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# ESTIMATION OF THE NUMBER OF ROOTS ON THE COMPLEMENT

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ABSTRACT. Let $f : (X, A) \to (Y, B)$ be a map of pairs of compact polyhedra. A surplus Nielsen root number $SN(f; X \setminus A, c)$ is defined which is lower bound for the number of roots on $X \setminus A$ for all maps in the homotopy class of $f$. It is shown that for many pairs this lower bound is the best possible one, as $SN(f; X \setminus A, c)$ can be realized without by-passing condition.

## 1. INTRODUCTION

Zhao considered the minimum number $MF[f; X \setminus A]$ of fixed points on the complement $X \setminus A$ and defined the Nielsen number on the complementary space of a given map $f : (X, A) \to (X, A)$, $N(f; X \setminus A)$ which is a lower bound for $MF[f; X \setminus A]$ and has the same basic properties as $N(f; X, A)$([8]). Zhao[9] introduced a new concept "surplus Nielsen number", $SN(f; X \setminus A)$, which is a lower bound for the number of fixed points on $X \setminus A$ for all maps in the homotopy class of $f$. And he showed that for many pairs this lower bound is the best possible one, as $SN(f; X \setminus A)$ can be realized without the by-passing condition.

This paper is a analogy of Zhao[9]. To determine the minimal number $MR[f; X \setminus A, c]$ of roots at $c \in B$ on $X \setminus A$ for all maps in the homotopy class of a given map $f : (X, A) \to (Y, B)$, the Nielsen root number on the complementary space $N(f; X \setminus A, c)$ is introduced in Yang[7], which is a lower bound for $MR[f; X \setminus A, c]$.

It is the purpose of this paper to introduce a better lower bound for $MR[f; X \setminus A, c]$, which can be realized without the hypothesis that $A$ can be by-passed. The method used here follows that of Zhao[9]. After some preparation in section 2, the

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surplus Nielsen root number of $f$ on $X \setminus A$, $SN(f; X \setminus A, c)$, is defined (Definition 3.1), $SN(f; X \setminus A, c) \geq N(f; X \setminus A, c)$. In section 4, we shall prove that $SN(f; X \setminus A, c) = MR[f; X \setminus A, c]$ if $X$ and every component of $X \setminus A$ is a manifold with dimension different from 2.

## 2. Root Classes on the Subspace

Let $f : X \to Y$ be a map of compact polyhedron, and let $U$ be a subset of $X$ which has finitely many arcwise connected components. A root class of $f|_U : U \to Y$ is said to be a root class of $f$ on $U$.

**Definition 2.1.** Two roots $x_0$ and $x_1$ of $f : X \to Y$ on $U$ are said to belong to the same root class of $f$ on $U$ if there exists a path $\alpha$ in $U$ from $x_0$ to $x_1$ such that $e_c \simeq f \cdot \alpha$ rel$\{0, 1\}$.

It is obvious that every root class of $f$ on $U$ belongs to a root class of $f$. The root classes of $f$ on $U$ have the same basic properties as original root classes. We repeat some basic properties of root classes, which can be found in [4].

**Proposition 2.2.** *The root set $\Gamma(f\ |_U)$ of $f$ on $U$ splits into a disjoint union of root classes on $U$.*

**Proposition 2.3.** *Every root class of $f$ on $U$ is an open subset of $\Gamma(f\ |_U)$.*

**Definition 2.4.** Let $H : f_0 \simeq f_1 : X \to Y$ be a homotopy. For $x_0 \in \Gamma(f_0\ |_U)$ and $x_1 \in \Gamma(f_1\ |_U)$, we say that $x_0$ and $x_1$ are *H-related* on $U$ if there exists a path $\beta$ in $U$ from $x_0$ to $x_1$ such that $H(\beta(t), t) \simeq e_c$ rel$\{0, 1\}$.

**Proposition 2.5.** *Let $H : f_0 \simeq f_1 : X \to Y$ be a homotopy. Let $x_0$ belong to a root class $\mathbb{R}_0$ of $f_0$ on $U$ and let $x_1$ belong to a root class $\mathbb{R}_1$ of $f_1$ on $U$. Let $x_0$ and $x_1$ be H-related on $U$. Then $x_0'$ and $x_1'$ are H-related on $U$ for any $x_0' \in \mathbb{R}_0$ and $x_1' \in \mathbb{R}_1$.*

**Definition 2.6.** For a root class $\mathbb{R}_0$ of $f_0$ on $U$ and a root class $\mathbb{R}_1$ of $f_1$ on $U$, we say that $\mathbb{R}_0$ and $\mathbb{R}_1$ are *H-related* on $U$ if there exist $x_0 \in \mathbb{R}_0$ and $x_1 \in \mathbb{R}_1$ such that $x_0$ and $x_1$ are H-related on $U$.

By Proposition 2.5, this definition is independent of the choice of $x_0$ and $x_1$.

For a subset $D \subset X \times I$, the subset $D_t = \{x \in X \mid (x, t) \in D\}$ of $X$ will be called the *t-slice* of $D$.

**Proposition 2.7.** *Let $H : f_0 \simeq f_1 : X \to Y$ be a homotopy, and let $\mathbb{R}_0$ and $\mathbb{R}_1$ be root classes of $f_0$ and $f_1$ on $U$ respectively. Then $\mathbb{R}_0$ and $\mathbb{R}_1$ are $H$-related on $U$ if and only if they are respectively the 0- and 1-slices of a single root class of $H$ on $U \times I$.*

## 3. Surplus Nielsen Root Number $SN(f; X \setminus A, c)$

Let $(X, A)$ be a pair of compact polyhedra, then $(X \setminus A)$ consists of finitely many components and every component of $X \setminus A$ is arcwise connected and semilocally 1-connected. Let us consider the map $f : (X, A) \to (Y, B)$, and a homotopy of the form $H : (X \times I, A \times I) \to (Y, B)$.

**Definition 3.1.** A root class $\mathbb{R}$ of $f$ on $X \setminus A$ is said to be a *nonsurplus* root class of $f$ on $X \setminus A$ if there is a point $x_0 \in \mathbb{R}$ and there is a path $\alpha : I, 0, I - \{1\}, 1 \to X, x_0, X \setminus A, A$ such that

$$f \cdot \alpha \simeq e_c : I, 0, 1 \to Y, c, B.$$

A root class of $f$ on $X \setminus A$ which is not a nonsurplus root class of $f$ on $X \setminus A$ is said to be a *surplus* root class of $f$ on $X \setminus A$.

By Definition 2.1 and Definition 3.1, we have

**Corollary 3.2.** *A root $x_0$ of $f$ on $X \setminus A$ belongs to a nonsurplus root class if and only if there exists a path from $x_0$ to $A$ satisfying the conditions of Definition 3.1.*

**Theorem 3.3.** *The number of surplus root classes of $f$ on $X \setminus A$ is finite, each of them is a compact subset of $X$.*

*Proof.* Let $\mathbb{R}$ be a surplus root class of $f$ on $X \setminus A$, we shall prove $\mathbb{R}$ is compact. Suppose $x_0 \in X - \mathbb{R}$, it suffices to find a neighborhood $V$ of $x_0$ in $X$ such that $V \cap \mathbb{R} = \varnothing$.

(i) If $x_0 \notin \Gamma(f, c)$, we can take $V = X - \Gamma(f, c)$.

(ii) If $x_0 \in \Gamma(f; X \setminus A, c)$, then $x_0$ belongs to a root class $\mathbb{R}'$ of $f$ on $X - A$. By Proposition 2.2 and 2.3, $\mathbb{R}' \cap \mathbb{R} = \varnothing$ and there is a neighborhood $V$ of $x_0$ in $X \setminus A$ such that $V \cap \Gamma(f; X \setminus A, c) \subset \mathbb{R}'$. Since $X \setminus A$ is an open subset of $X$, $V$ is also a neighborhood of $x_0$ in $X$ and $V \cap \mathbb{R} \subset \mathbb{R}' \cap \mathbb{R} = \varnothing$.

(iii) If $x_0 \in A \cap \Gamma(f, c)$. Let $C$ be the component of $X \smallsetminus A$ containing $\mathbb{R}$. Assume that $x_0 \in cl(C)$, otherwise we can take $V = X - cl(C)$. Pick a neighborhood $W$ of $c$ such that every loop in $W$ is trivial in $Y$. There is an arcwise connected neighborhood $V$ of $x_0$ such that $V \subset f^{-1}(W)$. Suppose $x_1 \in V \cap \Gamma(f\,|_C)$, take a path $\alpha$ in $V$ from $x_1$ to $x_0$ with $\alpha(I-1) \subset C$, then $f \cdot \alpha$ in $W$, hence

$$f \cdot \alpha \simeq e_c : I,\, 0,\, 1 \to Y,\, c,\, B$$

Thus, $x_1$ is in a nonsurplus root class of $f$ on $X \smallsetminus A$, and this implies that $V \cap \mathbb{R} = \varnothing$.

From the proof above, we also get that the union of all the surplus root classes of $f$ on $X \smallsetminus A$ is a compact set. As in [3, p. 7, Corollary 1.13], we get that the number of surplus root classes of $f$ on $X \smallsetminus A$ is finite. □

From this theorem, we can define the index of a surplus root class of $f$ on $X \smallsetminus A$ in the same way as in [4], which is a homomorphism from $H_*(X) \to H_*(Y, Y - \{c\})$.

**Definition 3.4.** A surplus root class $\mathbb{R}$ of $f$ on $X \smallsetminus A$ is *essential* if $\text{ind}(f, \mathbb{R}) \neq 0$; *inessential* if $\text{ind}(f, \mathbb{R}) = 0$. The number of essential surplus root classes of $f$ on $X \smallsetminus A$ is called the *surplus Nielsen root number* of $f$ on $X \smallsetminus A$, denoted $SN(f; X \smallsetminus A, c)$.

**Lemma 3.5.** *Let $H : f \simeq g : (X, A) \to (Y, B)$ be a homotopy, let $\mathbb{R}_0$ and $\mathbb{R}_1$ be root classes of $f$ and $g$ on $X \smallsetminus A$ respectively, and let $\mathbb{R}_0$ and $\mathbb{R}_1$ be H-related on $X \smallsetminus A$. Then the following conditions are equivalent:*

(i) *$\mathbb{R}_0$ is a surplus root class of $f$ on $X \smallsetminus A$,*
(ii) *$\mathbb{R}_1$ is a surplus root class of $g$ on $X \smallsetminus A$,*
(iii) *$\mathbb{R}_0$ and $\mathbb{R}_1$ are respectively the 0- and 1-slices of a single surplus root class of $H$ on $(X \times I) - (A \times I)$.*

*Proof.* Since $\mathbb{R}_0$ and $\mathbb{R}_1$ are $H$-related on $X \smallsetminus A$, we can assume, by Proposition 2.7, that $\mathbb{R}_0$ and $\mathbb{R}_1$ are respectively the 0- and 1-slices of a single root class $\mathbb{R}$ on $(X \times I) - (A \times I)$.

If $\mathbb{R}_0$ is a nonsurplus root class of $f$ on $X \smallsetminus A$, then there is a path

$$\alpha : I,\, 0,\, I - \{1\},\, 1 \to X,\, x_0,\, X \smallsetminus A,\, A$$

such that

$$e_c \simeq f \cdot \alpha : I,\, 0,\, 1 \to Y,\, c,\, B,$$

where $x_0 \in \mathbb{R}_0$. Define a map $i_0 : X \to X \times I$ and $j_0 : Y \to Y \times I$ by $i_0(x) = (x, 0)$ and $j_0(y) = (y, 0)$ respectively. Then we get a path

$$i_0 \cdot \alpha : I, 0, I - \{1\}, 1 \to X \times I, (x_0, 0), (X \times I) - (A \times I), A \times I$$

with

$$\mathbb{H} \cdot (i_0 \cdot \alpha) = j_0 \cdot (f \cdot \alpha) \simeq j_0 \cdot e_c : I, 0, 1 \to Y \times I, (c, 0), B \times I.$$

Note that $(x_0, 0) \in \mathbb{R}$, it follows that $\mathbb{R}$ is a nonsurplus root class of $H$ on $(X \times I) - (A \times I)$.

Furthermore, if $\mathbb{R}$ is a nonsurplus root class on $(X \times I) - (A \times I)$, then there is a path

$$\beta : I, 0, I - \{1\}, 1 \to X \times I, (x', s), (X \times I) - (A \times I), A \times I$$

with $(x', s) \in \mathbb{R}$ such that

$$e_{(c,s)} \simeq \mathbb{H} \cdot \beta : I, 0, 1 \to Y \times I, (c, s), A \times I.$$

Thus, we $x' \in \Gamma(f_s)$, where $f_s$ is the $s$-slice of $H$, i.e. $f_s(x) = H(x, s)$. For a $x_1 \in \mathbb{R}_1$, $x_1$ and $x'$ are respectively in the 1- and $s$-slices of $\mathbb{R}$, and then they are $H'$-related on $X - A$, where $H'(x, t) = H(x, 1 - t + s \cdot t)$ is a homotopy from $g$ to $f_s$. By Definition 2.4, there is a path $\gamma$ in $X \setminus A$ from $x_1$ to $x'$ such that

$$H'(\gamma(t), t) \simeq \gamma(t) rel\{0, 1\},$$

i.e.,

$$H(\gamma(t), 1 - t + s \cdot t) \simeq \gamma(t) rel\{0, 1\}.$$

Define maps $p : X \times I \to X$ and $q : Y \times I \to Y$ by $p(x, t) = x, q(y, t) = y$ respectively, then the product of $\gamma$ and $p \cdot \beta$

$$\gamma(p \cdot \beta) : I, 0, I - \{1\}, 1 \to X, x_1, X \setminus A, A$$

is a path from $x_1$ to $A$, and

$$g \cdot \gamma(t) = H(\gamma(t), 1) \simeq H(\gamma(t), 1 - t + s \cdot t) \simeq \gamma(t),$$

$$g \cdot (p \cdot \beta) = H(p \cdot \beta, 1) \simeq q \cdot (\mathbb{H} \cdot \beta) \simeq q \cdot e_{(c, s)}.$$

Moreover, we get

$$e_c \simeq g(\gamma(p \cdot \beta)) : I, 0, 1 \to Y, c, B.$$

Thus, $\mathbb{R}_1$ is a nonsurplus root class of $g$ on $X \setminus A$.

The converse is the same. □

By this lemma we have

**Theorem 3.6** (Homotopy invariance). *If two maps $f \simeq g : (X, A) \to (Y, B)$ are homotopic, then $SN(f; X \setminus A, c) = SN(g; X \setminus A, c)$.*

The next theorem follows directly from Theorem 3.3, Theorem 3.6 and the properties of the root index.

**Theorem 3.7.** *$SN(f; X \setminus A, c)$ is a nonnegative integer. Any map which is homotopic to $f : (X, A) \to (Y, B)$ has at least $SN(f; X \setminus A, c)$ roots on $X \setminus A$. Thus, $SN(f; X \setminus A, c) \leq MR[f; X \setminus A, c]$.*

**Theorem 3.8.** *Let $f : (X, A) \to (Y, B)$ be a map of pairs of compact polyhedra, then $SN(f; X \setminus A, c) \geq N(f; X \setminus A, c)$. If $A$ can be by-passed in $X$, then $SN(f; X \setminus A, c) = N(f; X \setminus A, c)$.*

*Proof.* By Definition 3.1 and [8, Theorem 2.3], a root class of $f$ on $X \setminus A$ which is contained in a weakly noncommon root class is a surplus one, all of them lie in an open subset $X \setminus A$ of $X$. By additivity of the root index, an essential weakly noncommon root class contains at least one essential surplus root class of $f$ on $X \setminus A$. Thus, $SN(f; X \setminus A, c) \geq N(f; X \setminus A, c)$.

If $A$ can be by-passed in $X$, then a root class of $f$ will contain at most one root class of $f$ on $X \setminus A$. By [8, Lemma 3.5], every surplus root class of $f$ on $X \setminus A$ is contained in a weakly noncommon root class, therefore $SN(f; X \setminus A, c) = N(f; X \setminus A, c)$. □

Following example shows that our new lower bound $SN(f; X \setminus A, c)$ can be greater strictly than $N(f; X \setminus A, c)$.

**Example 3.9.** Let $X = S^1 = \{e^{\theta i}\}$, and let $A = \{e^0, e^{\pi i}\}$. A map $f : (X, A) \to (X, A)$ is given by $f(e^{\theta i}) = (e^{-2|\theta - \pi| i})$. The point $c = e^{\pi i}$.

As a map on $X$, $f$ is homotopic to a map $g : X \to X$ given by $g(e^{\theta i}) = e^0$. Hence, $f$ has no essential root class. It follows that $N(f; X \setminus A, c) = 0$. But, two roots $\{e^{\frac{\pi}{2} i}, e^{\frac{3\pi}{2} i}\}$ at $c$ lie in different components of $X - A$ and has non-zero indices. Thus, $SN(f; X \setminus A, c) = 2$.

## 4. MINIMUM THEOREM FOR $SN(f; X \setminus A, c)$

**Lemma 4.1** *Let $(X, A)$ and $(Y, B)$ be pairs of compact polyhdra, where every component of $X \setminus A$ is a PL manifold with dimension greater than 2. Let $x_0$ be* *an isolated root of a root finite map $f : (X, A) \to (Y, B)$ on $X \smallsetminus A$. Suppose $\alpha : I, 0, I-\{1\}, 1 \to X, x_0, X\smallsetminus A, A$ is a path from $x_0$ to $A$ with $\Gamma(f)\cap\alpha(I) = \{x_0\}$ and*

$$e_c \simeq f \cdot \alpha : I, 0, 1 \to Y, c, B.$$

*Then $f$ is homotopic to a map $f' : (X, A) \to (Y, B)$ with*

$$\Gamma(f') = (\Gamma(f) - \{x_0\}) \cap \{\alpha(1)\}.$$

*Proof.* By a perturbation, we can assume that $\alpha$ is a $PL$ arc which containing no roots except for starting point. Let $H : (I \times I, \{1\} \times I) \to (Y, B)$ be the homotopy from $e_c$ to $f \cdot \alpha$, i.e. $H(t, 0) = c$ and $H(t, 1) = f \cdot \alpha(t)$. Since $(Y, B)$ is a simplicial pair, we may assume that $c$ is a vertex of $Y$ and $\{H(1, s)\}_{0\leq s\leq 1}$ is a $PL$ arc in $B$ from $c$ to $f(x_0)$. Choose a conic neighborhood $N(f(x_0), \varepsilon) = \{y \in Y | d(y, f(x_0)) < \varepsilon\}$ of $f(x_0)$ such that $N \cap H(\{1\}, I)$ is a line segment. We define a homotopy $G : (X \times I, A \times I) \to (Y, B)$ by

$$G(x, t) = \begin{cases} f(x) & \text{if } f(x) \notin N(c, \varepsilon t) \\ (\frac{2}{\varepsilon t} d(f(x), f(x_0)) - 1) f(x) & \\ +(2 - \frac{2}{\varepsilon t} d(f(x), f(x_0))) f(x_0) & \text{if } 0 < \frac{\varepsilon t}{2} < d(f(x), f(x_0)) \leq \varepsilon t \\ H(1, 1 - t + \frac{2}{\varepsilon} d(f(x), f(x_0))) & \text{if } 0 \leq d(f(x), f(x_0)) \leq \frac{\varepsilon t}{2} \end{cases}$$

(cf. [4]). We define $g : (X, A) \to (Y, B)$ by $g(x) = G(x, 1)$, then $g$ is homotopic to $f$ with $\Gamma(g) = \Gamma(f) \cup \{\alpha(1)\}$ and $e_c \simeq g \cdot \alpha$.

By using the method in [6] and [2], we can combine the root $x_0$ to $\alpha(1)$. □

**Theorem 4.2.** *Let $(X, A)$ and $(Y, B)$ be pairs of compact polyhedra such that*

(1) *$X$ and $Y$ are PL manifolds with same dimension,*

(2) *every component of $X \smallsetminus A$ is a PL manifold with dimension greater than 2.*

*Then every map $f : (X, A) \to (Y, B)$ is homotopic to a map $g : (X, A) \to (Y, B)$ with $SN(f; X \smallsetminus A, c)$ roots on $X \smallsetminus A$.*

*Proof.* By transversality and homotopy extension, we can assume that $f$ is root-finite and that all roots of $f$ on $X \smallsetminus A$ lie in maximal simplexes. We can unite roots belonging to the same root class of $f$ on $X \smallsetminus A$ as in [6]. Suppose $x_0$ lies in a nonsurplus root class, then there is a path

$$\alpha : I, 0, I - \{1\}, 1 \to X, x_0, X \smallsetminus A, A$$

such that

$$e_c \simeq f \cdot \alpha : I, 0, 1 \to Y, c, B.$$

By Lemma 4.1, we shall move the root $x_0$ to $\alpha(1) \in A$. As $X$ has no local cut point, we can take the paths with different terminal points which are not roots. Finally, delete root classes on $X \setminus A$ which consist of a single root of index zero by the usual method ([1, p. 123, Theorem 4]). Then we get a map $g : (X, A) \to (Y, B)$ with $SN(f; X \setminus A, c)$ roots on $X - A$. □

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