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AN OPTIMAL CONTROL FOR THE WAVE EQUATION WITH A LOCALIZED NONLINEAR DISSIPATION

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ABSTRACT. We consider the problem of an optimal control of the wave equation with a localized nonlinear dissipation. An optimal control is used to bring the state solutions close to a desired profile under a quadratic cost of control. We establish the existence of solutions of the underlying initial boundary value problem and of an optimal control that minimizes the cost functional. We derive an optimality system by formally differentiating the cost functional with respect to the control and evaluating the result at an optimal control.

1. Introduction

In this paper we consider the optimal control problem for the wave equations with a localized nonlinear dissipation;

$$
\begin{aligned}
&u_{tt} - \triangle u + a(x)u_t = f && \text{in } \Omega_T \equiv \Omega \times (0,T], \\
&u(x,0) = u_0(x),\ u_t(x,0) = u_1(x) && \text{in } \Omega, \\
&u(x,t) = 0 && \text{on } \partial\Omega \times [0,T],
\end{aligned} \tag{1.1}
$$

where $f : \Omega_T \to R$, $u_0, u_1 : \Omega \to R$ are given and $a : \bar{\Omega} \to R$, $u : \bar{\Omega}_T \to R$ are the unknown, $u = u(x,t)$. We set for $x_0 \in R^N$,

$$\Gamma(x_0) = \{x \in \partial\Omega \ : \ (x - x_0) \cdot \nu(x) \geq 0\}$$

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where $\nu(x)$ denotes the outward unit normal of the boundary $\partial\Omega$ at $x \in \partial\Omega$ and let

$$\omega = \big(\cup_{x \in \Gamma(x_0)} B_\delta(x) \big) \cap \bar{\Omega}$$

where $B_\delta(x) = \{y \in R^N \ : \ ||x-y|| < \delta \ \text{ for some } \delta > 0\}$. Given the control set:

$$(1.2) \qquad U_M = \{a \in L^\infty(\omega) \ : \ M \geq a \geq \epsilon_0 \quad \text{for some } \ M, \epsilon_0 > 0\}.$$

In here, the corresponding state variable $u = u(a)$ satisfies the state equation (1.1). We take as our objective functional:

$$(1.3) \qquad J(a) = \frac{1}{2} \int_0^T \int_\omega (u(a) - z_d)^2 dx dt + \frac{\beta}{2} \int_\omega a^2(x) dx$$

where $z_d \in L^2(\Omega_T)$ is a given target function and $a \in L^\infty(\bar{\Omega}), a(x) \geq 0$. We can find $a^* \in U_M$ such that

$$J(a^*) = \min_{a \in U_M} J(a).$$

For the background in control of PDEs, see Liang ([5]). Nakao ([6],[7]) developed decay of solutions of wave equation with a local degenerate dissipation. Bradley and Lenhart ([3]) treated Δ^2 type of bilinear control for the Kirchhoff plate equation. Park et al. ([8],[9],[10]) treated for optimal control of parameters and operators.

The goal of this work is to obtain an unique optimal control in terms of the solution to the optimal system, which will consist of the original wave problem coupled with an adjoint problem. In Section2, we show the well-posedness of our state problem in an appropriate solution space. Then we show the existence of an optimal control by a minimizing sequence argument. In Section3, the optimality systems is derived by differentiating the objective functional with respect to the control. The solution map $a \to u(a)$ is differentiated which is used in the differentiation of the objective functional. Then for sufficiently small time T, under some boundedness assumption, we prove uniqueness of the optimal systems, which characterizes the unique optimal control.

2. Existence of an Optimal Control

The following assumptions are made throughout this part : Ω is bounded domain in R^N and $\partial\Omega$ is C^2 smooth. $\Omega_T = \Omega \times (0,T]$. $f, f_t \in L^2(\Omega_T)$. $a(x) \in L^\infty(\bar{\Omega})$,$a(x) = 0$ in $\bar{\Omega} - \omega$,$a(x) > 0$ in ω and $0 \le a(x) \le M^*, M \le M^* < \infty$, M^* is a constant.

We present our definition of weak solution.

DEFINITION. *Given $a \in L^\infty(\bar{\Omega})$ and $u_0 \in H_0^1(\Omega), u_1 \in L^2(\Omega)$, $u \in C(0,T; H_0^1(\Omega))$ with $u_t \in C(0,T; L^2(\Omega)), u_{tt} \in C(0,T; H^{-1}(\Omega))$ is a weak solution of the problem* (1.1)*:*

$$(2.1)\ (a)\ \int_0^T \langle u_{tt}, \phi\rangle dt + \int_0^T \langle \nabla u, \nabla\phi\rangle dt + \int_0^T \langle a(x)u_t, \phi\rangle dt = \int_0^T \langle f, \phi\rangle dt \qquad \text{for any } \phi \in H_0^1(\Omega) \text{ and a.e. } 0 \le t \le T;$$

$$(b)\ u(x,0) = u_0(x);$$

$$(c)\ u_t(x,0) = u_1(x);$$

where $\langle , \rangle$ denotes the duality pairing of $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

For notational convenience, we set

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega),$$
$$u = u(a), \tilde{u} = (u, u_t).$$

LEMMA 2.1. *(Well-Posedness)(See* [11]*) For $\tilde{u}_0 = (u_0, u_1) \in \mathcal{H}$ and $a \in L^\infty(\bar{\Omega})$, the problem (1.1) has a unique weak solution u.*

Proof. We write (1.1) in the semigroup formulation

$$(2.2)\qquad \frac{d}{dt}\begin{pmatrix} u \\ u_t \end{pmatrix} = \begin{pmatrix} 0 & I \\ \triangle & 0 \end{pmatrix}\begin{pmatrix} u \\ u_t \end{pmatrix} + \begin{pmatrix} 0 \\ -au_t + f \end{pmatrix}$$
$$\tilde{u}(0) = \tilde{u}_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}.$$

Define the operator $A : [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega) \to \mathcal{H}$ by

$$A\tilde{u} = \begin{pmatrix} 0 & I \\ \triangle & 0 \end{pmatrix} \tilde{u}.$$

Its domain, $D(A) = [H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ is clearly dense in $\mathcal{H}$. Formulation (2.2) may be written as

$$\frac{d}{dt}\tilde{u}(t) = A\tilde{u}(t) + B_a(\tilde{u})(t) \tag{2.3}$$

where

$$B_a(\tilde{u}) = \begin{pmatrix} 0 \\ -au_t + f \end{pmatrix}.$$

Note that A is skew adjoint (See [1]), i.e. $A^* = -A$

$$A^*\tilde{u} = \begin{pmatrix} 0 & -I \\ -\triangle & 0 \end{pmatrix} \tilde{u}$$

and thus $D(A) = D(A^*)$. Thus we have that A generates a unitary group on $\mathcal{H}$ (See [1]). Motivated by the formulation (2.3), we seek a solution of the form:

$$\tilde{u}(t) = e^{At}\tilde{u}_0 + \int_0^t e^{A(t-\tau)} B_a(\tilde{u})(\cdot,\tau)d\tau$$

We will prove that the map T_a,

$$T_a\tilde{u}(t) = e^{At}\tilde{u}_0 + \int_0^t e^{A(t-\tau)} B_a(\tilde{u})(\cdot,\tau)d\tau$$

has a unique fixed point in $C([0,T_0];\mathcal{H})$.

Step1. We will prove that, if T_0 is small enough, there exists a unique fixed point such that

$$T_a(\tilde{u}(t)) = \tilde{u}(t) \quad \text{in } C([0,T_0];\mathcal{H}).$$

To use the contraction mapping theorem, we need to show that T_a is bounded and contractive.

Boundedness:

$$
\begin{aligned}
(2.4)\quad & ||T_a\tilde{u}||_{C([0,T_0];\mathcal{H})} \\
& \leq ||e^{At}\tilde{u_0}||_{C([0,T_0];\mathcal{H})} + \sup_{0\leq t\leq T_0}\int_0^t ||e^{A(t-\tau)}B_a(\tilde{u})(\cdot,\tau)||_{\mathcal{H}}d\tau \\
& \leq ||e^{At}\tilde{u_0}||_{C([0,T_0];\mathcal{H})} + \sup_{0\leq t\leq T_0}\int_0^t ||e^{A(t-\tau)}[a(\cdot)u_\tau(\cdot,\tau) \\
& \quad +f(\cdot,\tau)]||_{L^2(\Omega)}d\tau
\end{aligned}
$$

since A generates a unitary group and that $||e^{At}|| = 1$, here $||\cdot||$ denoting the operator norm. Since $||a||_\infty \leq M^*$, we obtain

$$
\begin{aligned}
(2.5)\quad & ||T_a\tilde{u}||_{C([0,T_0];\mathcal{H})} \\
& \leq ||\tilde{u_0}||_{\mathcal{H}} + M^*T_0||\tilde{u}||_{C([0,T_0];\mathcal{H})} + T_0||f||_{C([0,T_0];\mathcal{H})}
\end{aligned}
$$

and hence T_a is bounded.

Contractivity:
Similarly, for any $\tilde{v},\tilde{u}\in C([0,T_0];\mathcal{H})$,

$$||T_a\tilde{v} - T_a\tilde{u}||_{C([0,T_0];\mathcal{H})} \leq M^*T_0||\tilde{v}-\tilde{u}||_{C([0,T_0];\mathcal{H})}.$$

By choosing $T_0 < 1/M^*$, we have T_a is contractive for $t \leq T_0$. Thus, by the contraction mapping theorem we have the existence of a unique fixed point on $C([0,T_0];\mathcal{H})$.

Step2. Extend the above result to a solution on $[T_0, 2T_0]$ by selecting a new initial datum as $\tilde{u}_{T_0} = \tilde{u}(T_0) \in \mathcal{H}$. By a second contraction argument, we have a unique solution on $C([T_0, 2T_0];\mathcal{H})$. Repeating the process a finite number of time, we obtain the existence of a unique weak solution to (1.1), with $\tilde{u} \in C([0,T];\mathcal{H})$. ☐

LEMMA 2.2. *(Regularity) (See [11]). Assume that Ω is a bounded domain, $\partial\Omega$ is C^2 smooth, $a \in L^\infty(\bar{\Omega})$, $u_0 \in H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$ and $f_t \in L^2(\Omega_T)$. Then the weak solution $u = u(a)$ of (1.1) satisfies $u \in C(0,T;H^2(\Omega))$,$u_t \in C(0,T;H^1(\Omega))$ and $u_{tt} \in C(0,T;L^2(\Omega))$.*

LEMMA 2.3. *(A priori Estimate)(See, [2]). If $a \in U_M$, $u_0 \in H^2(\Omega)$, $u_1 \in H_0^1(\Omega)$, $f \in L^2(\Omega_T)$, $\partial\Omega$ is C^2, then the weak solution $u = u(a)$*

of (1.1) satisfies

$$\sup_{0\le t\le T}(\|u(t)\|_{H_0^1(\Omega)}+\|u_t(t)\|_{L^2(\Omega)})+\|u_{tt}\|_{L^2([0,T];H^{-1}(\Omega))}$$
$$\le C(\|f\|_{L^2(0,T;L^2(\Omega))}+\|u_0\|_{H_0^1(\Omega)}+\|u_1\|_{L^2(\Omega)})$$

where C is a constant which depends only on Ω and T.

Proof. There exist sequence $\{u_{0n}\}\subset H^2(\Omega),\{u_{1n}\}\subset H_0^1(\Omega)$ and $\{a_n\}\subset U_M$ such that

$$\begin{aligned}
&u_{0n}\to u_0 \quad \text{strongly in } H_0^1(\Omega),\\
&u_{1n}\to u_1 \quad \text{strongly in } L^2(\Omega),\\
&a_n\to a \quad \text{weakly in } L^2(\omega)\subset L^2(\bar\Omega).
\end{aligned}$$

Denote by u_n the weak solution of (1.1) corresponding to the initial datum u_{0n},u_{1n} with control a_n; then u_n satisfies the regularity of Lemma 2.2. Multiplying the PDE (1.1) by $(u_n)_t$, denoted by $(u_n)_t$, and integrating over $\Omega_t=\Omega\times(0,t]$ with $0\le t\le T$, we obtain

$$\begin{aligned}
0&=\int_{\Omega_t}[(u_n)_{\tau\tau}(u_n)_\tau-(\triangle u_n)(u_n)_\tau+a_n((u_n)_\tau)^2-f(u_n)_\tau]dxd\tau\\
&=\int_{\Omega_t}[\frac{1}{2}\frac{d}{d\tau}((u_n)_\tau)^2+|\nabla u_n|^2)+a_n((u_n)_\tau)^2-f(u_n)_\tau]dxd\tau
\end{aligned}$$

First, we have

$$\begin{aligned}
&\int_{\Omega\times\{t\}}[((u_n)_t)^2+|\nabla u_n|^2)]dx\\
&\quad=\int_\Omega[(u_{1n})^2+|\nabla u_{0n}|^2]dx+2\int_{\Omega_t}[-a_n((u_n)_\tau)^2+f(u_n)_\tau]dxd\tau\\
&\quad\le\|u_{1n}\|^2_{L^2(\Omega)}+\|\nabla u_{0n}\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(\Omega_T)}\\
&\qquad+(1+M)\|(u_n)_\tau\|^2_{L^2(\Omega_t)}\\
&\quad\le(\|u_{1n}\|^2_{L^2(\Omega)}+\|\nabla u_{0n}\|^2_{L^2(\Omega)}+\|f\|^2_{L^2(\Omega_T)})\\
&\qquad+(1+M)\int_{\Omega_t}[((u_n)_\tau)^2+|\nabla u_n|^2)]dxd\tau.
\end{aligned}\tag{2.6}$$

Using Gronwall's inequality, we obtain

$$(2.7)\qquad \int_{\Omega\times\{t\}} [((u_n)_t)^2 + |\nabla u_n|^2)]dx$$
$$\leq [1 + (1 + M)Te^{(1+M)T}]$$
$$\times(||u_{1n}||^2_{L^2(\Omega)} + ||\nabla u_{0n}||^2_{L^2(\Omega)} + ||f||^2_{L^2(\Omega_T)}).$$

Using Poincare's Inequality, we obtain

$$(2.8)\qquad \int_{\Omega\times\{t\}} [((u_n)_t)^2 + (u_n)^2)]dx$$
$$\leq C_1 \int_{\Omega_t} [((u_n)_\tau)^2 + (u_n)^2)]dx\, d\tau$$
$$+C_1(||u_{1n}||^2_{L^2(\Omega)} + ||\nabla u_{0n}||^2_{L^2(\Omega)} + ||f||^2_{L^2(\Omega_T)})$$

where C_1 is independent of u_n.

Combining (2.7), (2.8) and letting $n \to \infty$, we get at time t

$$(2.9)\qquad ||u_t(t)||^2_{L^2(\Omega)} + ||\nabla u(t)||^2_{L^2(\Omega)} + ||u(t)||^2_{L^2(\Omega)}$$
$$\leq C_2(||u_1||^2_{L^2(\Omega)} + ||\nabla u_0||^2_{L^2(\Omega)} + ||f||^2_{L^2(\Omega_T)})$$

where C_2 depends only on Ω, M and T.
Taking the supremum, gives

$$(2.10)\qquad \sup_{0\leq t\leq T}[||u_t(t)||^2_{L^2(\Omega)} + ||u(t)||^2_{H^1_0(\Omega)}]$$
$$\leq C_2(||u_1||^2_{L^2(\Omega)} + ||u_0||^2_{H^1_0(\Omega)} + ||f||^2_{L^2(0,T;L^2(\Omega))}).$$

Using the state equation, we obtain

$$||u_{tt}||_{L^2([0,T];H^{-1}(\Omega))} \leq C_3 \int_0^T \left(||f(t)||^2_{L^2(\Omega)} + ||u_t||^2_{H^1_0(\Omega)}\right)dt$$
$$\leq C_4T(||u_1||^2_{L^2(\Omega)} + ||u_0||^2_{H^1_0(\Omega)} + ||f||^2_{L^2(0,T;L^2(\Omega))}),$$

where C_3, C_4 are constants independent of a. □

We need to get higher order regularity of the weak solution of problem (1.1) in order to solve optimal problem . So, using Galerkin approximation method we obtain the higher order regularity of the weak solution.

LEMMA 2.4. *(Improved Regularity)(See, [2]). Assume that u is the weak solution of problem (1.1), if $a = a(x) \in U_M, u_0 \in H^2(\Omega), u_1 \in H_0^1(\Omega), f \in L^2(\Omega_T), f_t \in L^2(\Omega_T)$, then $u_t \in L^\infty(0,T;H^1(\Omega)), u_{tt} \in L^\infty(0,T;L^2(\Omega))$,$u_{ttt} \in L^2(0,T;H^{-1}(\Omega))$ and we have the estimate:*

$$
\begin{aligned}
&ess \sup_{0\le t\le T} \big(||u(t)||_{H^2(\Omega)} + ||u_t(t)||_{H_0^1(\Omega)} + ||u_{tt}(t)||_{L^2(\Omega)}\big) \\
&\quad + ||u_{ttt}||_{L^2([0,T];H^{-1}(\Omega))} \\
&\le C(||f||_{H^1(0,T;L^2(\Omega))} + ||u_0||_{H^2(\Omega)} + ||u_1||_{H_0^1(\Omega)})
\end{aligned}
\tag{2.11}
$$

with C depending only on Ω, T, ε_0, and M.

Proof. Construct a sequence of approximations by selecting smooth functions $\sigma_k = \sigma_k(x)$, $(k = 1, 2, ...)$ such that

$\{\sigma_k\}_{k=1}^\infty$ is a basis of $H_0^1(\Omega)$,
$\{\sigma_k\}_{k=1}^\infty$ is an orthonormal basis of $L^2(\Omega)$,

and $\{\sigma_k\}$ are eigenfunctions for $-\Delta$ on $H_0^1(\Omega)$ corresponding to the eigenvalue λ_k. For integer m, write

$$u_m(t) \equiv \sum_{k=1}^m d_k^m(t)\sigma_k(x) \tag{2.12}$$

where $d_k^m(t)$ satisfy

$$d_k^m(0) = (u_0(x), \sigma_k)_{L^2}, d_{kt}^m(0) = (u_1(x), \sigma_k)_{L^2}, k = 1, 2, ..., m \tag{2.13}$$

and

$$\langle u_{mtt}(t), \sigma_k\rangle + \langle \nabla u_m(t), \nabla\sigma_k\rangle + \langle a(x)u_{mt}(t), \sigma_k\rangle = \langle f(t), \sigma_k\rangle. \tag{2.14}$$

Using the orthogonality of $\{\sigma_k\}$ in $L^2(\Omega)$ and substituting the sum for u_m from (2.12) into (2.14), Eq.(2.14) becomes a system of ordinary differential equation (ODE)

(2.15)

$$d_{ktt}^m(t) + \sum_{l=1}^m \langle \nabla\sigma_l, \nabla\sigma_k\rangle d_l^m(t) + \sum_{l=1}^m \langle a(x)\sigma_l, \sigma_k\rangle d_{lt}^m(t) = \langle f(t), \sigma_k\rangle,$$

$0 \le t \le T, \quad k = 1, 2, ..., m$ with initial conditions (2.13). By standard existence theorem for ODE, there exists a unique absolutely continuous solution satisfying (2.13) and (2.14).

Note that each $d_k^m(t)$ has more regularity since the coefficients are time independent and $f_t \in L^2(\Omega_T)$, and we will use $d_{ktt}^m(t) \in L^2(0,T)$.

Step 1. In the proof of Lemma 2.3, we have already derived the bounds

$$\begin{aligned}(2.16)\quad &\sup_{0\le t\le T}(||u_m(t)||_{H_0^1(\Omega)} + ||u_{mt}(t)||_{L^2(\Omega)}) + ||u_{mtt}||_{L^2(0,T;H^{-1}(\Omega))} \\ &\qquad \le C(||f||_{L^2(0,T;L^2(\Omega))} + ||u_0||_{H_0^1(\Omega)} + ||u_1||_{L^2(\Omega)}).\end{aligned}$$

Passing to limits as $m = m_l \to \infty$, we deduce

$$\begin{aligned}(2.17)\quad &\sup_{0\le t\le T}(||u(t)||_{H_0^1(\Omega)} + ||u_t(t)||_{L^2(\Omega)}) + ||u_{tt}||_{L^2(0,T;H^{-1}(\Omega))} \\ &\qquad \le C(||f||_{L^2(0,T;L^2(\Omega))} + ||u_0||_{H_0^1(\Omega)} + ||u_1||_{L^2(\Omega)}).\end{aligned}$$

Step 2. Assume now the hypotheses of given initial conditions. Fix a positive integer m, and next differentiate the identity (2.14) with respect to t. Writing $\tilde{u}_m := u_{mt}$ we obtain

$$(2.18)\quad \langle \tilde{u}_{mtt}(t), \sigma_k\rangle + \langle \nabla\tilde{u}_m(t), \nabla\sigma_k\rangle + \langle a(x)\tilde{u}_{mt}(t), \sigma_k\rangle = \langle f_t(t), \sigma_k\rangle.$$

Multiplying (2.18) by $d_{ktt}^m(t)$ and adding for $k = 1, 2, \cdot, \cdot, \cdot, m$, we discover

$$\begin{aligned}(2.19)\qquad &\langle \tilde{u}_{mtt}(t), \tilde{u}_{mt}(t)\rangle + \langle \nabla\tilde{u}_m(t), \nabla\tilde{u}_{mt}(t)\rangle \\ &\quad + \langle a(x)\tilde{u}_{mt}(t), \tilde{u}_{mt}(t)\rangle = \langle f_t(t), \tilde{u}_{mt}(t)\rangle.\end{aligned}$$

Rewrite the above equality as

$$\begin{aligned}(2.20)\qquad &\frac{1}{2}\frac{d}{dt}(||\tilde{u}_{mt}(t)||^2_{L^2(\Omega)} + ||\nabla\tilde{u}_m(t)||^2_{L^2(\Omega)}) \\ &\quad + \langle a(x)\tilde{u}_{mt}(t), \tilde{u}_{mt}(t)\rangle = \langle f_t(t), \tilde{u}_{mt}(t)\rangle.\end{aligned}$$

Using $a(x) \ge \varepsilon_0 > 0$ and integrating to time t, we obtain

$$(2.21)\int_\Omega(\tilde{u}_{mt}(t))^2dx+\int_\Omega|\nabla\tilde{u}_m(t)|^2dx+\varepsilon_0\int_0^t\int_\Omega(\tilde{u}_{ms}(s))^2dxds$$
$$\leq\frac{1}{\varepsilon_0}\int_0^t\int_\Omega(f_s(x,s))^2dxds+\int_\Omega(\tilde{u}_{mt}(0))^2dx+\int_\Omega|\nabla\tilde{u}_m(0)|^2dx.$$

Thus we have

$$(2.22)\quad \sup_{0\leq t\leq T}(\int_\Omega(\tilde{u}_{mt}(t))^2dx+\int_\Omega|\nabla\tilde{u}_m(t)|^2dx)$$
$$+\varepsilon_0\int_0^t\int_\Omega(\tilde{u}_{ms}(s))^2dxds$$
$$\leq C(\varepsilon_0)(\int_0^T\int_\Omega(f_t(x,t))^2dxdt+\int_\Omega(\tilde{u}_{mt}(0))^2dx$$
$$+\int_\Omega|\nabla\tilde{u}_m(0)|^2dx).$$

Using the properties of σ_k, we have

$$(2.23)\quad \int_\Omega(\tilde{u}_{mt}(0))^2dx+\int_\Omega|\nabla\tilde{u}_m(0)|^2dx$$
$$\leq C_2(||u_0||^2_{H^2(\Omega)}+||u_1||^2_{H^1_0(\Omega)}).$$

Combining (2.22)-(2.23) and including $u_{mt}=\tilde{u}_m$, we conclude that

$$(2.24)\quad \sup_{0\leq t\leq T}(||u_{mtt}(t)||^2_{L^2(\Omega)}+||\nabla u_{mt}(t)||^2_{L^2(\Omega)})+||u_{mt}||^2_{L^2(0,T;L^2(\Omega))}$$
$$\leq C_3(||f_t||^2_{L^2(0,T;L^2(\Omega))}+||u_0||^2_{H^2(\Omega)}+||u_1||^2_{H^1_0(\Omega)}).$$

Passing to limits as $m=m_l\to\infty$, we deduce

$$(2.25)\quad \sup_{0\leq t\leq T}(||u_{tt}(t)||^2_{L^2(\Omega)}+||\nabla u_t(t)||^2_{L^2(\Omega)})+||u_t||^2_{L^2(0,T;L^2(\Omega))}$$
$$\leq C_3(||f_t||^2_{L^2(0,T;L^2(\Omega))}+||u_0||^2_{H^2(\Omega)}+||u_1||^2_{H^1_0(\Omega)}).$$

Step 3. Now

$$(2.26)\quad (-\Delta u_m,\sigma_k)=(f-u_{mtt}-au_{mt},\sigma_k)\ (k=1,2,\cdots,m).$$

Recall we are taking $\{\sigma_k)\}$ to be the complete collection of eigenfunctions for $-\Delta$ on $H_0^1(\Omega)$. Multiplying (2.26) by $\lambda_k d_k^m(t)$ and summing $k = 1, 2, \cdots, m$, we deduce

$$(2.27)\quad (-\Delta u_m, -\Delta u_m) = (f - u_{mtt} - au_{mt}, -\Delta u_m)\ (k = 1, 2, \cdots, m).$$

Using $a(x) \leq M$,we obtain

$$(2.28)\qquad ||\Delta u_m||^2_{L^2(\Omega)} \leq C_4(||f||^2_{L^2(\Omega)} + ||u_{mtt}||^2_{L^2(\Omega)} + ||u_{mt}||^2_{L^2(\Omega)}).$$

Since $\Delta u_m = 0$ on $\partial\Omega$ and $u \in H^2(\Omega) \cap H_0^1(\Omega)$ (see, [2]),we employ the inequality

$$(2.29)\qquad \beta||u||^2_{H^2(\Omega)} \leq ||\Delta u||^2_{L^2(\Omega)}$$

Combining (2.28) and (2.29), we obtain

$$(2.30)\qquad ||u_m||^2_{H^2(\Omega)} \leq C_5(||f||^2_{L^2(\Omega)} + ||u_{mtt}||^2_{L^2(\Omega)} + ||u_{mt}||^2_{L^2(\Omega)}).$$

Thus we deduce from (2.16), (2.24) and (2.30) that

$$(2.31)\qquad \sup_{0\leq t\leq T}(||u_m(t)||^2_{H^2(\Omega)} + ||u_{mt}(t)||^2_{H_0^1(\Omega)} + ||u_{mtt}(t)||^2_{L^2(\Omega)})$$
$$\leq C_6(||f||_{H^1(0,T;L^2(\Omega))} + ||u_0||^2_{H^2(\Omega)} + ||u_1||^2_{H_0^1(\Omega)}).$$

Here we estimated $||u_m(0)||_{H^2(\Omega)} \leq C||u_0||_{H^2(\Omega)}$.
Passing to limits as $m = m_l \to \infty$, we derived the same found for u.

Step 4. It remains to show $u_{ttt} \in L^2(0, T; H^{-1}(\Omega))$. To do so, take $v \in H_0^1(\Omega)$, with $||v||_{H_0^1(\Omega)} \leq 1$, and write $v := v_1 + v_2$ where $v \in span\{\sigma_k\}_{k=1}^m$ and $(v_2, \sigma_k) = 0$ $(k = 1, 2, ..., m)$. Since the functions $\{\sigma_k\}_{k=1}^\infty$ are orthogonal in $H_0^1(\Omega)$, $||v_1||_{H_0^1(\Omega)} \leq ||v||_{H_0^1(\Omega)} \leq 1$.

Utilizing (2.14), we deduce for a.e. $0 \leq t \leq T$ that

$$(u_{mttt}, v_1) + (-\Delta u_{mt}, v_1) + (au_{mtt}, v_1) = (f_t, v_1).$$

Then for a.e. $0 \leq t \leq T$

$$\begin{aligned}\langle u_{mttt}, v\rangle &= (u_{mttt}, v) = (u_{mttt}, v_1)\\ &= (f_t, v_1) + (\Delta u_{mt}, v_1) - (au_{mtt}, v_1).\end{aligned}$$

and

$$(2.32)\qquad |\langle u_{mttt}, v\rangle| \le C_7(||f_t||_{L^2(\Omega)} + ||\Delta u_{mt}||_{L^2(\Omega)} + ||u_{mtt}||_{L^2(\Omega)})$$

since $||v||_{H_0^1(\Omega)} \le 1$.
Combining (2.30) and (2.30) we obtain

$$\begin{aligned}(2.33)\, ||u_{mttt}||_{H^{-1}(\Omega)} &\le C_7(||f_t||_{L^2(\Omega)} + ||\Delta u_{mt}||_{L^2(\Omega)} + ||u_{mtt}||_{L^2(\Omega)}) \\ &\le C_7(||f||_{H^1(0,T;L^2(\Omega))} + ||u_0||^2_{H^2(\Omega)} + ||u_1||^2_{H_0^1(\Omega)}).\end{aligned}$$

And also u_{mttt} is bounded in $L^2(0,T;H^{-1}(\Omega))$. Passing to limits as $m = m_l \to \infty$, we deduce that $u_{ttt} \in L^2(0,T;H^{-1}(\Omega))$. Combining the estimates above yields the desired result. ☐

Now we obtain the existence of an optimal control.

THEOREM 2.1. *There exists an optimal control $a^* \in U_M$ which minimizes the objective functional $J(a)$ for $a \in U_M$.*

Proof. Let $\{a^n\} \subset U_M$ be a minimizing sequence such that

$$(2.34)\qquad \lim_{n\to\infty} J(a^n) = \inf_{a\in U_M} J(a).$$

Denote $u^n = u(a^n)$. By Lemma 2.4 we have

$$\begin{aligned}&ess \sup_{0\le t\le T} \big(||u(t)||_{H^2(\Omega)} + ||u_t(t)||_{H_0^1(\Omega)} + ||u_{tt}(t)||_{L^2(\Omega)}\big) \\ &+||u_{ttt}||_{L^2(0,T;H^{-1}(\Omega))} \le C(||f||_{H^1(0,T;L^2(\Omega))} + ||u_0||_{H^2(\Omega)} + ||u_1||_{H_0^1(\Omega)})\end{aligned}$$

On a subsequence, by weak compactness there exist u^* in $C([0,T];H^2(\Omega))$ such that

$$(2.35)\qquad \begin{aligned}&u^n \to u^* \text{ weakly* in } L^\infty(0,T;H^2(\Omega)) ,\\ &u_t^n \to u_t^* \text{ weakly* in } L^\infty(0,T;H_0^1(\Omega)),\\ &a^n \to a^* \text{ weakly in } L^2(\omega) \text{ and weakly in } L^2(\bar\Omega),\\ &u_{tt}^n \to u_{tt}^* \text{ weakly in } L^2(0,T;L^2(\Omega)).\end{aligned}$$

Since $H^2(\Omega), H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ are compact embedding, we have $u_t^n \to u_t^*$ strongly in $L^\infty(0,T;L^2(\Omega))$ and . By the definition of weak solution, we have

$$< u_{tt}^n, \phi >= -\int_{\Omega} [\nabla u^n \cdot \nabla \phi + a^n u_t^n \phi - f\phi] dx$$

for any $\phi \in H_0^1(\Omega)$ and a.e. $0 \leq t \leq T$. Since

$$u_t^n \to u_t^* \text{ strongly in } L^2(\Omega_T),$$

$$a^n \to a^* \text{ weakly in } L^2(\omega) \text{ and } L^2(\bar{\Omega}),$$

then

$$u_t^n a^n \to u_t^* a^* \text{ weakly in } L^2(\omega \times [0,T]) \text{ and } L^2(\Omega \times [0,T]).$$

Passing to the limit as $n \to \infty$ in the weak formulation of u^n, we obtain

$$< u_{tt}^*, \phi >= -\int_{\Omega} [\nabla u^* \cdot \nabla \phi + a^* u_t^* \phi - f\phi] dx.$$

Thus $u^* = u(a^*)$ is the solution of state Eq.(1.1) with control a^*. Since

$$J(a^*) = \frac{1}{2}\int_{\omega\times(0,T)} (u(a^*) - z_d)^2 dxdt + \frac{\beta}{2}\int_{\omega} (a^*)^2(x) dx,$$

using lower-semicontinuity of L^2 norm with respect to weak convergence, we have

$$\begin{aligned} J(a^*) &\leq \lim_{n\to\infty} \frac{1}{2}\int_{\omega\times[0,T]} (u(a^n) - z_d)^2 dxdt + \lim_{n\to\infty} \frac{\beta}{2}\int_{\omega} (a^n)^2(x) dx \\ &\leq \lim_{n\to\infty} \inf J(a^n) = \inf_{a\in U_M} J(a). \end{aligned}$$

Finally, we conclude that a^* is an optimal control. □

Remark (i) This problem is not related to the structure shape of domain Ω; (ii) the dimension of domain $\Omega \subset R^n$ are $n = 2, 3, 4$ and this problem is more generalized space than Boris and Lenhart([4]); (iii) optimal solution a^* has nonlinear in ω and linear property in $\Omega - \omega$ and also only be include important information data in regional area ω.

3. Characterization of the Optimal Control

We now derive the optimality system by differentiating the objective functional $J(a)$ with respect to the control a. Since $u = u(a)$ is involved in $J(a)$, we must first prove the appropriate differentiability of the mapping

$$a \to (u(a), u_t(a)) = \tilde{u}(a).$$

LEMMA 3.1. *The mapping*

$$a \in U_M \to \tilde{u}(a) = (u(a), u_t(a)) \in L^2(0,T;\bar{\mathcal{H}}), \bar{\mathcal{H}} = H^2(\Omega) \times H_0^1(\Omega)$$

is differentiable in the following sense:

$$\frac{\tilde{u}(a+\varepsilon l) - \tilde{u}(a)}{\varepsilon} \to \tilde{\psi}$$

weakly in $L^2(0,T;\bar{\mathcal{H}})$ as $\varepsilon \to 0$, for any a satisfying $a + \varepsilon l \in U_M$ for ε small and $l \in U_M$. Moreover $\tilde{\psi} = (\psi, \psi_t)$ in $L^2(0,T;\bar{\mathcal{H}})$ is a weak solution of the following problem:

$$(3.1) \qquad \begin{aligned} \psi_{tt} &= \Delta\psi - a\psi_t - lu_t, && \text{in } \Omega_T\ , \\ \psi &= 0, && \text{on } \partial\Omega \times [0,T], \\ \psi(0) &= \psi_t(0) = 0, && \text{in } x \in \Omega \end{aligned}$$

where $u = u(a)$.

Proof. Denote $u^\varepsilon = u(a + \varepsilon l)$ and $u = u(a)$, then $(u^\varepsilon - u)/\varepsilon$ is a weak solution of

$$\begin{aligned} (\tfrac{u^\varepsilon - u}{\varepsilon})_{tt} &= \Delta(\tfrac{u^\varepsilon - u}{\varepsilon}) - a(\tfrac{u^\varepsilon - u}{\varepsilon})_t - lu_t^\varepsilon && \text{in } \Omega_T \\ \tfrac{u^\varepsilon - u}{\varepsilon} &= 0 && \text{in } \partial\Omega \times [0,T]\ , \\ \tfrac{u^\varepsilon - u}{\varepsilon} &= 0 && \text{in } t = 0, x \in \Omega\ , \\ (\tfrac{u^\varepsilon - u}{\varepsilon})_t &= 0 && \text{in } t = 0, x \in \Omega\ . \end{aligned}$$

Using Lemma2.4, we get

$$
\begin{aligned}
ess \sup_{0\le t\le T} & [||(\frac{u^\varepsilon - u}{\varepsilon})(t)||_{H^2(\Omega)} + ||(\frac{u^\varepsilon - u}{\varepsilon})_t(t)||_{H_0^1(\Omega)} \\
& + ||(\frac{u^\varepsilon - u}{\varepsilon})_{tt}(t)||_{L^2(\Omega)}] \\
& \le C||l{u^\varepsilon}_t||_{L^2(0,T;L^2(\Omega))} \\
& \le CT||l||_{L^\infty} \sup_{0\le t\le T} ||{u^\varepsilon}_t(t)||_{L^2(\Omega)} \\
& \le C_1 T||l||_{L^\infty} \sup_{0\le t\le T} ||{u^\varepsilon}_t(t)||_{H_0^1(\Omega)} \\
& \le C_2 T||l||_{L^\infty}
\end{aligned}
$$

where C_2 depend on the L^∞ bound on l, but it is independent of ε. Hence on a subsequence, by weak compactness, we have

$$\frac{u^\varepsilon - u}{\varepsilon} \to \psi \text{ weakly* in } L^\infty(0,T;H^2(\Omega)),$$

$$\frac{u_t^\varepsilon - u_t}{\varepsilon} \to \psi_t \text{ weakly* in } L^\infty(0,T;H_0^1(\Omega)) \text{ and strongly in } L^2(\Omega)\ ,$$

and

$$\frac{u_{tt}^\varepsilon - u_{tt}}{\varepsilon} \to \psi_{tt} \text{ weakly* in } L^\infty(0,T;L^2(\Omega)).$$

By the definition of weak solution, we have

$$
\begin{aligned}
< \tfrac{u_{tt}^\varepsilon - u_{tt}}{\varepsilon}, \phi > \ & + \int_\Omega \nabla(\frac{u^\varepsilon - u}{\varepsilon}) \cdot \nabla\phi \, dx + \int_\Omega a(x)(\frac{u_t^\varepsilon - u_t}{\varepsilon})\ \phi \, dx \\
& = - \int_\Omega l u_t^\varepsilon \phi \, dx
\end{aligned}
$$

for any $\phi \in H_0^1(\Omega)$, and a.e. $0 \le t \le T$. Letting $\varepsilon \to 0$ and using $u_t^\varepsilon \to u_t$ strongly in $L^2(\Omega)$, we obtain that ψ is the weak solution of problem (3.1). □

Now we are ready to derive the necessary conditions that characterize an optimal control.

THEOREM 3.1. *Given an optimal control a and corresponding $\tilde{u} = \tilde{u}(a) = (u, u_t)$, there exists a weak solution $\tilde{p} = (p, p_t)$ in $\bar{\mathcal{H}}$ to the* *adjoint problem:*

$$
(3.2)\qquad \begin{aligned} p_{tt} &= \triangle p + ap_t + \chi_\omega(u - z_d), && in\ \Omega \times [0,T)\ , \\ p &= 0 && \partial\Omega \times [0,T] \\ p(T) &= p_t(T) = 0 && x \in \Omega. \end{aligned}
$$

Furthermore, a satisfies

$$
(3.3)\qquad a = \begin{cases} \max\Big(\varepsilon_0, \min\big(\frac{\int_0^T pu_t dt}{\beta}, M\big)\Big), & in\ \omega \\ 0, & in\ \bar{\Omega} - \omega \end{cases}
$$

Proof. Let $a \in U_M^*$ be an optimal control and $\tilde{u} = \tilde{u}(a)$ be the corresponding optimal solution. Let $a + \varepsilon l \in U_M^*$ for $\varepsilon > 0$ and $\tilde{u}^\varepsilon = \tilde{u}(a + \varepsilon l)$ be the corresponding weak solution of Eq.(1.1). We compute the directional derivative of the objective functional $J(a)$ with respect to a in the direction of l. Since J is supposed to attain its minimum for a, we have

$$
\begin{aligned}
0 &\le \lim_{\varepsilon\to 0^+} \frac{J(a+\varepsilon l) - J(a)}{\varepsilon} \\
&= \lim_{\varepsilon\to 0^+} \Big(\frac{1}{2\varepsilon}\int_0^T\int_\omega [(u^\varepsilon - z_d)^2 - (u - z_d)^2]dxdt + \frac{\beta}{2\varepsilon}\int_\omega [(a+\varepsilon l)^2 - a^2]dx\Big) \\
&= \lim_{\varepsilon\to 0^+} \Big(\int_0^T\int_\omega \frac{(u^\varepsilon - u)(u^\varepsilon + u - 2z_d)}{2\varepsilon}dxdt + \frac{\beta}{2}\int_\omega 2al + (\varepsilon l)^2 dx\Big) \\
&= \int_0^T\!\!(3.4)\!\!\int_\omega \psi(u - z_d)dxdt + \int_\omega \beta al\ dx,
\end{aligned}
$$

where ψ is defined in Lemma 3.1. Taking $u - z_d$ as source like "f" and using the similar argument in Lemma 2.1, we obtain the existence and uniqueness of solution of problem (3.2). Assume that p is the solution of problem (3.2) i.e., p satisfies

$$
(3.5)\qquad \begin{aligned} &< p_{tt}(t), \phi > + \int_\Omega \nabla p(t)\cdot\nabla\phi dx - \int_\Omega ap_t(t)\phi dx \\ &= \int_\Omega \chi_\omega(u - z_d)\phi dx \end{aligned}
$$

for any $\phi \in H_0^1(\Omega)$, and a.e. $0 \le t \le T$. Then

$$\begin{aligned} 0 &\le \int_0^T \int_\omega \psi(u - z_d)dxdt + \int_\Omega \beta al\chi_\omega dx \\ &= \int_0^T \Big(< p_{tt}, \psi > + \int_\Omega \nabla p \cdot \nabla \psi dx - \int_\Omega ap_t(t)\psi dx \Big) dt + \int_\Omega \beta al\chi_\omega dx \\ &= \int_0^T \Big(< \psi_{tt}, p > + \int_\Omega \nabla p \cdot \nabla \psi dx + \int_\Omega ap(t)\psi_t dx \Big) dt + \int_\Omega \beta al\chi_\omega dx \end{aligned}$$

by integration by parts twice in time and using $\psi(0) = \psi_t(0) = 0$ and $p(T) = P_t(T) = 0$. Then from Eq.(3.1) for ψ, we obtain

$$0 \le \int_\Omega l(x) \Big(\int_0^T -p(x,t)u_t(x,t)dt + \beta a(x)\chi_\omega(x) \Big) dx.$$

Note that $l = l(x)$ is an arbitrary function with $a + \varepsilon l \in U_M$ for all small ε. By a standard control argument involving the sign of the variation l depending on the size of a, we obtain the desired characterization of a, namely,

$$a = \begin{cases} \max\Big(\varepsilon_0, \min\big(\frac{\int_0^T pu_t dt}{\beta}, M\big)\Big), & \text{in } \omega \\ 0, & \text{in } \Omega - \omega \end{cases} \tag{3.6}$$

Thus proof is complete. □

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