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ON THE CONVERGENCE OF THE MODIFIED NOOR ITERATION PROCESS WITH ERRORS IN THE CLASS OF QUASI-CONTRACTIVE OPERATORS

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ABSTRACT. We establish a general theorem to approximate fixed points of quasi-contractive operators on a normed space through the Noor iteration process with errors in the sense of Liu [9]. Our result generalizes and improves upon, among others, the corresponding result of Berinde [1].

1. INTRODUCTION AND PRELIMINARIES

Throughout this note, $\mathbb{N}$ will denote the set of all positive integers. Let C be a nonempty convex subset of a normed space E and $T : C \to C$ be a mapping. Let $\{b_n\}$ and $\{b'_n\}$ be two sequences in $[0, 1]$.

The Mann iteration process is defined by the sequence $\{x_n\}_{n=0}^{\infty}$ (see [10])

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n T x_n, \ n \in \mathbb{N}. \end{cases} \tag{1.1}$$

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n T y_n, \\ y_n = (1 - b'_n)\, x_n + b'_n T x_n, \ n \in \mathbb{N} \end{cases} \tag{1.2}$$

is known as the Ishikawa iteration process [5].

In 1995, Liu [9] introduced error terms in the sequence (1.2) defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1 - b_n)\, x_n + b_n T y_n + u_n, \\ y_n = (1 - b'_n)\, x_n + b'_n T x_n + v_n, \ n \in \mathbb{N}, \end{cases} \tag{1.3}$$

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where $\{u_n\},\{v_n\}$ are two summable sequences in C. Also it contains the Mann process with error terms

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = (1-b_n)\,x_n + b_n T y_n + u_n, \; n \in \mathbb{N}. \end{cases} \tag{1.4}$$

The sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = b_n T y_n + (1-b_n)x_n, \\ y_n = b'_n T z_n + (1-b'_n)x_n\,, \\ z_n = b''_n T x_n + (1-b''_n)x_n, \;\; n \in \mathbb{N}, \end{cases} \tag{1.5}$$

where $\{b_n\}$ $\{b'_n\}$ and $\{b''_n\}$ are appropriate sequences in $[0,1]$ is known as Noor iteration process [18].

Inspired and motivated by these facts, we suggest the following three-step iterative process with errors in the sense of Liu [9] and define the sequence $\{x_n\}_{n=0}^{\infty}$ as follows

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = b_n T y_n + (1-b_n)x_n + u_n\,, \\ y_n = b'_n T z_n + (1-b'_n)x_n + v_n, \\ z_n = b''_n T x_n + (1-b''_n)x_n + w_n, \;\; n \in \mathbb{N}, \end{cases} \tag{1.6}$$

where $\{u_n\},\{v_n\},\{w_n\}$ are three summable sequences in C.

The iteration processes (1.1-1.5) can be viewed as the special cases of the iteration process (1.6).

We recall the following definitions in a metric space (X,d). A mapping $T : X \to X$ is called an a-contraction if

$$d(Tx,Ty) \le ad(x,y) \text{ for all } x,y \in X, \tag{1.7}$$

where $0 < a < 1$.

The map T is called Kannan mapping [6] if there exists $b \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \le b[d(x,Tx) + d(y,Ty)] \;\text{ for all } x,y \in X. \tag{1.8}$$

A similar definition is due to Chatterjea [3]: there exists a $c \in (0,\frac{1}{2})$ such that

$$d(Tx,Ty) \le c[d(x,Ty) + d(y,Tx)] \text{ for all } x,y \in X. \tag{1.9}$$

Combining these three definitions, Zamfirescu [19] proved the following important result.

Theorem 1. *Let (X,d) be a complete metric space and $T : X \to X$ a mapping for which there exists the real numbers a,b and c satisfying $0 < a < 1$, $b,c \in (0,\frac{1}{2})$ such that for each pair $x,y \in X$, at least one of the following conditions holds:*

(z_1) *$d(Tx,Ty) \leq ad(x,y)$ for all $x,y \in X$*

(z_2) *$d(Tx,Ty) \leq b[d(x,Tx) + d(y,Ty)]$ for all $x,y \in X$*

(z_3) *$d(Tx,Ty) \leq c[d(x,Ty) + d(y,Tx)]$ for all $x,y \in X$.*

Then T has a unique fixed point p and the Picard iteration $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = Tx_n, \quad n \in \mathbb{N}$$

converges to p for any arbitrary but fixed $x_1 \in X$.

One of the most, general contraction condition for which the unique fixed point can be approximated by means of Picard iteration, has been obtained by Ciric [4] in 1974: there exists $0 < h < 1$ such that

$$d(Tx,Ty) \leq h\{d(x,y),\ d(x,Tx),\ d(y,Ty),\ d(x,Ty),\ d(y,Tx)\}, \tag{QC}$$

$\forall x,\ y \in X$.

Remark. 1. A mapping satisfying (QC) is commonly called quasi contraction. It is obvious that each of the conditions $(1.7 - 1.9)$ and $(z_1 - z_3)$ implies (QC).

2. An operator T satisfying the contractive conditions $(z_1) - (z_3)$ in the above theorem is called $Z-$operator.

In 2004, Berinde [1] introduced a new class of operators on an arbitrary Banach space E_1 satisfying

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|Tx - x\| \tag{1.10}$$

for any $x,y \in E_1$, $0 \leq \delta < 1$.

He proved that this class is wider than the class of Zamfiresu operators and used the Ishikawa iteration process (1.2) to approximate fixed points of this class of operators in an arbitrary Banach space given in the form of following theorem:

Theorem 2. *Let K be a nonempty closed convex subset of an arbitrary Banach space E_1. Let $T: K \to K$ be an operator satisfying (1.10). Let $\{x_n\}_{n=0}^{\infty}$ be defined through the iterative process (1.2). If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Following Berinde [1], we also have

$$\|Tx - Ty\| \leq \delta \|x - y\| + 2\delta \|Ty - y\|. \tag{1.11}$$

From (1.10) and (1.11), we have

$$\|Tx - Ty\| \leq \delta \|x - y\| + 4\delta \|Ty - y\| \|Tx - x\|; \tag{AR}$$

$x, y \in E_1$, $0 \leq \delta < 1$.

In this paper, we approximate the common fixed points of the class of mappings defined in (AR) by using the iteration process (1.6). Our result generalizes and improves upon, among others, the corresponding result of Berinde [1].

We shall need the following lemma.

Lemma 1 ([9]). *Let $\{r_n\}, \{s_n\}, \{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying*

$$r_{n+1} \leq (1 - s_n)r_n + s_n t_n + k_n \quad \textit{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n\to\infty} t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$ hold, then $\lim_{n\to\infty} r_n = 0$.

2. MAIN RESULT

Usually iterative results are obtained by making use of a fixed point theorem. Following Berinde [1], we obtain such a result without employing any fixed point theorem.

Theorem 3. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (AR). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.6). *If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\lim_{n\to\infty} \|v_n\| = 0 = \lim_{n\to\infty} \|w_n\|$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Proof. Assume that $F(T) \neq \phi$ and $w \in F(T)$, then using (1.6), we have

$$\begin{aligned} \|x_{n+1} - w\| &= \|b_n T y_n + (1 - b_n)x_n + u_n - w\| \\ &= \|(1 - b_n)(x_n - w) + b_n (Ty_n - w) + u_n\| \\ &\leq (1 - b_n) \|x_n - w\| + b_n \|Ty_n - w\| + \|u_n\|. \end{aligned} \tag{2.1}$$

Now for $x = y_n$ and $y = w$, (AR) gives

$$\|Ty_n - w\| \leq \delta \|y_n - w\|. \tag{2.2}$$

In a similar fashion, we can get

$$\begin{aligned}\|y_n - w\| &= \left\|b'_n T z_n + (1-b'_n)x_n + v_n - w\right\| \\ &= \left\|(1-b'_n)(x_n - w) + b'_n (Tz_n - w) + v_n\right\| \\ &\le (1-b'_n)\|x_n - w\| + b'_n \|Tz_n - w\| + \|v_n\| . \end{aligned} \tag{2.2}$$

Using (AR), for $x = z_n$ and $y = w$, we get

$$\|Tz_n - w\| \le \delta \|z_n - w\| . \tag{2.4}$$

Also

$$\begin{aligned}\|z_n - w\| &= \left\|b''_n T x_n + (1-b''_n)x_n + w_n - w\right\| \\ &= \left\|(1-b''_n)(x_n - w) + b''_n (Tx_n - w) + w_n\right\| \\ &\le (1-b''_n)\|x_n - w\| + b''_n \|Tx_n - w\| + \|w_n\| . \end{aligned} \tag{2.3}$$

Again by (AR), if $x = x_n$ and $y = w$, we get

$$\|Tx_n - w\| \le \delta \|x_n - w\| . \tag{2.6}$$

From (2.1-2.6), we obtain

$$\begin{aligned}\|x_{n+1} - w\| &\le (1-b_n)\|x_n - w\| + \delta b_n[(1-b'_n)\|x_n - w\| \\ &\quad + \delta b'_n \|z_n - w\| + \|v_n\|] + \|u_n\| \\ &= \left[1 - b_n[1 - \delta(1-b'_n)]\right]\|x_n - w\| + \delta^2 b_n b'_n \|z_n - w\| \\ &\quad + \delta b_n \|v_n\| + \|u_n\| \\ &\le \left[1 - b_n[1 - \delta(1-b'_n)]\right]\|x_n - w\| \\ &\quad + \delta^2 b_n b'_n \left[\left[1 - (1-\delta)b''_n\right]\|x_n - w\| + \|w_n\|\right] + \delta b_n \|v_n\| + \|u_n\| \\ &= \left[1 - (1-\delta)b_n[1 + \delta b'_n(1 + \delta b''_n)]\right]\|x_n - w\| \\ &\quad + \delta^2 b_n b'_n \|w_n\| + \delta b_n \|v_n\| + \|u_n\| . \end{aligned} \tag{2.4}$$

It may be noted that for $\delta \in [0,1)$ and $\{\eta_n\} \in [0,1]$, the following inequality is always true

$$1 \le 1 + \delta\eta_n \le 1 + \delta. \tag{2.8}$$

From (2.7) and (2.8), we get

$$\|x_{n+1} - w\| \le [1 - (1-\delta)b_n]\|x_n - w\| + \delta b_n(\delta b'_n \|w_n\| + \|v_n\|) + \|u_n\| .$$

By Lemma 1, with $\sum_{n=1}^{\infty} b_n = \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\lim_{n\to\infty} \|v_n\| = 0 = \lim_{n\to\infty} \|w_n\|$, we get that $\lim_{n\to\infty} \|x_n - w\| = 0$. Consequently $x_n \to w \in F$ and this completes the proof. □

Corollary 1. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (AR). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.5). *If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Corollary 2. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (AR). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.3). *If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$, $\sum_{n=1}^{\infty} \|u_n\| < \infty$ and $\lim_{n\to\infty} \|v_n\| = 0$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Corollary 3. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (AR). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.4). *If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$ and $\|u_n\| = 0(b_n)$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Corollary 4. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (AR). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.2). *If $F(T) \neq \phi$ and $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the unique fixed point of T.*

Corollary 5. *Let C be a nonempty closed convex subset of a normed space E. Let $T : C \to C$ be an operator satisfying (AR). Let $\{x_n\}_{n=0}^{\infty}$ be defined by the iterative process* (1.1). *If $F(T) \neq \phi$, $\sum_{n=1}^{\infty} b_n = \infty$, then $\{x_n\}_{n=0}^{\infty}$ converges strongly to a fixed point of T.*

Remark. 1. The contractive condition (1.7) makes T a continuous function on X while this is not the case with the contractive conditions $(1.8 - 1.9)$ and (AR).

2. The Chatterjea's and the Kannan's contractive conditions (1.9) and (1.8) are both included in the class of Zamfirescu operators and so their convergence theorems for the Ishikawa iteration process are obtained in Corollary 4.

3. Theorem 4 of Rhoades [13] in the context of Mann iteration on a uniformly convex Banach space has been extended in Corollary 5.

4. In Corollary 4, Theorem 8 of Rhoades [14] is generalized to the setting of normed spaces.

5. Our result also generalizes Theorem 5 of Osilike [11] and Theorem 2 of Osilike [12].

REFERENCES

1. V. Berinde: On the convergence of the Ishikawa iteration in the class of quasi contractive operators, *Acta Math Univ. Comenianae*, LXXIII, **1** (2004), 119-126.
2. Y. J. Cho, H. Zhou & G. Guo: weak and strong covergence theorems for three-step iterations with errors for Asymptotically nonexpansive mappings, *Com. and Math. with Appl.* **47** (2004), 707-717.
3. S. K. Chatterjea: Fixed point theorems, *C.R. Acad.Bulgare Sci.*, **25** (1972), 727-730.
4. L. B. Ciric: Ageneralization of Banach privciple, *Proc. Amer. Math. Soc.* **45** (1974), 727-730.
5. S. Ishikawa: Fixed points by a new iteration method, *Proc.Amer.Math.Soc.* **44** (1974), 147-150.
6. R. Kannan: Some results on fixed points, *Bull. Calcutta Math. Soc.* **10** (1968), 71-76.
7. R. Kannan: Some results on fixed points III, *Fund. Math.* **70** (1971), 169-177.
8. R. Kannan: Construction of fixed points of class of nonlinear mappings, *J. Math. Anal. Appl.* **41** (1973), 430-438.
9. L. S. Liu: Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces, *J. Math. Anal. Appl.* **194** (1995), no. 1, 114–125.
10. W. R. Mann: Mean value methods in iterations, *Proc. Amer. Math. Soc.* **4** (1953), 506-510.
11. M. O. Osilike: Short proofs of stability results for fixed point iteration procedures for a class of contractive-type mappings, *Indian J. Pure and Appl. Math.* **30** (1999), no. 12, 1229-1234.
12. M. O. Osilike: Stability results for fixed point iteration procedures, *J. Nigerian Math. Soc.* **14/15**, 1995/1996, 17-29.
13. B. E. Rhoades: Fixed point iteration using infinite matrices, *Trans. Amer. Math. Soc.* **196** (1974), 161-176.
14. B. E. Rhoades: Comments on two fixed point iteration method, *J. Math. Anal. Appl.* **56** (1976), no. 2, 741-750.
15. W. Takahashi: Iterative methods for approximation of fixed points and thier applications, *J. Oper. Res. Soc. Jpn.* **43** (2000), no. 1, 87 -108.
16. W. Takahashi & T. Tamura: Convergence theorems for a pair of nonexpansive mappings, *J. Convex Analysis*, **5** (1995), no. 1, 45-58.
17. Y. Xu: Ishikawa and Mann iteration process with errors for nonlinear strongly accretive operator equations, *J. Math. Anal. Appl.* **224** (1998), 91-101.
18. B. L. Xu & M. Aslam Noor: Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, *J. Math. Anal. Appl.* **267** (2002), 444-453.
19. T. Zamfirescu: Fix point theorems in metric spaces, *Arch. Math.*(Basel), **23** (1972), 292-298.

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