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GENERALIZED ANTI FUZZY SUBGROUPS

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Abstract. Using the notion of *anti fuzzy points* and its *besideness* to and *non-quasi-coincidence* with a fuzzy set, new concepts of an anti fuzzy subgroup are introduced and their inter-relations are investigated.

1. Introduction

The concept of fuzzy sets was first initiated by Zadeh [3]. Since then it has become a vigorous area of research in engineering, medical science, social science, physics, statistics, graph theory, etc. Biswas [2] gave the idea of anti fuzzy subgroups, and obtained some results. In this paper, we introduce new concepts of an anti fuzzy subgroup by using the notion of *anti fuzzy points* and its *besideness* to and *non-quasi-coincidence* with a fuzzy set, and investigate their inter-relations

2. Anti fuzzy subgroups

Given a fuzzy set Φ in a set G and for every $\alpha \in [0,1]$, the subsets

$$C(\Phi;\alpha) := \{x \in G \mid \Phi(x) \leq \alpha\} \text{ and } O(\Phi;\alpha) := \{x \in G \mid \Phi(x) < \alpha\}$$

are called the *closed α-cut* and the *open α-cut* of Φ, respectively.

Proposition 2.1. *Let Φ and Ψ be fuzzy sets in a set G. Then*

(i) $C(\Phi;1) = G$.

(ii) $(\forall \alpha, \beta \in [0,1])$ $(\alpha \leq \beta \Rightarrow C(\Phi;\alpha) \subseteq C(\Phi;\beta))$.

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(iii) $(\Phi \subseteq \Psi \Rightarrow (\forall \alpha \in [0,1])\, C(\Psi;\alpha) \subseteq C(\Phi;\alpha))$.
(iv) $(\forall \alpha, \beta \in [0,1])\ (C(\Phi;\alpha) = \cap_{\alpha<\beta} O(\Phi;\beta))$.
(v) $(\forall \alpha \in [0,1])\ (C(\Phi \cup \Psi;\alpha) = C(\Phi;\alpha) \cap C(\Psi;\alpha))$.
(vi) $(\forall \alpha \in [0,1])\ (C(\Phi \cap \Psi;\alpha) = C(\Phi;\alpha) \cup C(\Psi;\alpha))$.

Proof. Straightforward. □

In what follows, let G denote a group with e as the identity element unless otherwise specified.

DEFINITION 2.2. [2] A fuzzy set Φ in G is called an *anti fuzzy subgroup* of G if it satisfies the following assertions:

$$(\forall x, y \in G)(\Phi(xy) \leq \max\{\Phi(x), \Phi(y)\}) \tag{1}$$

$$(\forall x \in G)(\Phi(x) = \Phi(x^{-1})) \tag{2}$$

If Φ is an anti fuzzy subgroup of G, then $\Phi(e) \leq \Phi(x)$ for all $x \in G$ (see [2]).

PROPOSITION 2.3. [2] *Let Φ be a fuzzy set in G. Then Φ is an anti fuzzy subgroup of G if and only if each closed α-cut $C(\Phi;\alpha)$ of Φ is a subgroup of G.*

The $C(\Phi;\alpha)$'s are called *level subgroups,* which can be empty and form a chain in the subgroup lattice $(S, \subseteq)$ of G. By Hausdorff's axiom (or Zorn's lemma) there is a maximal chain C in S.

The order of the set of membership values of a fuzzy set Φ is called the *order* of Φ and is denoted by $|\Phi|$.

THEOREM 2.4. *Let G be with $|G| \geq 2$. Let Φ be a fuzzy set in G defined by*

$$\Phi(x) := \begin{cases} \alpha_1 \text{ if } x = e, \\ \alpha_2 \text{ otherwise,} \end{cases} \tag{3}$$

for any $x \in G$ and $0 \leq \alpha_1 < \alpha_2 \leq 1$. Then Φ is an anti fuzzy subgroup of G of order 2.

Proof. Let $x, y \in G$. Note that $x \neq e$ implies $x^{-1} \neq e$, and $x = e$ implies $x^{-1} = e$. Hence $\Phi(x) = \Phi(x^{-1})$. Assume that $y \neq x^{-1}$. If $x \neq e$ or $y \neq e$, then $xy \neq e$. Thus $\Phi(xy) = \alpha_2 = \max\{\Phi(x), \Phi(y)\}$. If $x = e$ and

$y = e$, then $xy = e$ and so $\Phi(xy) = \alpha_1 = \max\{\Phi(x), \Phi(y)\}$. Now suppose $y = x^{-1}$. Then $xy = e$ and hence $\Phi(xy) = \alpha_1 \leq \max\{\Phi(x), \Phi(y)\}$. Therefore Φ is an anti fuzzy subgroup of G of order 2. □

THEOREM 2.5. *Let G be a finite cyclic group of prime order. Then an anti fuzzy subgroup Φ of G with $|\Phi| = 2$ must be of the form* (3).

Proof. If $|G| = 2$, say $G = \{e, x\}$, then

$$\Phi(e) = \Phi(xx) \leq \max\{\Phi(x), \Phi(x)\} = \Phi(x)$$

together with $x^{-1} = x$ and $e^{-1} = e$ proves the assertion. Assume that $|G| > 2$ and that an anti fuzzy subgroup Φ of G with $|\Phi| = 2$ is not of the form (3). Then we have the following two cases:

Case 1. $\Phi(e) = \alpha_2$ and $\Phi(x) = \alpha_1$ for some $x \in G$ where $0 \leq \alpha_1 < \alpha_2 \leq 1$. Since Φ is an anti fuzzy subgroup of G, it follows from (2) that $\alpha_1 = \Phi(x) = \Phi(x^{-1})$ so that

$$\Phi(xx^{-1}) = \Phi(e) = \alpha_2 > \alpha_1 = \max\{\Phi(x), \Phi(x^{-1})\}$$

which contradicts (1).

Case 2. $\Phi(e) = \alpha_1$, $\Phi(x) = \alpha_1$ and $\Phi(y) = \alpha_2$ for some $x, y \in G \setminus \{e\}$ where $0 \leq \alpha_1 < \alpha_2 \leq 1$. Let $H := \{z \in G \mid \Phi(z) = \Phi(e)\}$. Obviously $e \in H$. If $x \in H$, then $\Phi(e) = \Phi(x) = \Phi(x^{-1})$ and hence $x^{-1} \in H$. If $x, z \in H$, then

$$\Phi(e) \leq \Phi(xz) \leq \max\{\Phi(x), \Phi(z)\} = \Phi(e).$$

It follows that $\Phi(xz) = \Phi(e)$ so that $xz \in H$. Hence H is a subgroup of G and it is proper since $y \notin H$. This contradicts our assumption, and the proof is complete. □

By a *proper anti fuzzy subgroup* we mean an anti fuzzy subgroup which is neither of order 1 nor of order 2 with the form (3).

THEOREM 2.6. *A group G with no proper subgroups cannot have proper anti fuzzy subgroups.*

Proof. Suppose that Φ is an anti fuzzy subgroup of G which has the form given in either Case 1 or Case 2 in the proof of Theorem 2.5. In Case 1, the closed α-cut $C(\Phi; \alpha_1)$ is a subgroup of G by Proposition 2.3, but $e \notin C(\Phi; \alpha_1)$ since $\alpha_1 < \alpha_2$. This is a contradiction. In Case 2, the closed α-cut $C(\Phi; \alpha_1)$ is a subgroup of G by Proposition

2.3, and it is proper by having at least two elements x and e. This contradicts our assumption. Now let Φ be an anti fuzzy subgroup of G with $|\Phi| \geq 3$, and of course with $|G| \geq 3$. Then there must exist three different elements, say x, y, z, with different membership values, say $\Phi(x) = \alpha_1 < \Phi(y) = \alpha_2 < \Phi(z) = \alpha_3$. The closed α-cuts $C(\Phi; \alpha_1)$, $C(\Phi; \alpha_2)$, and $C(\Phi; \alpha_3)$ are different subgroups of G and $C(\Phi; \alpha_1) \subset C(\Phi; \alpha_2) \subset C(\Phi; \alpha_3)$. At least one of them is proper subgroup of G. This completes the proof. ☐

THEOREM 2.7. *Let Φ be an anti fuzzy subgroup of G. Then a maximal subgroup chain $C = \{H_i \mid i \in \Lambda\}$ of G, where Λ is any index set, fulfils:*

(i) $(\forall y \in H_{i+1} \setminus H_i)\ (\exists \alpha_i \in [0,1])\ (\Phi(y) = \alpha_i)$
(ii) $(\forall x \in H_i \setminus H_{i-1})\ (\forall y \in H_{i+1} \setminus H_i)\ (\Phi(x) \leq \Phi(y))$.

Proof. Consider a maximal subgroup chain C of G containing all different closed α-cuts $C(\Phi; \alpha)$ of Φ. Such chain always exists since every chain can be extended to a maximal one.

(i) If $|H_{i+1} \setminus H_i| = 1$ for some i, then the statement is trivial. In the case $|H_{i+1} \setminus H_i| \geq 2$, we suppose $\Phi(y_1) \neq \Phi(y_2)$, say $\Phi(y_1) = \alpha_1 < \Phi(y_2) = \alpha_2$ for some $y_1, y_2 \in H_{i+1} \setminus H_i$. Take the closed α-cuts $C(\Phi; \alpha_1)$ and $C(\Phi; \alpha_2)$. By Proposition 2.3, they form different nonempty level subgroups of G with $H_i \subset C(\Phi; \alpha_1) \subset C(\Phi; \alpha_2) \subsetneq H_{i+1}$, which contradicts the maximality of the given chain.

(ii) By (i), $\Phi(y) = \alpha_i$ for all $y \in H_{i+1} \setminus H_i$, and $\Phi(x) = \alpha_{i-1}$ for all $x \in H_i \setminus H_{i-1}$ with $0 \leq \alpha_i \leq \alpha_{i-1} \leq 1$. Suppose $\alpha_i < \alpha_{i-1}$. For the corresponding level subgroups we get

$$H_i = C(\Phi; \alpha_i) \subset C(\Phi; \alpha_{i-1}) = H_{i-1},$$

which is a contradiction. ☐

3. Redefined anti fuzzy subgroups

DEFINITION 3.1. A fuzzy set Φ in G of the form

$$\Phi(y) = \begin{cases} \alpha \in [0,1) & \text{if } y = x, \\ 1 & \text{otherwise,} \end{cases}$$

is called an *anti fuzzy point* with support x and value α and is denoted by x^{α}.

DEFINITION 3.2. Let Φ be a fuzzy set in a group G. An anti fuzzy point x^{α} is said to *beside* to Φ, written $x^{\alpha} \lessdot \Phi$, if $\Phi(x) \leq \alpha$. An anti fuzzy point x^{α} is said to *non-quasi-coincident* with Φ, written $x^{\alpha} \upharpoonleft \Phi$, if $\Phi(x) + \alpha < 1$. If $x^{\alpha} \lessdot \Phi$ and (resp. or) $x^{\alpha} \upharpoonleft \Phi$, then we write $x^{\alpha} \lessdot \wedge \upharpoonleft \Phi$ (resp. $x^{\alpha} \lessdot \vee \upharpoonleft \Phi$).

LEMMA 3.3. *For any fuzzy set Φ in G, the condition* (1) *is equivalent to the condition:*

(4) $(\forall x, y \in G)(\forall \alpha, \beta \in [0,1))(x^{\alpha} \lessdot \Phi,\ y^{\beta} \lessdot \Phi \Rightarrow (xy)^{\max\{\alpha,\beta\}} \lessdot \Phi)$.

Proof. Assume that (1) is valid and let $x, y \in G$ and $\alpha, \beta \in [0,1)$ be such that $x^{\alpha} \lessdot \Phi$ and $y^{\beta} \lessdot \Phi$. Then $\Phi(x) \leq \alpha$ and $\Phi(y) \leq \beta$. It follows from (1) that

$$\Phi(xy) \leq \max\{\Phi(x), \Phi(y)\} \leq \max\{\alpha, \beta\}$$

so that $(xy)^{\max\{\alpha,\beta\}} \lessdot \Phi$. Now suppose that (4) is true. Note that $x^{\Phi(x)} \lessdot \Phi$ and $y^{\Phi(y)} \lessdot \Phi$ for all $x, y \in G$. Using (4), we have $(xy)^{\max\{\Phi(x),\Phi(y)\}} \lessdot \Phi$ and so $\Phi(xy) \leq \max\{\Phi(x), \Phi(y)\}$. This completes the proof. $\square$

In what follows Ω and $\mho$ denote any one of $\lessdot$, $\upharpoonleft$, $\lessdot \vee \upharpoonleft$, or $\lessdot \wedge \upharpoonleft$ unless otherwise specified.

DEFINITION 3.4. A fuzzy set Φ in G is called an $(\Omega, \mho)$-*anti fuzzy subgroup* of G, where $\Omega \neq \lessdot \wedge \upharpoonleft$, if it satisfies:

(b1) $(\forall x, y \in G)$ $(\forall \alpha, \beta \in [0,1))$ $(x^{\alpha}\Omega\Phi, y^{\beta}\Omega\Phi \Rightarrow (xy)^{\max\{\alpha,\beta\}}\mho\Phi)$
(b2) $(\forall x \in G)$ $(\Phi(x) = \Phi(x^{-1}))$.

In Definition 3.4, the case $\Omega = \lessdot \wedge \upharpoonleft$ is omitted because there exists a fuzzy set Φ in G such that $\{x^{\alpha} \mid x^{\alpha} \lessdot \wedge \upharpoonleft \Phi\}$ is empty. In fact, if $\Phi(x) \geq 0.5$ for all $x \in G$, then Φ is such a fuzzy set.

EXAMPLE 3.5. (I) Consider the Klein's 4-group $G = \{e, a, b, c\}$ with the following multiplication table.

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Let $\Phi : G \to [0, 1]$ be defined by $\Phi(e) = 0.4$, $\Phi(a) = 0.3$ and $\Phi(b) = \Phi(c) = 0.6$. Then Φ is a $(\lessdot, \lessdot\vee\upharpoonright)$-anti fuzzy subgroup of G. We note that Φ is not an $(\Omega, \mho)$-anti fuzzy subgroup of G for every $(\Omega, \mho) \in \{(\lessdot, \lessdot), (\upharpoonright, \lessdot\vee\upharpoonright), (\lessdot\vee\upharpoonright, \lessdot\vee\upharpoonright)\}$ since

(i) $a^{0.3} \lessdot \Phi$ and $a^{0.33} \lessdot \Phi$, but $(aa)^{\max\{0.3,0.33\}} = a^{0.33} \overline{\lessdot}\, \Phi$,
(ii) $a^{0.55} \upharpoonright \Phi$ and $b^{0.35} \upharpoonright \Phi$, but $(ab)^{\max\{0.55,0.35\}} = c^{0.55} \overline{\lessdot\vee\upharpoonright}\, \Phi$,
(iii) $a^{0.55} \lessdot\vee\upharpoonright \Phi$ and $c^{0.33} \lessdot\vee\upharpoonright \Phi$, but $(ac)^{\max\{0.55,0.33\}} = b^{0.55} \overline{\lessdot\vee\upharpoonright}\, \Phi$.

(II) Let $G = \{e, a, b\}$ be the group defined by the multiplication table

	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

Let $\Phi : G \to [0, 1]$ be defined by $\Phi(e) = 0.3$, $\Phi(a) = 0.2$ and $\Phi(b) = 0.1$. Then Φ satisfies the following implication:

(5)
$$(\forall x, y \in G)(\forall \alpha, \beta \in [0, 1])(x^{\alpha} \lessdot \Phi,\ y^{\beta} \lessdot \Phi \Rightarrow (xy)^{\max\{\alpha,\beta\}} \lessdot\vee\upharpoonright \Phi).$$

But since $\Phi(a^{-1}) \neq \Phi(a)$, Φ is not a $(\lessdot, \lessdot\vee\upharpoonright)$-anti fuzzy subgroup of G.

THEOREM 3.6. *If Φ is a non-constant $(\Omega, \mho)$-anti fuzzy subgroup of G, then*

(i) $\Phi(e) \neq 1$,
(ii) $G^* := \{x \in G \mid \Phi(x) \neq 1\}$ *is a subgroup of* G.

Proof. (i) Assume that $\Phi(e) = 1$. Since Φ is non-constant, there exists $x \in G$ such that $\Phi(x) = \alpha < 1$. If $\Omega \in \{\lessdot, \lessdot\vee\upharpoonright\}$, then $x^{\alpha}\Omega\Phi$

and $(x^{-1})^{\alpha}\Omega\Phi$. But

$$\Phi(xx^{-1}) = \Phi(e) = 1 > \alpha = \max\{\alpha, \alpha\}$$

and

$$\Phi(xx^{-1}) + \max\{\alpha, \alpha\} = \Phi(e) + \alpha = 1 + \alpha > 1.$$

Thus $(xx^{-1})^{\max\{\alpha,\alpha\}}\overline{\mho}\Phi$, a contradiction. Note that $x^0 \lessdot \Phi$ and $(x^{-1})^0 \lessdot \Phi$. Since

$$\Phi(xx^{-1}) = \Phi(e) = 1 > \max\{0, 0\} \quad \text{and} \quad \Phi(xx^{-1}) + \max\{0, 0\} = 1,$$

we have $(xx^{-1})^{\max\{0,0\}}\overline{\mho}\Phi$, a contradiction. Therefore $\Phi(e) \neq 1$.

(ii) Let $x, y \in G^*$. Then $\Phi(x) \neq 1$ and $\Phi(y) \neq 1$. Assume that $\Phi(xy) = 1$. If $\Omega \in \{\lessdot, \lessdot \vee \Rsh\}$, then $x^{\Phi(x)}\Omega\Phi$ and $y^{\Phi(y)}\Omega\Phi$. Since $\Phi(xy) \nleq \max\{\Phi(x), \Phi(y)\}$ and

$$\Phi(xy) + \max\{\Phi(x), \Phi(y)\} = 1 + \max\{\Phi(x), \Phi(y)\} \nless 1,$$

we have $(xy)^{\max\{\Phi(x),\Phi(y)\}}\overline{\mho}\Phi$. Note that $x^1 \lessdot \Phi$ and $y^1 \lessdot \Phi$, but

$$\Phi(xy) + \max\{\Phi(x), \Phi(y)\} = 1 + \max\{\Phi(x), \Phi(y)\} \nless 1$$

and $\Phi(xy) = 1 \nleq \max\{\Phi(x), \Phi(y)\}$. Hence $(xy)^{\max\{\Phi(x),\Phi(y)\}}\overline{\mho}\Phi$. Consequently $\Phi(xy) \neq 1$ and thus $xy \in G^*$. Obviously $x \in G^*$ implies $x^{-1} \in G^*$. Therefore G^* is a subgroup of G. □

THEOREM 3.7. *Every non-constant $(\lessdot, \Rsh)$-anti fuzzy subgroup Φ of G is the characteristic anti function of G^*, that is,*

$$\Phi(x) = \begin{cases} 0 & \text{if } x \in G^*, \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Assume that there exists $x \in G^*$ such that $\Phi(x) \neq 0$. Let $\alpha \in [0, 1)$ be such that $\alpha > \max\{\Phi(e), \Phi(x), 1 - \Phi(x)\}$. Then $x^{\alpha} \lessdot \Phi$ and $e^{\alpha} \lessdot \Phi$, but

$$\Phi(xe) + \max\{\alpha, \alpha\} = \Phi(x) + \alpha < 1,$$

that is, $(xe)^{\max\{\alpha,\alpha\}}\overline{\Rsh}\Phi$. This is a contradiction. Hence $\Phi(x) = 0$ for all $x \in G^*$, and therefore Φ is the characteristic anti function of G^*. □

COROLLARY 3.8. *Every non-constant $(\lessdot, \lessdot \wedge \Rsh)$-anti fuzzy subgroup Φ of G is the characteristic anti function of G^*.*

THEOREM 3.9. *Every non-constant* $(\Lsh, \Lsh)$*-anti fuzzy subgroup* Φ *of* G *is constant on* G^*.

Proof. Assume that there exists $b \in G^*$ such that $\alpha = \Phi(b) \neq \Phi(e) = \beta$. Suppose $\alpha > \beta$ and choose $\alpha_1, \alpha_2 \in [0, 1)$ such that $\alpha_2 < 1 - \alpha < \alpha_1 < 1 - \beta$. Then $e^{\alpha_1} \Lsh \Phi$ and $b^{\alpha_2} \Lsh \Phi$, but $(eb)^{\max\{\alpha_1, \alpha_2\}} = b^{\alpha_1} \overline{\Lsh} \Phi$. This is a contradiction. If $\alpha < \beta$, then $b^{1-\beta} \Lsh \Phi$ and $(b^{-1})^{1-\beta} \Lsh \Phi$. On the other hand, $(bb^{-1})^{\max\{1-\beta, 1-\beta\}} = e^{1-\beta} \overline{\Lsh} \Phi$. This is impossible. Hence $\Phi(b) = \Phi(e)$ for all $b \in G^*$, and therefore Φ is a constant on G^*. □

THEOREM 3.10. *Let* H *be a subgroup of* G *and let* $\Phi : G \to [0, 1]$ *be such that*

(i) $(\forall x \in G \setminus H)\ (\Phi(x) = 1)$,
(ii) $(\forall x \in H)\ (\Phi(x) = \Phi(x^{-1}) \leq 0.5)$.

Then Φ *is a* $(<, < \vee \Gamma)$*-anti fuzzy subgroup of* G.

Proof. Let $x, y \in G$ and $\alpha, \beta \in [0, 1)$ be such that $x^\alpha < \Phi$ and $y^\beta < \Phi$. Then $\Phi(x) \leq \alpha$ and $\Phi(y) \leq \beta$. Moreover $x, y \in H$ by (i), and hence $xy \in H$. If $\max\{\alpha, \beta\} < 0.5$, then

$$\Phi(xy) + \max\{\alpha, \beta\} < 0.5 + 0.5 = 1$$

and so $(xy)^{\max\{\alpha,\beta\}}\ \Gamma\ \Phi$. If $\max\{\alpha, \beta\} \geq 0.5$, then $\Phi(xy) \leq 0.5 \leq \max\{\alpha, \beta\}$ and thus $(xy)^{\max\{\alpha,\beta\}} < \Phi$. Consequently, we have $(xy)^{\max\{\alpha,\beta\}} < \vee \Gamma\ \Phi$. This completes the proof. □

THEOREM 3.11. *Let* Φ *be a* $(\Gamma, < \vee \Gamma)$*-anti fuzzy subgroup of* G *such that* Φ *is non-constant on* G^*. *Then* $\Phi(x) \leq 0.5$ *for all* $x \in G^*$.

Proof. We first prove that there exists $x \in G$ such that $\Phi(x) \leq 0.5$. If possible, let $\Phi(x) > 0.5$ for all $x \in G$. Since Φ is non-constant on G^*, there is $x \in G^*$ such that $\alpha = \Phi(x) \neq \Phi(e) = \beta$. Assume that $\alpha < \beta$ and choose $\delta < 0.5$ such that $\alpha + \delta < 1 < \beta + \delta$. Then $x^\delta \Lsh \Phi$ and $(x^{-1})^\delta \Lsh \Phi$, but $(xx^{-1})^{\max\{\delta,\delta\}} = e^\delta \overline{< \vee \Lsh} \Phi$, a contradiction. If $\alpha > \beta$, then we can choose $\delta < 0.5$ such that $\beta + \delta < 1 < \alpha + \delta$. Then $e^\delta \Lsh \Phi$ and $x^0 \Lsh \Phi$, but $(ex)^{\max\{\delta, 0\}} = x^\delta \overline{< \vee \Gamma} \Phi$, which is a contradiction. Therefore $\Phi(x) \leq 0.5$ for some $x \in G$. Now we prove $\Phi(e) \leq 0.5$. If not, then $\gamma = \Phi(e) > 0.5$. Since there exists $x \in G$ such that $\alpha = \Phi(x) \leq 0.5$, we have $\alpha < \gamma$. Choose $\beta < \gamma$ such that

$\alpha+\beta < 1 < \gamma+\beta$. Then $x^{\beta} \upharpoonright \Phi$ and $(x^{-1})^{\beta} \upharpoonright \Phi$, but $(xx^{-1})^{\max\{\beta,\beta\}} = e^{\beta}\overline{\lessdot\vee\upharpoonright}\Phi$. This is impossible, and therefore $\Phi(e) \leq 0.5$. Finally suppose that $\alpha = \Phi(x) > 0.5$ for some $x \in G^*$. Let us choose $\alpha_1 > 0$ such that $\alpha > 0.5 + \alpha_1$. Then $x^0 \upharpoonright \Phi$ and $e^{0.5-\alpha_1} \upharpoonright \Phi$, but

$$(ex)^{\max\{0,0.5-\alpha_1\}} = x^{0.5-\alpha_1}\overline{\lessdot\vee\upharpoonright}\Phi.$$

This is a contradiction. Hence $\Phi(x) \leq 0.5$ for all $x \in G^*$. □

PROPOSITION 3.12. *Let Φ be a $(\lessdot, \lessdot\vee\upharpoonright)$-anti fuzzy subgroup of G. Then*

(i) *Φ satisfies the following inequality*

$$(\forall x, y \in G)\,(\Phi(xy) \leq \max\{\Phi(x), \Phi(y), 0.5\}).$$

(ii) *If there exists $x \in G$ such that $\Phi(x) \leq 0.5$, then $\Phi(e) \leq 0.5$.*

Proof. (i) Let $x, y \in G$. If $\max\{\Phi(x), \Phi(y)\} > 0.5$, then $\Phi(xy) \leq \max\{\Phi(x), \Phi(y)\}$. For, assume that $\Phi(xy) > \max\{\Phi(x), \Phi(y)\}$ and choose $\alpha \in [0,1]$ such that $\max\{\Phi(x), \Phi(y)\} < \alpha < \Phi(xy)$. Then $x^{\alpha} \lessdot \Phi$ and $y^{\alpha} \lessdot \Phi$, but

$$(xy)^{\max\{\alpha,\alpha\}} = (xy)^{\alpha}\,\overline{\lessdot\vee\upharpoonright}\,\Phi,$$

a contradiction. Hence $\Phi(xy) \leq \max\{\Phi(x), \Phi(y)\}$ whenever $\max\{\Phi(x), \Phi(y)\} > 0.5$. Now suppose that $\max\{\Phi(x), \Phi(y)\} \leq 0.5$. Then $x^{0.5} \lessdot \Phi$ and $y^{0.5} \lessdot \Phi$ which imply that

$$(xy)^{\max\{0.5,0.5\}} = (xy)^{0.5} \lessdot\vee\upharpoonright\ \Phi,$$

that is, $(xy)^{0.5} \lessdot \Phi$ or $(xy)^{0.5} \upharpoonright \Phi$. If $\Phi(xy) > 0.5$, then $\Phi(xy) + 0.5 > 0.5 + 0.5 = 1$ and so $(xy)^{0.5}\overline{\upharpoonright}\Phi$. This is a contradiction. Hence $\Phi(xy) \leq 0.5$. Consequently, $\Phi(xy) \leq \max\{\Phi(x), \Phi(y), 0.5\}$ for all $x, y \in G$.

(ii) Let $x \in G$ be such that $\Phi(x) \leq 0.5$. Then

$$\Phi(e) = \Phi(xx^{-1}) \leq \max\{\Phi(x), \Phi(x^{-1}), 0.5\} = 0.5.$$

This completes the proof. □

PROPOSITION 3.13. *Let $G = \langle a \rangle$ be a cyclic group of finite order and let Φ be a $(\lessdot, \lessdot\vee\upharpoonright)$-anti fuzzy subgroup of G such that $\Phi(a) \leq 0.5$. Then $\Phi(x) \leq 0.5$ for all $x \in G$.*

Proof. Let $x \in G$. Since $G = \langle a \rangle$, there exists $m \in \mathbb{N}$ such that $x = a^m$. Thus $\Phi(a^2) \leq \max\{\Phi(a), 0.5\} = 0.5$, $\Phi(a^3) \leq \max\{\Phi(a^2), \Phi(a), 0.5\} = 0.5$, and so on. It follows that $\Phi(x) = \Phi(a^m) \leq \max\{\Phi(a^{m-1}), \Phi(a), 0.5\} = 0.5$. This completes the proof. □

THEOREM 3.14. *Every $(\lessdot \vee \Gamma, \lessdot \vee \Gamma)$-anti fuzzy subgroup of G is a $(\lessdot, \lessdot \vee \Gamma)$-anti fuzzy subgroup of G.*

Proof. Let Φ be a $(\lessdot \vee \Gamma, \lessdot \vee \Gamma)$-anti fuzzy subgroup of G. For any $x, y \in G$, let $\alpha_1, \alpha_2 \in [0, 1)$ be such that $x^{\alpha_1} \lessdot \Phi$ and $y^{\alpha_2} \lessdot \Phi$. Then $x^{\alpha_1} \lessdot \vee \Gamma\ \Phi$ and $y^{\alpha_2} \lessdot \vee \Gamma\ \Phi$, which imply that $(xy)^{\max\{\alpha_1, \alpha_2\}} \lessdot \vee \Gamma\ \Phi$. Hence Φ is a $(\lessdot, \lessdot \vee \Gamma)$-anti fuzzy subgroup of G. □

THEOREM 3.15. *Every $(\lessdot, \lessdot)$-anti fuzzy subgroup of G is a $(\lessdot, \lessdot \vee \Gamma)$-anti fuzzy subgroup of G.*

Proof. Straightforward. □

Example 3.5 shows that the converses of Theorems 3.14 and 3.15 need not be true.

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REFERENCES

[1] S. K. Bhakat and P. Das, $(\in, \in \vee q)$-*fuzzy subgroup,* Fuzzy Sets and Systems **80** (1996), 359–368.
[2] R. Biswas, *Fuzzy subgroups and anti fuzzy subgroups,* Fuzzy Sets and Systems **35** (1990), 121–124.
[3] L. A. Zadeh, *Fuzzy sets,* Inform. and Control **8** (1965), 338–353.

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