq \Gamma(u) \otimes \Gamma(v)(\rho \wedge \gamma)\\ &\leq \Gamma(u)(\rho \wedge \gamma) \wedge \Gamma(v)(\rho \wedge \gamma) \leq \Gamma(u)(\rho) \wedge \Gamma(v)(\gamma)\\ &\leq \lambda \wedge \mu.\end{aligned}$$

Thus, $\mathbf{I_U}(\lambda \wedge \mu) \geq \rho \wedge \gamma$. It is a contradiction. Thus (I4) holds.

(T) Suppose there exists $\lambda \in L^X$ such that $\mathbf{I_D}(\mathbf{I_D}(\lambda)) \not\geq \mathbf{I_D}(\lambda)$. By the definition of $\mathbf{I_D}(\lambda)$, there exist $\rho \in L^X$, $u \in \mathbf{D}$ with $\Gamma(u)(\rho) \leq \lambda$ such that $\mathbf{I_D}(\mathbf{I_D}(\lambda)) \not\geq \rho$.

On the other hand, since $u \in \mathbf{D}$, there exists $v \in \mathbf{D}$ with $v \circ v \leq u$. It implies, by Theorem 2.7(6),

$$\Gamma(v) \circ \Gamma(v)(\rho) = \Gamma(v \circ v)(\rho) \leq \Gamma(u)(\rho) \leq \lambda.$$

By the definition of $\mathbf{I_D}(\lambda)$, we have $\Gamma(v)(\rho) \leq \mathbf{I_D}(\lambda)$. By the definition of $\mathbf{I_D}(\mathbf{I_D}(\lambda))$, it follows that $\mathbf{I_D}(\mathbf{I_D}(\lambda)) \geq \rho$. It is a contradiction. Hence, $\mathbf{I_D}(\mathbf{I_D}(\lambda)) \geq \mathbf{I_D}(\lambda)$.

(E) For $\alpha \in L$ and $\lambda \in L^X$,

$$\begin{aligned}&\alpha \odot \mathbf{I_D}(\lambda)\\ &= \alpha \odot \bigvee\{\rho \in L^X \mid \exists u \in \mathbf{D}, \Gamma(u)(\rho) \leq \lambda\}\\ &\leq \bigvee\{(\alpha \odot \rho) \in L^X \mid \exists u \in \mathbf{D}, \alpha \odot \Gamma(u)(\rho) \leq \alpha \odot \lambda\}\\ &\leq \bigvee\{(\alpha \odot \rho) \in L^X \mid \exists u \in \mathbf{D}, \Gamma(u)(\alpha \odot \rho) \leq \alpha \odot \lambda\}\\ &\leq \mathbf{I_D}(\alpha \odot \lambda).\end{aligned}$$

(2) Since $\Gamma(u) \in \Omega(X)$, by $\mathbf{I}_3$ of Theorem 3.4(2), we have

$$\begin{aligned}\Gamma(u)(\alpha \odot 1_{\{y\}}) &= \bigvee_{x \in X}(\alpha \odot 1_{\{y\}})(x) \odot u(x, -)\\ &= \alpha \odot u(y, -).\end{aligned}$$

It implies

$$\begin{aligned}&\mathbf{I_D}(\lambda)\\ &= \bigvee\{\alpha \odot 1_{\{y\}} \mid \exists u \in \mathbf{D}, \Gamma(u)(\alpha \odot 1_{\{y\}}) \leq \lambda\}\\ &= \bigvee\{\alpha \odot 1_{\{y\}} \mid \exists u \in \mathbf{D}, \alpha \odot u(y, -) \leq \lambda\}\end{aligned}$$

(3) By Theorem 3.3 and (1), $\mathbf{T_{I_D}}$ is an enriched $(L, \odot)$-topology induced by $\mathbf{D}$.

For $u \in \mathbf{D}$ with $\Gamma(u) \leq \phi$, $\phi(\rho) \leq \lambda$ implies $\Gamma(u)(\rho) \leq \phi(\rho) \leq \lambda$. Thus, $\mathbf{I_{U_D}} \leq \mathbf{I_D}$.

For $u \in \mathbf{D}$ with $\Gamma(u)(\rho) \leq \lambda$, since $\Gamma(u) \in \mathbf{U_D}$, $\mathbf{I_{U_D}} \geq \mathbf{I_D}$.

(4) For $\phi \in \mathbf{U}$ with $\phi(\rho) \leq \lambda$, $\Lambda(\phi) \in \mathbf{D_U}$. So, $\Gamma(\Lambda(\phi))(\rho) = \phi(\rho) \leq \lambda$. Thus, $\mathbf{I_{D_U}} \geq \mathbf{I_U}$.

For $u \in \mathbf{D}$ with $\Gamma(u)(\rho) \leq \lambda$, by the definition of $\mathbf{D_U}$, there exists $\phi \in \mathbf{U}$ such that $\Lambda(\phi) \leq u$. So, $\Gamma(\Lambda(\phi))(\rho) = \phi(\rho) \leq \Gamma(u)(\rho) \leq \lambda$. Hence $\mathbf{I_{D_U}} \leq \mathbf{I_U}$.

Example 3.6 Let $X = \{x, y, z\}$ be a set and $([0,1], \odot)$ an biquantale defined by $x \odot y = \max\{0, x + y - 1\}$ (ref.[4-6,14]).

Define $\phi \in \Omega(X)$ as follows:

$$\phi(1_{\{x\}}) = \phi(1_{\{y\}}) = 1_{\{x,y\}}, \ \ \phi(1_{\{z\}}) = \phi(1_{\{z\}})$$

Since

$$\phi \otimes \phi(1_{\{x\}}) = \phi \otimes \phi(1_{\{y\}}) = 1_{\{x,y\}}, \phi \otimes \phi(1_{\{z\}}) = 1_{\{z\}},$$

by Lemma 2.3(2), $\phi \otimes \phi = \phi$. We have $\phi \circ \phi = \phi$ because

$$\phi \circ \phi(1_{\{x\}}) = \phi \circ \phi(1_{\{y\}}) = 1_{\{x,y\}}, \phi \circ \phi((1_{\{z\}})) = 1_{\{z\}}.$$

Since

$$\phi^{-1}(1_{\{x\}}) = \phi^{-1}(1_{\{y\}}) = 1_{\{x,y\}}, \phi^{-1}(1_{\{z\}}) = 1_{\{z\}},$$

Hence $\phi^{-1} = \phi$.

(1) Define $\mathbf{U} = \{\psi \in \Omega(X) \mid \phi \leq \psi\}$. Then $\mathbf{U}$ is a Hutton $(L, \otimes)$-uniformity on X.

For each $\lambda \in L^X$, by $\mathbf{I}_3$ of Theorem 3.4,

$$\mathbf{I_U}(\lambda)(x) = \mathbf{I_U}(\lambda)(y) = \lambda(x) \wedge \lambda(y), \ \ \mathbf{I_U}(\lambda)(z) = \lambda(z).$$

We obtain

$$\begin{aligned}\mathbf{T_{I_U}} &= \{\alpha \odot 1_X, \lambda \in L^X \mid \lambda(x) = \lambda(y) = a,\\ &\qquad \lambda(y) = b, \forall a, b, \alpha \in L\}.\end{aligned}$$

(2) We obtain $\mathbf{D_U} = \{u \in E(X \times X) \mid \Lambda(\phi) \leq u\}$. Since $\phi \circ \phi = \phi$ and $\phi^{-1} = \phi$, by Theorem 2.8(7), $\Lambda(\phi)$ is an $\odot$-equivalence relation such that

$$\Lambda(\phi)(x,x) = 1, \ \Lambda(\phi)(x,y) = 1, \ \Lambda(\phi)(x,z) = 0$$

$$\Lambda(\phi)(y,x) = 1, \ \Lambda(\phi)(y,y) = 1, \ \Lambda(\phi)(y,z) = 0$$

$$\Lambda(\phi)(z,x) = 0, \ \Lambda(\phi)(z,y) = 0, \ \Lambda(\phi)(z,z) = 1$$

Furthermore, $\Lambda(\phi) \circ \Lambda(\phi) = \Lambda(\phi)$, $\Lambda(\phi^{-1}) = \Lambda(\phi)^s = \Lambda(\phi)$ and $\Lambda(\phi) \odot \Lambda(\phi) = \Lambda(\phi \otimes \phi) = \Lambda(\phi)$. Hence $\mathbf{D_U}$ is an $(L, \odot)$-uniformity on X.

(3) For each $\lambda \in L^X$, by Theorem 3.5(2),

$$\mathbf{I_{D_U}}(\lambda) = (\alpha \odot 1_{\{x\}}) \vee (\alpha \odot 1_{\{y\}}) \vee (\beta \odot 1_{\{z\}}).$$

where $\alpha = \lambda(x) \wedge \lambda(y)$, $\beta = \lambda(z)$. Hence $\mathbf{I_{D_U}} = \mathbf{I_U}$.

Theorem 3.7 *Let, $(X, \mathbf{U})$ and $(Y, \mathbf{V})$ be $(L, \otimes)$-Hutton - uniform spaces. Let $f : (X, \mathbf{U}) \to (Y, \mathbf{V})$ be H-uniformly continuous. Then we have the following properties.*

(1) $f^{\leftarrow}(\mathbf{I_V}(\lambda)) \leq \mathbf{I_U}(f^{\leftarrow}(\lambda)))$, *for each* $\lambda \in L^Y$.

(2) $f : (X, \mathbf{T_{I_U}}) \to (Y, \mathbf{T_{I_V}})$ *is L-continuous.*

Proof. (1) Since $f^{\Leftarrow}(\psi)(f^{\leftarrow}(\rho)) \leq f^{\leftarrow}(\psi(\rho))$, we have

$$\begin{aligned}
& f^{\leftarrow}(\mathbf{I}_{\mathbf{V}}(\lambda)) \\
& = f^{\leftarrow}\Big(\bigvee\{\rho \mid \exists \psi \in \mathbf{V},\ \psi(\rho) \leq \lambda\}\Big) \\
& \leq \bigvee\{f^{\leftarrow}(\rho) \mid f^{\leftarrow}(\psi(\rho)) \leq f^{\leftarrow}(\lambda),\ \ f^{\Leftarrow}(\psi) \in \mathbb{U}\} \\
& \leq \bigvee\{f^{\leftarrow}(\rho) \mid f^{\Leftarrow}(\psi)(f^{\leftarrow}(\rho)) \leq f^{\leftarrow}(\lambda),\ \ f^{\Leftarrow}(\psi) \in \mathbb{U}\} \\
& \leq \mathbf{I}_{\mathbf{U}}(f^{\leftarrow}(\lambda))).
\end{aligned}$$

(2) Since $\lambda \leq \mathbb{I}_{\mathbf{V}}(\lambda)$ implies $f^{\leftarrow}(\lambda) \leq \mathbb{I}_{\mathbf{U}}(f^{\leftarrow}(\lambda))$ from (1), for each $\lambda \in \mathbb{T}_{\mathbf{I}_{\mathbf{V}}}$, we have $f^{\leftarrow}(\lambda) \in \mathbb{T}_{\mathbf{I}_{\mathbf{U}}}$.

Theorem 3.8 *Let $(X, \mathbf{D}_1)$ and $(Y, \mathbf{D}_2)$ be $(L, \odot)$-uniform spaces. Let $f : (X, \mathbf{D}_1) \to (Y, \mathbf{D}_2)$ be L-uniformly continuous. Then:*

(1) $\mathbf{I}_{\mathbf{D}_1}(f^{\leftarrow}(\rho)) \geq f^{\leftarrow}(\mathbf{I}_{\mathbf{D}_2}(\rho))$, *for each* $\rho \in L^Y$.

(2) $f : (X, \mathbf{T}_{\mathbf{I}_{\mathbf{D}_1}}) \to (Y, \mathbf{T}_{\mathbf{I}_{\mathbf{D}_2}})$ *is L-continuous.*

Proof. (1) Put $\lambda = f^{\leftarrow}(\gamma)$ from Lemma 2.12(1), $\Gamma(v)(\gamma) \leq \rho$ implies

$$\begin{aligned}
\Gamma((f \times f)^{\leftarrow}(v)(f^{\leftarrow}(\gamma)) & = f^{\leftarrow}(\Gamma(v)(f^{\rightarrow}(f^{\leftarrow}(\gamma))) \\
& \leq f^{\leftarrow}(\Gamma(v)(\gamma)) \leq f^{\leftarrow}(\rho).
\end{aligned}$$

Since $(f \times f)^{\leftarrow}(v) \in \mathbf{U}$ for $v \in \mathbf{V}$, we have

$$\begin{aligned}
& f^{\leftarrow}(\mathbf{I}_{\mathbf{D}_2}(\rho)) \\
& = f^{\leftarrow}(\bigvee\{\gamma \in L^Y \mid \Gamma(v)(\gamma) \leq \rho,\ \ v \in \mathbf{V}\}) \\
& = \bigvee\{f^{\leftarrow}(\gamma) \in L^X \mid \Gamma(v)(\gamma) \leq \rho,\ \ v \in \mathbf{V}\} \\
& \leq \bigvee\{f^{\leftarrow}(\gamma) \in L^X \mid \Gamma((f \times f)^{\leftarrow}(v)(f^{\leftarrow}(\gamma)) \leq f^{\leftarrow}(\rho), \\
& \quad (f \times f)^{\leftarrow}(v) \in \mathbf{U}\} \\
& \leq \mathbf{I}_{\mathbf{D}_1}(f^{\leftarrow}(\rho)).
\end{aligned}$$

(2) From (1)and Theorem 3.3, $\mathbf{I}_{\mathbf{D}_2}(\rho) \geq \rho$ implies $\mathbf{I}_{\mathbf{D}_1}(f^{\leftarrow}(\rho)) \geq \phi^{\leftarrow}(\rho)$. It is easily proved.

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Yong Chan Kim

He received the M.S and Ph.D. degrees in Department of Mathematics from Yonsei University, in 1984 and 1991, respectively. From 1991 to present, he is a professor in the Department of Mathematics, Kangnung University. His research interests are fuzzy topology and fuzzy logic.

Young Sun Kim

He received the M.S and Ph.D. degrees in Department of Mathematics from Yonsei University, in 1985 and 1991, respectively. From 1988 to present, he is a professor in the Department of Applied Mathematics, Pai Chai University. His research interests are fuzzy topology and fuzzy logic.