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INTEGRAL GEOMETRY ON PRODUCT OF SPHERES II

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1. Introduction and Result

Let G be a Lie group and H a closed subgroup of G. We assume that G has a left invariant Riemannian metric that is also right invariant under elements of H. Then G/H is a homogeneous space with an invariant Riemannian metric. Consider now two submanifolds M and N of G/H, one fixed and the other moving under the action of $g \in G$. We always assume that M and N are in generic positions. This means that the dimension of the intersection $M \cap gN$ is nonnegative for almost all $g \in G$. Let $\mathrm{vol}(M \cap gN)$ be an *integral invariant* of the submanifold $M \cap gN$. One of the basic problems in integral geometry is to find explicit formulas for integral of $\mathrm{vol}(M \cap gN)$ over G with respect to the invariant measure $d\mu_G(g)$ on G in terms of known integral invariants of M and N. Especially R. Howard [1] obtained a generalized Poincaré formula for Riemannian homogeneous spaces as follows:

Let M and N be submanifolds of G/H with $\dim M + \dim N = \dim(G/H)$. Assume that G is unimodular. Then

$$(1.1) \quad \int_G \sharp(M \cap gN)\, d\mu_G(g) = \iint_{M \times N} \sigma_H(T_x^{\perp} M,\, T_y^{\perp} N)\, d\mu_{M \times N}(x, y),$$

where $\sharp(X)$ denotes the number of elements in a set X and $\sigma_H(T_x^{\perp} M, T_y^{\perp} N)$ is defined by (2.1) in Section 2.

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The formula (1.1) holds under the general situation. However, it is difficult to give an explicit description through the concrete computation of $\sigma_H(T_x^{\perp}M,\, T_y^{\perp}N)$, and only a little is known about it. In this paper, we attempt to explicitly describe this formula for two dimensional submanifolds in the product of unit sphere S^2. More precisely,

THEOREM 1.1. *Let M and N be submanifolds of $S^2 \times S^2$ of dimension 2. Assume that for almost all $g \in G$, M and gN intersect transversely. For any point $x \in M$ and $y \in N$, ξ_x and η_y denote the unit vector of T_xM and T_yN, respectively. Then we have*

$$\int_{SO(3)\times SO(3)} \sharp(M\cap gN)\, d\mu_{SO(3)\times SO(3)}(g) = \iint_{M\times N} \sigma(\xi,\, \eta)\, d\mu_{M\times N}(x,\, y).$$

Here $\sigma(\xi,\, \eta)$ was introduced by the Gauss hypergeometric function in the Section 3.

2. Preliminaries

Here we shall review the generalized Poincaré formula on Riemannian homogeneous spaces given by R. Howard [1] and recall the Gauss hypergeometric function and the elliptic integrals.

Let E be a finite dimensional real vector space with an inner product, and let V and W be two vector subspaces of E with orthonormal bases $v_1, \dots, v_p$ and $w_1, \dots, w_q$ respectively. The *angle* between subspaces V and W is defined by

$$\sigma(V, W) = \|v_1 \wedge \cdots \wedge v_p \wedge w_1 \wedge \cdots \wedge w_q\|\,,$$

where

$$\|x_1 \wedge \cdots \wedge x_k\|^2 = |\det[\langle x_i \wedge x_j \rangle]|\,.$$

This definition is independent of the choice of orthonormal bases. It is obvious that if $p + q = \dim E$ then

$$\sigma(V,\, W) = \sigma(V^{\perp},\, W^{\perp}).$$

Let G be a Lie group and H a closed subgroup of G. We assume that G has a left invariant Riemannian metric that is also invariant under the right actions of elements of H. This metric induces a G-invariant Riemannian metric on G/H. We denote by o the origin of

G/H. If $x, y \in G/H$ and V is a vector subspace of $T_x(G/H)$ and W is a vector subspace of $T_y(G/H)$ then define $\sigma_H(V, W)$ by

$$\sigma_H(V, W) = \int_H \sigma((dg_x)_o^{-1}V, dh_o^{-1}(dg_y)_o^{-1}W)\, d\mu_H(h) \tag{2.1}$$

where g_x and g_y are elements of G such that $g_x o = x$ and $g_y o = y$. This definition is independent of the choice of g_x and g_y in G such that $g_x o = x$ and $g_y o = y$.

We list here the basic properties of the Gauss hypergeometric function that are needed in this paper only. For further details see [4].

The Gauss hypergeometric series, convergent for $|z| < 1$, is given by the power series

$$F(a, b, c; z) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)}{\Gamma(a)} \cdot \frac{\Gamma(b+n)}{\Gamma(b)} \cdot \frac{\Gamma(c)}{\Gamma(c+n)} \cdot \frac{z^n}{n!}, \tag{2.2}$$

where Γ is the gamma function. By analytic continuation $F(a, b, c; z)$ can be extended to define a function analytic and single-valued in the complex z plane cut along the positive real axis from 1 to ∞. We remark that above series reduces to a polynomial of degree n in x when a or b is equal to $-n, (n = 0, 1, 2, \cdots)$. The series (2.2) is not defined when c is equal to $-m, (m = 0, 1, 2, \cdots)$, provided a or b is not a negative integer n with $n < m$. The hypergeometric equation

$$z(1-z)\frac{d^2u}{dz^2} + (c - (a+b+1)z)\frac{du}{dz} - abu = 0$$

has the solution $u = F(a, b, c; z)$.

The six functions $F(a \pm 1, b, c; z)$, $F(a, b \pm 1, c; z)$ and $F(a, b, c \pm 1; z)$ are called *contiguous* to $F(a, b, c; z)$. Relations between $F(a, b, c; z)$ and any two contiguous functions have been given by Gauss. By repeated application of these relations the function $F(a+m, b+n, c+l; z)$ with integer m, n, l can be expressed as a linear combination of $F(a, b, c; z)$ and one of its contiguous functions with coefficients which are rational functions of a, b, c, z. For examples,

$$\begin{aligned} azF(a+1, b+1, c+1; z) &= c[F(a, b+1, c; z) - F(a, b, c; z)], \\ (c-1)F(a, b, c-1; z) &= (c-a-1)F(a, b, c; z) + aF(a+1, b, c; z). \end{aligned}$$

Among the special cases are

$$(2.3)\qquad (1-z)^t = F(-t, b, b; z),$$

$$(2.4)\qquad \arcsin z = zF\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; z^2\right).$$

Furthermore C. F. Gauss evaluated, for $\Re(c-a-b) > 0$,

$$(2.5)\qquad F(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.$$

In this paper, we may consider only when z is a real number.

We now recall that *the incomplete elliptic integrals of the first* and *second kind* are defined by, for $0 < k < 1$,

$$F(\psi, k) = \int_0^\psi \frac{1}{\sqrt{1-k^2\sin^2\theta}}\, d\theta, \qquad E(\psi, k) = \int_0^\psi \sqrt{1-k^2\sin^2\theta}\, d\theta,$$

respectively. If $\psi = \pi/2$ then the integrals are called *the complete elliptic integral of the first* and *second kind*, and are denoted by $K(k)$ and $E(k)$ or simply K and E respectively.

3. Proof of the Theorem 1.1

Let S^2 be the standard sphere of dimension 2. Throughout this section, to simplify notation, we will regard G and H as $SO(3) \times SO(3)$ and $SO(2) \times SO(2)$. The special orthogonal group $SO(3)$ acts transitively on S^2. The isotropy subgroup of $SO(3)$ at a point in S^2 is $SO(2)$. Thus $S^2 \times S^2$ can be realized as a homogeneous space G/H. Let $\mathfrak{so}(3) \times \mathfrak{so}(3)$ be the Lie algebra of G. Define an inner product on $\mathfrak{so}(3) \times \mathfrak{so}(3)$ by

$$\langle X, Y\rangle = -\frac{1}{2}\operatorname{Trace}(XY) \qquad (X, Y \in \mathfrak{so}(3) \times \mathfrak{so}(3)).$$

We extend this inner product $\langle\cdot, \cdot\rangle$ on $\mathfrak{so}(3)\times\mathfrak{so}(3)$ to the left invariant Riemannian metric on G. Then we obtain a bi-invariant Riemannian metric on G. This bi-invariant Riemannian metric on G induces a G-invariant Riemannian metric on G/H.

Let M and N be submanifolds of $S^2 \times S^2$ of dimension 2. By the formula (1.1), we have

$$\text{(3.1)} \quad \int_G \sharp(M \cap gN)\, d\mu_G(g) = \iint_{M\times N} \sigma_H(T_xM,\, T_yN)\, d\mu_{M\times M}(x,\, y).$$

For any point $x = (x_1, x_2) \in M$,

$$T_xM = T_{(x_1,x_2)}M \subset T_{x_1}S^2 \oplus T_{x_2}S^2.$$

Thus u_x can be realized as an unit vector of T_xM just as follows:

$$u_x = (u_1, u_2) \in T_{x_1}S^2 \oplus T_{x_2}S^2 = \mathbb{R}^2 \times \mathbb{R}^2 \cong \mathbb{R}^4.$$

We here can transport u_x to $(\cos\theta_1,\, 0,\, \sin\theta_1,\, 0)$, since the action of H preserves the length of vectors. Thus we can take

$$(\cos\theta_1,\, 0,\, \sin\theta_1,\, 0)\,,\ (-\sin\theta_1\cos\theta_2,\, \sin\theta_2\cos\theta_3,\, \cos\theta_1\cos\theta_2,\, \sin\theta_2\sin\theta_3)$$

as an orthonormal basis of T_xM. Similarly we have

$$(\cos\tau_1,\, 0,\, \sin\tau_1,\, 0)\,,\ (-\sin\tau_1\cos\tau_2,\, \sin\tau_2\cos\tau_3,\, \cos\tau_1\cos\tau_2,\, \sin\tau_2\sin\tau_3)$$

as an orthonormal basis of T_yN.
In this choice of orthonormal bases, we can easily take one. But it is too much variables to calculate the $\sigma_H(\cdot\,,\cdot)$.

Now let $Gr_2^o(\mathbb{R}^4)$ be an oriented Grassmann manifold as a submanifold of $\bigwedge_2 \mathbb{R}^4$. We take an orientation on $\mathbb{R}^4$ such that $\mathbf{e_1}$, $\mathbf{e_2}$, $\mathbf{e_3}$, $\mathbf{e_4}$ is a positive basis of $\mathbb{R}^4$ and the inner product on $\bigwedge_2 \mathbb{R}^4$ induced by that on $\mathbb{R}^4$. Let $*$ be the *Hodge star operator* on $\bigwedge_2 \mathbb{R}^4$. Put

$$\bigwedge_+^2 = \left\{\xi \in \bigwedge_2 \mathbb{R}^4 |\, *\xi = \xi\right\}, \qquad \bigwedge_-^2 = \left\{\xi \in \bigwedge_2 \mathbb{R}^4 |\, *\xi = -\xi\right\}.$$

Then we have an orthogonal direct sum decomposition

$$\bigwedge_2 \mathbb{R}^4 = \bigwedge_+^2 \oplus \bigwedge_-^2 .$$

We define orthonormal bases A_i and B_i of $\bigwedge_+^2$ and $\bigwedge_-^2$ by

$$A_1 = \frac{1}{\sqrt{2}}(\mathbf{e_1}\wedge\mathbf{e_2}+\mathbf{e_3}\wedge\mathbf{e_4}), \qquad B_1 = \frac{1}{\sqrt{2}}(\mathbf{e_1}\wedge\mathbf{e_2}-\mathbf{e_3}\wedge\mathbf{e_4}),$$
$$A_2 = \frac{1}{\sqrt{2}}(\mathbf{e_1}\wedge\mathbf{e_3}-\mathbf{e_2}\wedge\mathbf{e_4}), \qquad B_2 = \frac{1}{\sqrt{2}}(\mathbf{e_1}\wedge\mathbf{e_3}+\mathbf{e_2}\wedge\mathbf{e_4}),$$
$$A_3 = \frac{1}{\sqrt{2}}(\mathbf{e_1}\wedge\mathbf{e_4}+\mathbf{e_2}\wedge\mathbf{e_3}), \qquad B_3 = \frac{1}{\sqrt{2}}(\mathbf{e_1}\wedge\mathbf{e_4}-\mathbf{e_2}\wedge\mathbf{e_3}).$$

Then we obtain

$$\bigwedge_+^2 = \text{Span}\{A_1,\, A_2,\, A_3\}, \qquad \bigwedge_-^2 = \text{Span}\{B_1,\, B_2,\, B_3\}.$$

By a simple calculation, we have

$$Gr_2^o(\mathbf{R}^4) = S^2\left(\frac{1}{\sqrt{2}}\right)\times S^2\left(\frac{1}{\sqrt{2}}\right).$$

Hence we can easily take orthonormal bases ξ and η as follows:

$$\xi = \frac{1}{\sqrt{2}}(\cos\theta_1 A_1 + \sin\theta_1 A_2) + \frac{1}{\sqrt{2}}(\cos\theta_2 B_1 + \sin\theta_2 B_2),$$
$$\eta = \frac{1}{\sqrt{2}}(\cos\tau_1 A_1 + \sin\tau_1 A_2) + \frac{1}{\sqrt{2}}(\cos\tau_2 B_1 + \sin\tau_2 B_2),$$

where $0 \le \theta_1,\, \theta_2,\, \tau_1,\, \tau_2 \le \pi$. We can simply write

$$\sigma_H(T_xM,\, T_yN) = \sigma_H(\xi,\, \eta),$$

since $\sigma_H(T_xM,\, T_yN)$ is dependent only on ξ and η, that is, θ and τ.

Now we work on the following integral

$$\sigma_H(\xi,\, \eta) = \int_H |\xi\wedge k\eta|\, d\mu_H(h).$$

We have set, to simplify notation,

$$\cos\theta_i\cos\tau_i = c_{ii}, \quad \sin\theta_i\sin\tau_1 = s_{ii}, \quad (i = 1, 2).$$

Then we immediately obtain

$$|\xi\wedge h\eta| = \frac{1}{2}\left|c_{11} + s_{11}\cos(\alpha+\beta) - c_{22} - s_{22}\cos(\alpha-\beta)\right|,$$

since

$$\begin{aligned} h\eta &= \frac{1}{\sqrt{2}}\left(\cos\tau_1 A_1 + \sin\tau_1\cos(\alpha+\beta)\, A_2 + \sin\tau_1\sin(\alpha+\beta)_3\right) \\ &+ \frac{1}{\sqrt{2}}\left(\cos\tau_2 B_1 + \sin\tau_2\cos(\alpha-\beta)\, B_2 + \sin\tau_2\sin(\alpha-\beta)\, B_3\right) \end{aligned}$$

for

$$h = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 & 0 \\ \sin\alpha & \cos\alpha & 0 & 0 \\ 0 & 0 & \cos\beta & -\sin\beta \\ 0 & 0 & \sin\beta & \cos\beta \end{bmatrix} \in SO(2)\times SO(2).$$

Hence we have to evaluate the following integral.

$$\begin{aligned} \sigma_H(\xi,\eta) &= \int_0^{2\pi}\int_0^{2\pi}\frac{1}{2}\mid c_{11} + s_{11}\cos(\alpha+\beta) \\ &\quad -c_{22} - s_{22}\cos(\alpha-\beta)\mid d\alpha\, d\beta, \end{aligned}$$

namely,

$$\begin{aligned} \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \mid c_{11} - c_{22} + (s_{11} &- s_{22})\cos\alpha\cos\beta \\ &- (s_{11}+s_{22})\sin\alpha\sin\beta \mid d\alpha\, d\beta. \end{aligned}$$

Since $0 \le \theta_1, \theta_2, \tau_1, \tau_2 \le \pi$, we have $0 \le s_{11}, s_{22} \le 1$ and $-1 \le c_{11}, c_{22} \le 1$. And put $a = s_{11}+s_{22}$, $b = s_{11}-s_{22}$, $c = c_{11}-c_{22}$ then

$$0 \le a \le 2, \quad -1 \le b \le 1, \quad -2 \le c \le 2, \quad |a| \ge |b|.$$

Having set up these notations, we can now give lemma that is needed to calculate our result.

LEMMA 3.1. Let $S^1(r)$ be a circle with radius r. If $|a| \le 1$ then

$$\int_{S^1(r)} |ra + x_1|\, d\mu_{S^1(r)}(x) = 4r^2\left(a\arcsin a + \sqrt{1-a^2}\right).$$

We can easily show this lemma and omit its proof.

At first, we shall prove the case where $c = 0$.

In this case, we will assume that $a = 0$. Then we have $b = 0$ since $s_{11} = s_{22} = 0$. Therefore we have

$$\sigma_H(\xi,\,\eta) = 0.$$

We suppose that $a > 0$. Then we have

$$\begin{aligned}\sigma_H(\xi,\,\eta) &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\left|\sqrt{b^2\,\cos^2\alpha + a^2\,\sin^2\alpha}\,\cos\beta\right|\,d\beta\,d\alpha\\ &= 2\int_0^{2\pi}\sqrt{b^2\,\cos^2\alpha + a^2\,\sin^2\alpha}\,d\alpha\\ &= 8a\int_0^{\pi/2}\sqrt{1-k^2\,\sin^2\alpha}\,d\alpha \qquad \left(\text{put } k := \sqrt{1-(b/a)^2}\right)\\ &= 8aE(k).\end{aligned}$$

Now we shall prove the case where $c \neq 0$.

Case I. The case where $0 \le |b| \le a \le |c|$.

In this case, we shall compute the following:

$$\begin{aligned}\sigma_H(\xi,\,\eta) &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}|c + b\cos\alpha\cos\beta - a\sin\alpha\sin\beta|\,d\beta\,d\alpha\\ &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\left|\,c - \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}\,\sin(\beta+\phi)\right|\,d\beta\,d\alpha.\end{aligned}$$

Here if $a \neq |b|$ then we have $\sin^2\alpha \le \dfrac{c^2-b^2}{a^2-b^2}$, since $c^2 \ge a^2$. Hence we have $|c| \ge \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$, for all $\alpha \in [0,\,2\pi]$. Therefore we obtain

(3.2)

$$\int_0^{2\pi}\left|\,c - \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}\,\sin(\beta+\phi)\right|\,d\beta = \int_0^{2\pi}|c|\,d\beta = 2\pi|c|.$$

If $a = |b|$ then we have

$$\int_0^{2\pi}|c - a\sin(\beta+\phi)|\,d\beta = 2\pi|c|. \tag{3.3}$$

In this case, from (3.2) and (3.3), we immediately obtain

$$\sigma_H(\xi,\,\eta) = \frac{1}{2}\int_0^{2\pi}2\pi|c|\,d\alpha = 2\pi^2|c|.$$

Case II. The case where $0 < |c| \leq |b| \leq a$.
If $a = |b|$ then, by Lemma 3.1, we have

$$\begin{aligned} \sigma_H(\xi, \eta) &= \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} |\, c - a\sin(\beta+\phi)|\, d\beta\, d\alpha \\ &= \frac{1}{2}\int_0^{2\pi}\left(4c\arcsin\left(\frac{c}{a}\right) + 4\sqrt{a^2-c^2}\right) d\alpha \\ &= 4\pi c\arcsin\left(\frac{c}{a}\right) + 4\pi\sqrt{a^2-c^2}. \end{aligned} \tag{3.4}$$

Here if $a \neq |b|$ then we have $\sin^2\alpha \geq \dfrac{c^2-b^2}{a^2-b^2}$, since $c^2 < a^2$. Hence we have $|c| \leq \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$, for all $\alpha \in [0, 2\pi]$. Therefore, by Lemma 3.1, we get

$$\begin{aligned} &\int_0^{2\pi}\left|\, c - \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}\,\sin(\beta+\phi)\right| d\beta \\ &= 4c\arcsin\left(\frac{c}{\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}}\right) + 4\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha - c^2}. \end{aligned} \tag{3.5}$$

Let us integrate on $[0, 2\pi]$ both term of (3.5). The integral of the second part of the right-hand side of (3.5) gives

$$\begin{aligned} &\int_0^{2\pi} \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha - c^2}\, d\alpha \\ &= \int_0^{2\pi}\sqrt{(a^2-c^2) - (a^2-b^2)\cos^2\alpha}\, d\alpha \\ &= 4\sqrt{a^2-c^2}\,E\left(\sqrt{\frac{a^2-b^2}{a^2-c^2}}\right). \end{aligned} \tag{3.6}$$

Now we compute the first part of the right-hand side of (3.5). To do this, we prepare the following lemma and formulas (3.7) and (3.8).

LEMMA 3.2. For integer m, we have

$$\int \sin^{2m} x\, dx = -\cos x\, F\left(\frac{1}{2}, \frac{1}{2} - m, \frac{3}{2}\,;\, \cos^2 x\right).$$

We can easily show the above lemma, using the binomial theorem, the details are left to the reader.
From Lemma 3.2, it is obvious that

$$\int_0^{\pi/2} \sin^{2m} x\,dx = \frac{(2m-1)!!}{(2m)!!}\cdot\frac{\pi}{2}. \tag{3.7}$$

where

$$m!! = \begin{cases} m(m-2)\cdots 4\cdot 2, & m: \text{even};\\ m(m-2)\cdots 3\cdot 1, & m: \text{odd}.\end{cases}$$

And, by a simple calculation and the binomial theorem, we obtain the following equality:

$$\left(\frac{1}{1+k^2\sin^2 x}\right)^{2n+1} = \sum_{m=0}^{\infty}\frac{(2n+2m-1)!!}{(2m)!!(2n-1)!!}(-k^2)^m\sin^{2m} x. \tag{3.8}$$

From the Taylor expansion of $\arcsin f(x)$ and (3.8), (3.7), we have

$$\begin{aligned}
&\int_0^{2\pi}\arcsin\left(\frac{c}{\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}}\right)d\alpha\\
=\ &\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}\int_0^{2\pi}c^{2n+1}\left(\frac{1}{\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}}\right)^{2n+1}d\alpha\\
=\ &\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{c^{2n+1}}{2n+1}\frac{1}{|b|^{2n+1}}\int_0^{2\pi}\left(\frac{1}{1+k^2\sin^2\alpha}\right)^{2n+1}d\alpha\\
=\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n-1)!!(2n+2m-1)!!}{(2n)!!(2n+1)(2m)!!(2n-1)!!}\left(\frac{c}{|b|}\right)^{2n+1}(-k^2)^m\int_0^{2\pi}\sin^{2m}\alpha\,d\alpha\\
=\ &\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(2n-1)!!(2n+2m-1)!!}{(2n)!!(2n+1)(2m)!!(2n-1)!!}\left(\frac{c}{|b|}\right)^{2n+1}(-k^2)^m\frac{(2m-1)!!}{(2m)!!}2\pi\\
=\ &\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}\left(\frac{c}{|b|}\right)^{2n+1}\sum_{m=0}^{\infty}\frac{2\sqrt{\pi}\,\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(n+m+\frac{1}{2}\right)}{m!\,\Gamma\left(n+\frac{1}{2}\right)}\frac{(-k^2)^m}{m!}\\
=\ &2\pi\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}\left(\frac{c}{|b|}\right)^{2n+1}F\left(\frac{1}{2},\,n+\frac{1}{2},\,1;\,-k^2\right)
\end{aligned}$$

where the step going from the second to third line used putting $k^2 = (a^2 - b^2)/b^2$, and the fifth to sixth line used

$$(2n + 2m - 1)!! = \frac{2^{n+m}}{\sqrt{\pi}}\Gamma\left(n + m + \frac{1}{2}\right).$$

Summarizing, we obtain

$$\begin{aligned}\sigma_H(\xi, \eta) &= 4\pi c \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}\left(\frac{c}{|b|}\right)^{2n+1} \\ &\quad \times F\left(\frac{1}{2}, n + \frac{1}{2}, 1\,; \frac{b^2 - a^2}{b^2}\right) \\ &\quad + 8\sqrt{a^2 - c^2}E\left(\sqrt{\frac{a^2 - b^2}{a^2 - c^2}}\right)\end{aligned}$$

REMARK 3.3. It is trivial that the case where $a = |b|$ in just above equality goes to (3.4).

Case III. The case where $0 \le |b| \le |c| \le a$.
In particular, if $0 < |b| = |c| = a$ then we have

$$\sigma_H(\xi, \eta) = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} |c - a\sin(\beta + \phi)|\, d\beta\, d\alpha.$$

Since $|c/a| = 1$, we obtain

$$\sigma_H(\xi, \eta) = \frac{1}{2}\int_0^{2\pi} 2\pi|c|\, d\alpha = 2\pi^2|c| = 2\pi^2 a.$$

It is sufficient to calculate the following:

$$\sigma_H(\xi, \eta) = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \left| c - \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}\,\sin(\beta + \phi)\right| d\beta\, d\alpha,$$

where the case is $|b| \le |c| < a$ or $|b| < |c| \le a$.
In these cases, we immediately know that $0 \le \dfrac{c^2 - b^2}{a^2 - b^2} \le 1$. The inequality

$$|c| \ge \sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}$$

is satisfied whenever $0 \le \alpha \le \theta$, $\pi - \theta \le \alpha \le \pi + \theta$, $2\pi - \theta \le \alpha \le 2\pi$, where

$$\theta = \arcsin \sqrt{\frac{c^2 - b^2}{a^2 - b^2}}. \tag{3.9}$$

Then we obtain

$$\int_0^{2\pi} \left| c - \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \sin(\beta + \phi) \right| d\beta = 2\pi |c|.$$

Therefore we have

$$\frac{1}{2} \left\{ \int_0^{\theta} 2\pi |c| \, d\alpha + \int_{\pi-\theta}^{\pi+\theta} 2\pi |c| \, d\alpha + \int_{2\pi-\theta}^{2\pi} 2\pi |c| \, d\alpha \right\} = 4\pi |c| \, \theta \tag{3.10}$$

On the other hand, the inequality

$$|c| \le \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}$$

holds for $\theta \le \alpha \le \pi - \theta$, $\pi + \theta \le \alpha \le 2\pi - \theta$. Then, by Lemma 3.1, we have

$$\begin{aligned} &\int_0^{2\pi} \left| c - \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha} \, \sin(\beta + \phi) \right| \, d\beta \\ = \;& 4c \arcsin \left(\frac{c}{\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha}} \right) + 4\sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha - c^2}. \end{aligned} \tag{3.11}$$

We first integrate the second part of right-hand side of (3.11) on $[\theta, \pi - \theta]$. Then we have

$$\begin{aligned} &\int_{\theta}^{\pi-\theta} \sqrt{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha - c^2} \, d\alpha \\ = &\int_{\theta}^{\pi-\theta} \sqrt{(a^2 - b^2) \sin^2 \alpha - (c^2 - b^2)} \, d\alpha. \end{aligned}$$

Here we put $(a^2 - b^2) \sin^2 \alpha - (c^2 - b^2) = (a^2 - c^2) \sin^2 \psi$. Then, using the coordinate transformation, above integral is as follows:

$$2(a^2 - c^2) \int_0^{\pi/2} \frac{1 - \cos^2 \psi}{\sqrt{(a^2 - c^2) \sin^2 \psi + (c^2 - b^2)}} \, d\psi. \tag{3.12}$$

From

$$\begin{aligned}
&\int_0^{\pi/2} \frac{d\psi}{\sqrt{(a^2-c^2)\sin^2\psi+(c^2-b^2)}} \\
&= \frac{1}{\sqrt{a^2-b^2}} \int_0^{\pi/2} \frac{d\psi}{\sqrt{1-\sin^2\theta\cos^2\psi}} \\
&= \frac{1}{\sqrt{a^2-b^2}} K(\sin\theta),
\end{aligned}$$

and, using putting $\cos\psi = t$,

$$\begin{aligned}
&\int_0^{\pi/2} \frac{\cos^2\psi \; d\psi}{\sqrt{(a^2-c^2)\sin^2\psi+(c^2-b^2)}} \\
&= \int_0^1 \frac{t^2 \; dt}{\sqrt{(a^2-b^2)-(a^2-c^2)\,t^2}\sqrt{1-t^2}} \\
&= \frac{1}{\sqrt{a^2-b^2}} \int_0^1 \frac{t^2}{\sqrt{1-\sin^2\theta\, t^2}\sqrt{1-t^2}}\, dt \\
&= -\frac{\sqrt{a^2-b^2}}{a^2-c^2}\left\{E(\sin\theta)-K(\sin\theta)\right\},
\end{aligned}$$

we know that (3.12) becomes the following:

$$2\sqrt{a^2-b^2}\left\{E(\sin\theta)-\frac{c^2-b^2}{a^2-b^2}K(\sin\theta)\right\}. \tag{3.13}$$

Next we compute the first part of right-hand side of (3.11) on $[\theta,\, \pi-\theta]$. Since

$$\int_\theta^{\pi-\theta} \sin^{2m}\alpha\, d\alpha = 2\cos\theta\, F\left(\frac{1}{2},\, \frac{1}{2}-m,\, \frac{3}{2};\, \cos^2\theta\right), \tag{3.14}$$

we have

$$\begin{aligned}
&\int_{\theta}^{\pi-\theta} \arcsin\left(\frac{c}{\sqrt{a^2\sin^2\alpha + b^2\cos^2\alpha}}\right) d\alpha \\
= \;& \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(2n-1)!!(2n+2m-1)!!}{(2n)!!(2n+1)(2m)!!(2n-1)!!}\left(\frac{c}{|b|}\right)^{2n+1}(-k^2)^m \\
&\times \int_{\theta}^{\pi-\theta} \sin^{2m}\alpha\, d\alpha \\
= \;& 2\cos\theta \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}\left(\frac{c}{|b|}\right)^{2n+1} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(2n+2m-1)!!}{(2m)!!(2n-1)!!}(-k)^m F\left(\frac{1}{2}, \frac{1}{2}-m, \frac{3}{2}; \cos^2\theta\right).
\end{aligned}$$

Summarizing, we obtain

$$\begin{aligned}
\sigma_H(\xi, \eta) \; = \;& 4\pi|c|\theta + 4\sqrt{a^2-b^2}\left\{E(\sin\theta) - \frac{c^2-b^2}{a^2-b^2}K(\sin\theta)\right\} \\
+ \;& 8c\cos\theta \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!}\frac{1}{2n+1}\left(\frac{c}{|b|}\right)^{2n+1} \\
&\quad \times \sum_{m=0}^{\infty} \frac{(2n+2m-1)!!}{(2m)!!(2n-1)!!}(-k)^m F\left(\frac{1}{2}, \frac{1}{2}-m, \frac{3}{2}; \cos^2\theta\right).
\end{aligned}$$

These equalities bring the proof to a conclusion.

Last of all we here give our result in the following table:

a, b, c	$\sigma_H(\xi, \eta)$
$0 \le \lvert b\rvert \le a \le \lvert c\rvert$	$2\pi^2\lvert c\rvert$
$0 \le \lvert c\rvert \le \lvert b\rvert \le a$	$4\pi c \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} \left(\frac{c}{\lvert b\rvert}\right)^{2n+1} \times F\left(\frac{1}{2}, n+\frac{1}{2}, 1\,;\, \frac{b^2-a^2}{b^2}\right) + 4\pi\sqrt{a^2-c^2}\, E\left(\sqrt{\frac{a^2-b^2}{a^2-c^2}}\right)$
$0 < \lvert b\rvert = \lvert c\rvert = a$	$2\pi^2\lvert c\rvert$
$0 \le \lvert b\rvert \le \lvert c\rvert \le a$ $(a \neq \lvert b\rvert)$	$4\pi\lvert c\rvert\theta + 4\sqrt{a^2-b^2}\left\{E(\sin\theta) - \frac{c^2-b^2}{a^2-b^2} K(\sin\theta)\right\} + 8c\cos\theta \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} \frac{1}{2n+1} \left(\frac{c}{\lvert b\rvert}\right)^{2n+1} \times \sum_{m=0}^{\infty} \frac{(2n+2m-1)!!}{(2m)!!(2n-1)!!} (-k)^m \times F\left(\frac{1}{2}, \frac{1}{2}-m, \frac{3}{2}\,;\, \cos^2\theta\right)$

where $\theta := \arcsin\sqrt{\dfrac{c^2-b^2}{a^2-b^2}}$.

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