East Asian Math. J. **22** (2006), No. 2, pp. 215–221

A NOTE ON THE AUSTIN'S GROUPOIDS

JUNG R. CHO AND JÓZEF DUDEK

ABSTRACT. On a groupoid satisfying the Austin's identity, every n-ary *linear* term is *essentially* n-ary. That is, if a term has no variables appearing more than once, then the term depends on every variable it involves.

1. Introduction

A *groupoid* is a pair $(G, \cdot)$ of a set G and a binary operation '$\cdot$' defined on G. A *term* or a *word* in a set $X = \{x_1, x_2, \cdots\}$ of symbols is an expression built up from X using the groupoid operation. We use the notation x^2 for the term xx. Thus x^2x, xx^2 and x^2x^2 represent $(xx)x$, $x(xx)$ and $(xx)(xx)$, respectively.

A term is called *n-ary* if it involves n distinct variables in its expression, and *linear* if each variable appears at most once in the expression. On a groupoid $(G, \cdot)$, an n-ary term $f(x_1, x_2, \cdots, x_n)$ defines a mapping of G^n into G by substitution. A mapping defined by a term in this way is called a *term function.* An n-ary term is called *essentially n-ary* over a groupoid $(G, \cdot)$ if, as a term function, it depends on each x_i for $i = 1, 2, \cdots, n$. That is,

$$f(a_1, \cdots, a_{i-1}, b, a_{i+1}, \cdots, a_n) \neq f(a_1, \cdots, a_{i-1}, c, a_{i+1}, \cdots, a_n)$$

for some elements $a_1, \cdots, a_{i-1}, a_{i+1}, \cdots, a_n, b, c$ in G.

Received October 25, 2006.

2000 Mathematics Subject Classification: 08A40, 20N02.

Key words and phrases: Austin's identity, groupoids, term, essentially n-ary, term-infinite.

For the first author, this work was supported for two years by Pusan National University Research Grant.

By $p_n(G,\cdot)$, we denote the number of all essentially n-ary terms over $(G,\cdot)$ for all $n \geq 0$. We say that a groupoid $(G,\cdot)$ is *term infinite* if $p_n(G,\cdot)$ is infinite for all $n \geq 2$. Of course, term infinite algebras are infinite but not conversely.

A groupoid $(G,\cdot)$ is called *nontrivial* if G has more than one element, and *proper* if the basic operation xy is essentially binary. In another word, a groupoid which is neither a left-zero semigroup nor right-semigroup is proper.

The groupoid identity

$$((y^2y)x)((y^2(y^2y)z) = x \tag{A}$$

is called the *Austin's identity*, and a groupoid $(G,\cdot)$ satisfying this identity is called an *Austin's groupoid.* Since its appearance in [1], this identity appeared in many papers ([3], [4], [5], [6], [8]), because the identity initiated the research on identities which have no nontrivial finite models.

A nontrivial Austin's groupoid has the following interesting properties.

THEOREM 1. ([1], [2]) *Every nontrivial Austin's groupoid is infinite.*

THEOREM 2. ([6]) *Every nontrivial Austin's groupoid is term-infinite.*

In [7], in comparison with the Austin's identity, it was shown that the identity $((y^2y)x)(y^2z) = x$ is the shortest groupoid identity which has no nontrivial finite models.

In this paper, we show the following theorem.

THEOREM 3. *On a non-trivial Austin's groupoid, every n-ary linear term is essentially n-ary for all $n \geq 1$.*

2. Some properties of Austin's groupoids

An element a of a groupoid is called *idempotent* if $a^2 = a$.

For every $n \geq 1$, define two special terms f_n and g_n by

$$f_n(x_1, x_2, \cdots, x_n) = ((\cdots((x_1x_2)x_3)\cdots)x_{n-1})x_n$$
$$\text{and} \quad g_n(x_1, x_2, \cdots, x_n) = x_1(x_2(\cdots(x_{n-2}(x_{n-1}x_n))\cdots)).$$

With a groupoid $(G, \cdot)$ and an element a of G, we define a mapping $T_a : G \to G$ by $T_a(x) = (a^2a)x$ for all x in G.

LEMMA 2.1. *If $(G, \cdot)$ is a nontrivial Austin's groupoid, then we have the following.*

i. *For each a in G, the mapping T_a is injective.*
ii. *$(G, \cdot)$ is proper.*
iii. *$(G, \cdot)$ has no idempotent element.*
iv. *The terms x, x^2 and x^2x are essentially unary and pairwise distinct.*
v. *The terms f_n and g_n are essentially n-ary for all $n \geq 1$.*

Proof. (i) If $T_a(x) = T_a(y)$ then, by the Austin's identity,

$$x = ((a^2a)x)((a^2(a^2a)z) = (T_a(x))((a^2(a^2a)z)$$
$$= (T_a(y))((a^2(a^2a))z) = ((a^2a)y)((a^2(a^2a)z) = y.$$

(ii) Assume that xy does not depend on x, then we have $xy = y^2$. Putting $(u^2u)v$ for x and $(u^2(u^2u))z$ for y in this identity, we get

$$v = ((u^2u)v)((u^2(u^2u))z) = xy = y^2 = (u^2(u^2u))z)^2,$$

which is impossible in a nontrivial groupoid. Assume now that xy does not depend on y, then we have $xy = x^2$ and so $T_a(b) = (a^2a)b = (a^2a)^2$ for all b in G. That is, T_a is constant, which contradicts (i). Therefore, xy is essentially binary. (iii) Suppose to the contrary that $(G, \cdot)$ has an idempotent element, say a. Note that $a^2a = a^2(a^2a) = a$ and so $(ax)(ay) = ((a^2a)x)(a^2(a^2a))y = x$. In particular, $a(ay) = (aa)(ay) = a$. Putting au for x in $x = (ax)(ay)$, we obtain that $au = (a(au))(ay) = a(ay) = a$ and hence $x = (ax)(ay) = aa = a$, a contradiction. (iv) By (iii), x^2 is essentially unary and $x^2 \neq x$. Assume $x^2x = c$, a constant. Then $x = ((y^2y)x)((y^2(y^2y))z) = (cx)((y^2c)z)$.

Putting $y = c$, we get $x = (cx)((c^2c)z) = (cx)(cz)$. Putting c for x and y, we have $c = c^2c^2$. Putting c^2 for x in $c = x^2x$ we have $c = (c^2)^2c^2 = cc^2$. Now putting c^2 for x in $x = (cx)(cz)$, we have $c^2 = (cc^2)(cz) = c(cz)$ and so $c^2 = cc^2 = c$, a contradiction to (iii). Thus x^2x is essentially unary. Now we show that $x^2x \neq x$. Assume $x^2x = x$, then $x = ((y^2y)x)((y^2(y^2y))z) = (yx)((y^2y)z) = (yx)(yz)$. Putting x^2 for y, we have $x = (x^2x)(x^2z)$ and hence $xx = (x^2x)(x^2x) = x$, a contradiction to (iii). Thus $x^2x \neq x$. Now assume $x^2x = x^2$, then $x = ((y^2y)x)((y^2(y^2y))z) = (y^2x)((y^2y^2)z)$ and so

$$(*) \qquad x^2 = (y^2x^2)((y^2y^2)z).$$

Putting x for y and x^2 for z in $(*)$, we have $x^2 = (x^2x^2)((x^2x^2)x^2) = (x^2x^2)((x^2)^2x^2) = (x^2x^2)(x^2)^2$. Put x for y and $(x^2)^2$ for z in $(*)$, then $x^2 = (x^2x^2)((x^2x^2)(x^2)^2) = (x^2x^2)x^2 = (x^2)^2x^2 = (x^2)^2$. That is, x^2 is an idempotent element, which is a contradiction to (iii). Thus $x^2x \neq x^2$. (v) We use induction on n. It is clear for $n = 1, 2$ as $(G, \cdot)$ is proper. Let $n \geq 3$ and assume that g_k are essentially k-ary for $1 \leq k \leq n-1$. By (A), we have

$$\begin{aligned} g_{n-1}(x_2 \cdots, x_{n-1}) &= x_2(x_3(\cdots(x_{n-1}x_n)\cdots)) \\ &= [(y^2y)(x_2(x_3(\cdots(x_{n-1}x_n)\cdots)))]((y^2(y^2y))z) \\ &= g_n(y^2y, x_2, \cdots, x_n)((y^2(y^2y))z). \end{aligned}$$

By induction hypothesis, g_{n-1} and hence g_n depends on $x_2, \cdots, x_n$. We also have

$$\begin{aligned} g_{n-1}(x_1, x_2, \cdots, x_{n-1}) &= x_1(x_2(\cdots(x_{n-2}x_{n-1})\cdots)) \\ &= x_1(x_2(\cdots(x_{n-2}[((y^2y)x_{n-1})((y^2(y^2y))z)])\cdots)) \\ &= g_n(x_1, x_2, \cdots, x_{n-2}, (y^2y)x_{n-1}, (y^2(y^2y))z). \end{aligned}$$

By induction hypothesis, g_{n-1} and hence g_n depends on x_1. Thus g_n depends on all its variables. To prove f_n is essentially n-ary for $n \geq 3$, we first show that $f_3(x, y, z) = (xy)z$ is essentially ternary. Assume f_3 does not depend on x. Then $(xy)z = (uy)z$. Putting $(x^2x)y$ for x and $(x^2(x^2x))z$ for y in this identity, we have $[((x^2x)y)((x^2(x^2x))z)]z = [u((x^2(x^2x^2))z)]z$, and so $yz = [u((x^2(x^2x^2))z)]z$. This implies that yz does not depends on y, which is a contradiction to (ii). Thus, f_3 depends on x. Since $y = ((x^2x)y)(x^2(x^2x))z) = f_3(x^2x, y, x^2(x^2x)z)$ we infer that f_3 depends on y. Using (A) again, we infer that f_3 depends on z by the identity $uz = (((x^2x)u)((y^2(y^2y))v))z = f_3((x^2x)u, (y^2(y^2y))v, z)$. Thus we have proved that f_3 is essentially ternary. Now, suppose $n > 3$ and assume that f_k is essentially k-ary for $2 \leq k \leq n-1$. Observe that, by the identity (A), we have

$$f_{n-1}(x_2, x_3, \cdots, x_n) = f_n((x^2x)x_2, (x^2(x^2x))z, x_3, \cdots, x_n)$$
$$\text{and} \quad f_{n-2}(x_2, x_4, \cdots, x_n) = f_n(x^2x, x_2, (x^2(x^2x))z, x_4, \cdots, x_n).$$

Thus, by induction hypothesis, we deduce that f_n depends on all variables. That is, f_n is essentially n ary for all $n \geq 2$. □

COROLLARY 2.2. *For a nontrivial Austin's groupoid $(G, \cdot)$, we have $p_n(G, \cdot) \geq 2$ for all $n \geq 1$.*

LEMMA 2.3. *Let $(G, \cdot)$ be a nontrivial Austin's groupoid. Then we have*

i. *For mappings $\phi_1, \phi_2 : G \to G$, we have $\phi_1 = \phi_2$ if and only if $\phi_1(xy) = \phi_2(xy)$ for all $x, y \in G$. Here, xy can be replaced by f_n or g_n for any $n \geq 1$.*
ii. *For any a in G, the mappings T_a is not the identity mapping.*
iii. *The mapping $n \mapsto T_a^n$ is injective or there exists an integer m such that $T_a^m(x) = x$ for all x in G.*
iv. *The term $(x^2x)y$ depends on y and $(x^2x)y \neq y$.*

Proof. (i) Assume that $\phi_1(xy) = \phi_2(xy)$. Putting $(y^2y)x$ for x and $(y^2(y^2y))z$ for y, we have $\phi_1(x) = \phi_2(x)$. Further proof proceeds by induction on the arity of the terms. (ii) Assume that $T_a(x) = x$ for some a and all $x \in G$. Then we have $(a^2a)x = x$. Putting $x = a^2a$ we see that a^2a is idempotent, which contradicts Lemma 2.1(iii). Thus $T_a \neq Id$. (iii) Suppose the mapping $n \mapsto T_a^n$ is not injective, then $T_a^j = T_a^k$ for some $j < k$. Then $T_a^{k-j}(T_a^j(b)) = T_a^k(b) = T_a^j(b)$ for all b in G. Since T_a and hence T_a^j is injective by Lemma 2.1(i), we see that $T_a^{k-j}(x) = x$ for all x in G. That is, $T_a^{k-j} = Id$. (iv) The fact that $(x^2x)y$ depends on y follows from Lemma 2.1(i). If $(x^2x)y = y$, then $(x^2x)(x^2x) = x^2x$, which contradicts Lemma 2.1(iii). Thus $(x^2x)y \neq y$. □

3. Proof of Theorem

In this section, we prove Theorem 3 by induction on the arity of linear terms.

For $n = 1, 2, 3$, the conclusion follows by Lemma 2.1. Let $n \geq 4$ and assume that the assertion is true for all k-ary linear terms for $1 \leq k \leq n-1$. Let f be an n-ary linear term. So, all variables in f are mutually distinct. We have two cases: (1) f contains at least two subterms of the form $x_i x_j$ and (2) f contains only one subterm of the form $x_i x_j$. Assume case (1) and so f contains subterms $x_1 x_2$ and $x_3 x_4$ after relabeling of variables if needed. Then, there are $(n-1)$-ary linear terms g and h such that

$$f(x_1, x_2, \cdots, x_n) = g(x_1 x_2, x_3, \cdots, x_n) = h(x_1, x_2, x_3 x_4, x_5, \cdots, x_n).$$

Then using (A) we have

$$f((y^2 y)x_1, (y^2(y^2 y))z, x_3, \cdots, x_n) = g(x_1, x_3, \cdots, x_n)$$

$$\text{and} \quad f(x_1, x_2, (y^2 y)x_3, (y^2(y^2 y))z, x_5, \cdots, x_n) = h(x_1, x_2, x_3, x_5 \cdots, x_n).$$

Since g and h are essentially by induction hypothesis, these identities show that f depends on all $x_1, x_2, \cdots, x_n$, i.e., f is essentially n-ary. Now, consider case (2). Since $n \geq 4$, f contains a subterm of the form $(x_1 x_2)x_3$ or $x_3(x_1 x_2)$. There is a $(n-1)$-ary linear term g such that $f(x_1, x_2, \cdots, x_n) = g(x_1 x_2, x_3, \cdots, x_n)$, and then we have $f((y^2 y)x_1, (y^2(y^2 y))z, x_3, \cdots, x_n) = g(x_1, x_3, \cdots, x_n)$. By induction g depends on $x_1, x_3, \cdots, x_n$, and hence so does f. If f contains $(x_1 x_2)x_3$ as a subterm, let $f(x_1, \cdots, x_n) = h((x_1 x_2)x_3, x_4, \cdots, x_n)$ for some $(n-2)$-ary linear term h. Then $f(y^2 y, x_2, (y^2(y^2 y))z, x_4, \cdots, x_n) = h(x_2, x_4, \cdots, x_n)$ and so f also depends on x_2. If f contains $x_3(x_1 x_2)$ as a subterm, since g_n is essentially n-ary by Lemma 2.1(v), we may assume that f is not of the form of g_n. Then f is of the form

$$f(x_1, x_2, \cdots, x_n) = \cdots([x_k(\cdots(x_3(x_1 x_2))\cdots)]x_{k+1})\cdots$$

for some $k \geq 3$. Putting $y^2 y$ for x_k and $(y^2(y^2 y))z$ for x_{k+1}, we have by (A) that

$$\begin{aligned} &f(x_1, x_2, \cdots, x_{k-1}, y^2 y, (y^2(y^2 y))z, x_{k+2}, \cdots, x_n) \\ &= \cdots(\cdots(x_3(x_1 x_2))\cdots)\cdots, \end{aligned}$$

where the right-hand side is a linear term without the variables x_k and x_{k+1}. By induction, f depends on each variables appearing on the right-hand side, in particular on x_2 as well. Consequently, f depends on every variable it involves. This completes the proof.

REFERENCES

[1] A.K. Austin, *A note on models of identities*, Proc. Amer. Math. Soc. **16**(1965), 522-523.

[2] A.K. Austin, *Finite models of laws in two variables*, Proc. Amer. Math. Soc. **17**(1966), 1410-1412.

[3] J. Cho and J. Dudek, *Affine spaces as models for regular identities*, Colloquium Mathematicum, **91/1**(2002), 29-38.

[4] J. Dudek, *A note on models of identities*, Algebra Universalis, **25**(1988), 400-401.

[5] J. Dudek and A. Kisielewicz, *On finite models of regular identities*, Notre Dame J. Formal Logic, **30/2**(1989), 624-628.

[6] A. Kisielewicz, *Varieties of algebras with non nontrivial finite members*, Lattice, Semigroups, and Universal Algebra (J. Almeida et al., eds.), Plenum Press, New York, 1990, 129-136.

[7] A. Kisielewicz, *Austin identities*, Algebra Universalis, **38**(1997), 324-328.

[8] W. Taylor, *Some interesting identities*, An. Inst. Mat. Univ. Nac. Autónoma México, **20**(1980), 127-156.

Department of Mathematics
Pusan National University
Busan 609-735, Korea
E-mail: jungcho@pusan.ac.kr

Mathematical Institute
University of Wrocław
pl. Grunwaldzki 2/4
50-384 Wrocław, Poland
E-mail: dudek@math.uni.wroc.pl