Journal of the Korea Society of Mathematical Education Series D:
Research in Mathematical Education
Vol. 10, No. 1, March 2006, 1–11

韓國數學教育學會誌 시리즈 D:
<數學教育研究>
제 10 권 제 1 호 2006 년 3 월, 1–11

Developing Mathematical Promise and Creativity[1]

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(Received February 27, 2006)

In today's world, it is not enough to be proficient at computation or at memorizing rote procedures to solve routine problems. These skills are important, but even more important are the abilities to recognize and define problems, generate multiple solutions or paths toward solution, reason, justify conclusions, and communicate results. These are not abilities that one is born with and they do not generally develop on their own. For students to become gifted, promising, and creative mathematicians, these talents must be cultivated and nurtured.

Keywords: mathematical creativity, creative mathematical thinking.
ZDM Classification: C40, U40
MSC2000 Classification: 97D50

WHAT IS MATHEMATICAL CREATIVITY?

Creativity is often not associated with the traditional image of school mathematics that students may think of as a static body of knowledge that someone discovered long ago that must be memorized and repeated. This portrayal of school mathematics has led to lessons where students tediously learn a collection of techniques by following predetermined rules. Hence it might be difficult to understand or even attempt to define what *mathematical creativity* is about for the students in this context. However, for students to become powerful, inventive mathematicians, they must view mathematics as a human activity that involves reasoning, strategizing and problem solving – activities that

[1] This paper will be presented as a Plenary Lecture at the Eleventh International Seminar of Mathematics Education on Creativity Development at the Chonam National University, Gwangju, Korea, April 7, 2006.
This paper has been adapted from "Developing Mathematical Creativity in Young Children," a series of workshops (SGA Workshops-39) presented with Prof. Hartwig Meissner of Muenster, Germany and Prof. Pui Yee Foong of Singapore, at the International Congress on Mathematical Education, Denmark, July 2004.

encourage mathematization, a process that involves *creative mathematical thinking.* Students should not only know the concepts and procedures, but also should know how mathematics is created and used to learn new concepts and to solve problems in original and numerous ways. For mathematical creativity to develop, students must be confronted with rich tasks or problems. The creative teacher in such a classroom moves away from the image as an expert who knows all to someone who should spur students to see themselves as autonomous learners who can think critically and make decisions for themselves.

RICH TASKS FOR CREATIVE MATHEMATICIANS

Innovative products, individuality and creativity are characteristics that often are not associated with mathematics learning and teaching, but a focus on these could go a long way toward increasing the mathematical power of children and teachers alike. In order to encourage students to make sense of the world around them and to be creative in investigating mathematical concepts, teachers should use rich, interesting problems that can be explored on a variety of levels in a variety of ways. Several of these can be found in problems and patterns that mathematicians have investigated for hundreds of years such as the Pythagorean Theorem, Pascal's triangle and magic squares as well as in more recent areas of investigation such as computer logic and fractals.

Teachers should expect problems to be solved in a variety of ways and should give students a chance to explain their reasoning to each other. They should use one problem as a springboard for several others and work on these problems with colleagues before trying them with children to see how many solutions, patterns, generalizations, and related problems they can find.

Good tasks or problems for encouraging mathematical creativity should meet the following criteria:

1. Tasks should ask questions that make students think deeply, even about simple concepts, not questions that make them guess what the teacher is thinking.
2. Tasks should have an entry point for all students and lend themselves to challenging even the most advanced students.
3. Tasks should enable children to build on previous knowledge and to discover previously unknown mathematical principles and concepts.
4. Tasks should be connected to core standards and benchmarks in mathematics.
5. Tasks should be rich, with a wide range of opportunities for children to explore, reflect, extend, and branch out into new related areas.
6. Tasks should give children the opportunity to demonstrate abilities in a variety

of ways, verbally, geometrically, graphically, algebraically, numerically, etc.

7. Tasks should allow children to use their abilities to question, reason, communicate, solve problems, and make connections to other areas of mathematics as well as to other subject areas and "real world" problems.
8. Tasks should make full use of tools including technology such as calculators and computers as well as mathematical manipulatives and models.
9. Tasks should give time for individual reflection and problems solving as well as time for group exploration and discovery.
10. Tasks should be interesting and should actively involve the child building a variety of thinking and learning styles.
11. Tasks should be open with more than one right answer and/or more than one path to solution.
12. Tasks should encourage students to question the answers and not just answer the questions to develop deep mathematical sense.

EXERCISES OR PROBLEMS: OPEN OR CLOSED?

Mathematical exercises that one typically sees in textbooks are those that have one right answer and often expect one algorithmic method of solution. These might include exercises to practice computation or simple word problems with obvious one-step or two-step solutions. These differ from a problem that is commonly defined as a situation with no immediate answer or immediate method of solution that students accept the challenge of solving. Problems may be open or closed. Closed-ended problems also have one right answer but may include multi-step or non-routine solutions. Some researchers distinguish between problems with an open middle and those problems that are open-ended. Problems with an open middle may be solved in a variety of ways that are determined by the students and not directed by the teacher. These may have a single or multiple solutions. Open-ended problems do not have a single right answer. These may be problems that are not completely defined and have missing or ambiguous data. Many real world problems fall in this category. It is through problem solving that students begin to develop their mathematical power and creativity. (Foong 1999) Problem solving alone is not enough, however. Students need to go beyond solving problems to posing new problems and investigations.

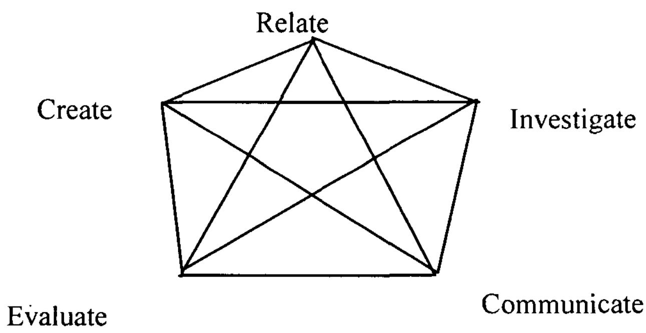
We need to help all our students move along the following continuum:

innumerates — doers — computers — consumers — problem solvers — problem posers — creators

Too often, students are satisfied with getting an answer to a problem and not looking at it any further. In this way, they miss the excitement of thinking deeply about mathematical ideas and discovering new concepts. The questions that students raise once the original problem has been solved are often the beginning of the real mathematics.

QUESTIONS, QUESTIONS, QUESTIONS

The following heuristic to can be used to encourage students to think like creative investigative mathematicians.



Using this heuristic, students may start at any point on the diagram and proceed in any order. One possible order might be:

- Relate the problem to other problems that you have solved. How is this similar to other mathematical ideas that you have seen? How is it different?
- Investigate the problem. Think deeply and ask questions.
- Evaluate your findings. Did you answer the question? Does the answer make sense?
- Communicate your results. How can you best let others know what you have discovered?
- Create new questions to explore. What else would you like to find out about this topic? Start a new investigation.

To assist students in their creation of new mathematical insights, some suggested questions for creative mathematical investigations are:

- What or what if?
  What patterns do I see in this data? What generalizations might I make from the patterns? What proof do I have? What are the chances? What is the best answer, the best method of solution, the best strategy to begin with...? What if I change one or more parts of the problem?
- When?
  When does this work? When does this not work?
- Where?
  Where did that come from? Where should I start? Where might I go for help?
- Why or why not?
  Why does that work? If it does not work, why not?
- How?
  How is this like other mathematical problems or patterns that I have seen? How does it differ? How does this relate to "real-life" situations or models? How many solutions are possible? How many ways might I use to represent, simulate, model, or visualize these ideas? How many ways might I sort, organize, and present this information?

ASSESSMENT CRITERIA

If you wish students to develop deeper understanding of concepts and become creative investigative mathematicians, you should use criteria for assessment that encourage depth and creativity such as:

Depth of understanding - the extent to which core concepts are understood, explored and developed.
Fluency - the number of different correct answers, methods of solution, or new questions formulated.
Flexibility - the number of different categories of answers, methods, or questions.
Originality - solutions, methods or questions that are unique and show insight.
Elaboration or elegance - clarity and quality of expression of thinking, including charts, graphs, drawings, models, and words.
Generalizations - patterns that are noted, hypothesized, and verified for larger categories.

Extensions - related questions that are asked and explored, especially those involving why and what if.

Help students learn to move from questions with one right answer to those that require reasoning and justification and to open-ended problems and explorations that have several solutions or related problems that will deepen and extend the concepts being learned. **Remember that the real learning frequently begins after the original problem has been solved.**

SCORING RUBRIC TO ENCOURAGE THINKING LIKE A CREATIVE MATHEMATICIAN[2]

See the table on p.7

[2] Adapted from Sheffield (2000).

Assessment Criteria	1 Novice	2 Apprentice	3 Proficient	4 Distinguished
Depth of Understanding	Little or no understanding	Partial understanding; minor mathematical errors	Good understanding; mathematically correct	In-depth understanding; well-developed ideas
Fluency	One incomplete or unworkable strategy or technique	At least one appropriate solution with strategy or technique shown.	At least two appropriate solutions, may use the same strategy or technique	Several appropriate solutions, may use the same strategy or technique
Flexibility	No method apparent	At least one method (e. g., all graphs, all algebraic equations and so on)	At least two methods of solution (e. g., geometric, graphical, algebraic, physical modeling)	Three or more methods of solution (e. g., geometric, graphical, algebraic, physical modeling)
Originality	Method may be different but does not lead to a solution	Method will lead to a solution but is fairly common	Unusual, workable method used by only a few students, or uncommon solution	Unique, insightful method or solution used only by one or two students
Elaboration or Elegance	Little or no appropriate explanation given	Explanation is understandable but is unclear in some places	Clear explanation using correct mathematical terms	Clear, concise, precise explanations making good use of graphs, charts, models, or equations
Generalizations and Reasoning	No generalizations made, or they are incorrect and reasoning is unclear	At least one correct generalizations made; but not well-supported with clear reasoning	At least one well-made, supported generalization, or more than one correct but unsupported generalization	Several well-supported generalizations; clear reasoning
Extensions	No related mathematical question explored	At least one related Mathematical question appropriately explored	One related question explored in-depth, or more than one question appropriately explored	More than one related question explored in-depth

COMMUNICATION SUGGESTIONS[3]

See the figure blow.

Role of a Student Mathematician

⇛ Repeat/rephrase

⇛ Agree/disagree...and tell why

⇛ Add on to...

⇛ Wait, think, and go deeper

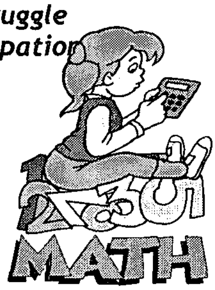
⇛ Talk to a partner

⇛ Record reasoning and questions

Our role as teachers is to:

✓ *ask questions that encourage mathematical creativity and reasoning*
✓ *elicit, engage and challenge each student's thinking*
✓ *listen carefully to students' ideas*
✓ *ask students to clarify and justify their ideas*
✓ *attach mathematical notation and language to students' ideas*
✓ *decide when to provide information, clarify, model, lead or let students struggle*
✓ *monitor and encourage participation*

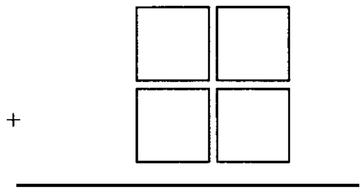
Adapted from Project M³:
Mentoring Mathematical Minds



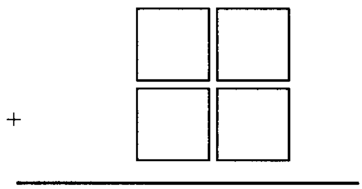
[3] ProjectM³ : http://www.projectm3.org

A FEW EXAMPLES OF ADAPTING PROBLEMS TO ENCOURAGE MATHEMATICAL CREATIVITY

I. Original Closed Exercise – Choose four cards from a set of cards with the numbers 0 - 9 on them and arrange them to make an addition problem. What is the total?



Opening the Problem – Choose four cards from a set of cards with the numbers 0 – 9 on them and arrange them to make an addition problem adding two two-digit numbers with the largest possible total. What is the total? How many different ways can you get this same total?



The Real Math Begins – What if you change the problem to subtraction? How many ways can you get the least (greatest) possible difference? Can you find a general rule for always getting the least (or greatest) difference? What if the problem is changed to multiplication or division?

II. Original Closed Exercise – You have two red blocks and one blue block. Draw all possible three-block towers using these three blocks.

Opening the Problem – Using three blocks that are either red or blue, draw all the possible three-block towers. Explain how you know you have found all the possible towers.

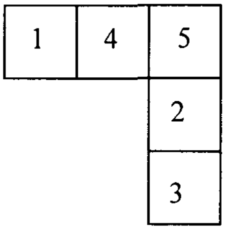
The Real Math Begins – What if you …

- add more colors?
- add more blocks?
- arrange the blocks in different configurations other than in a line for the tower?

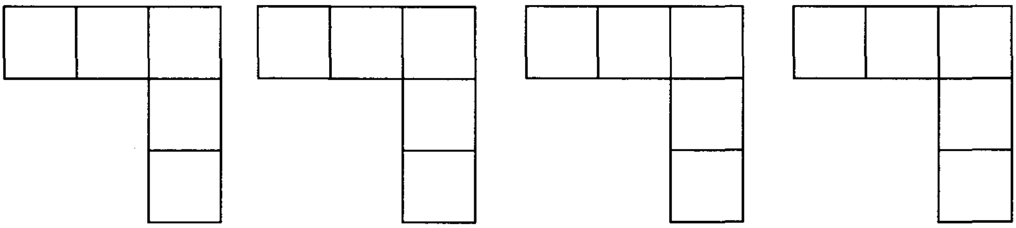
What patterns do you notice?

Can you generalize to any number of blocks?

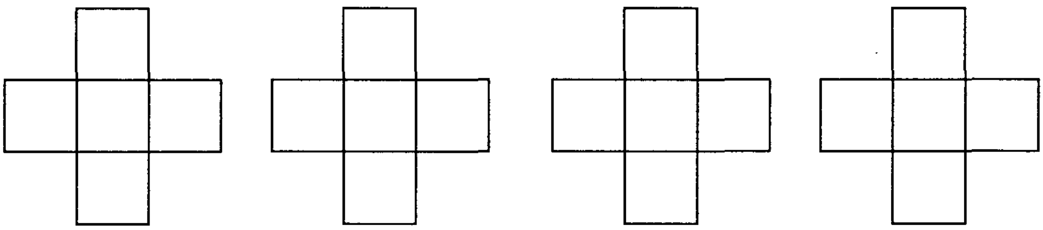
III. Original Closed Exercise: Find the sum of the row and the column. What do you notice?



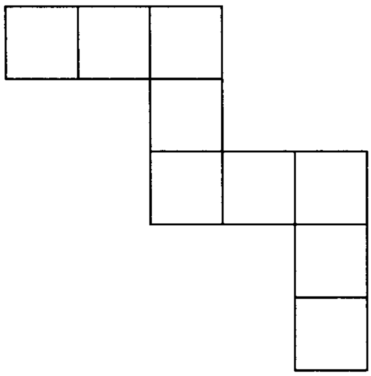
Opening the Problem: Put the numbers from 1 – 5 in the squares below so the row and the column have the same sum. How many dif ferent ways can you do this?

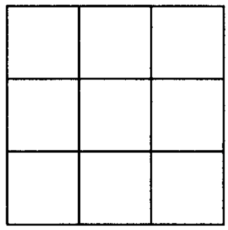


The Real Math Begins: Put the numbers from 1 – 5 in the squares below so the row and the column have the same sum. How many different ways can you do this? How does this compare to the problem above? What if you use numbers from 6 – 10? 100 – 104? Even numbers from 2 – 10? Multiples of 5 from 5 – 25?



What if you change the pattern and number of squares? Use the numbers from 1 – 9 (is N through N + 8, or even numbers from 2N to 2N + 16, etc.) to make each row and column equal in the following:





Make other diagrams and puzzles of your own.

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