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RELATIONS BETWEEN DECOMPOSITION SERIES AND TOPOLOGICAL SERIES OF CONVERGENCE SPACES

Sang Ho Park

Abstract. In this paper, we will show some relations between decomposition series $\{\pi^\alpha q : \alpha$ is an ordinal $\}$ and topological series $\{\tau_\alpha q : \alpha$ is an ordinal $\}$ for a convergence structure q and the formular $\pi^\beta(\tau_\alpha q) = \pi^{\omega^\alpha \beta} q$, where ω is the first limit ordinal and α and $\beta(\geq 1)$ are ordinals.

I. Introduction and Preliminaries

A *convergence structure* q on a set X defined by [1] in 1964 is a function from the set $F(X)$ of all filters on X into the set $P(X)$ of all subsets of X, satisfying the following conditions:

(1) $x \in q(\dot{x})$ for all $x \in X$;

(2) $\mathcal{F} \leq \mathcal{G}$ implies $q(\mathcal{F}) \subseteq q(\mathcal{G})$;

(3) $x \in q(\mathcal{F})$ implies $x \in q(\mathcal{F} \cap \dot{x})$,

where $\dot{x}$ denotes the principal ultrafilter containing $\{x\}$; $\mathcal{F}$ and $\mathcal{G}$ are in $F(X)$. Then the pair (X, q) is called a *convergence space*. If $x \in q(\mathcal{F})$, then we say that $\mathcal{F}$ *q-converges* to x. The filter $\mathcal{V}_q(x)$ obtained by intersecting all filters which q-converge to x is called the *q-neighborhood filter* at x. If $\mathcal{V}_q(x)$ q-converges to x for each $x \in X$, then q is said to be *pretopological* and the pair (X, q) is called a *pretopological convergence space*.

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Let $C(X)$ be the set of all convergence structures on X, partially ordered as follows:

$$q_1 \le q_2 \text{ iff } q_2(\mathcal{F}) \subseteq q_1(\mathcal{F}) \text{ for all } \mathcal{F} \in F(X).$$

If $q_1 \le q_2$, then we say that q_1 is *coarser* than q_2, and q_2 is *finer* than q_1. By [2], we know that if q_1 is pretopological, then

$$q_1 \le q_2 \text{ iff } \mathcal{V}_{q_1}(x) \le \mathcal{V}_{q_2}(x) \text{ for all } x \in X.$$

For any $q \in C(X)$, we define a related convergence structure $\pi(q)$, as follows:

$$x \in \pi(q)(\mathcal{F}) \text{ iff } \mathcal{V}_q(x) \le \mathcal{F}.$$

In this case, $\pi(q)$ is called the *pretopological modification* of q.

In 1973, Kent and Richardson [3] introduced the associated decomposition series $\{\pi^\alpha q : \alpha \text{ is an ordinal }\}$ defined by

$$\pi^\alpha q(\mathcal{F}) \xrightarrow{q} x \iff \mathcal{V}_q^\alpha(x) \le \mathcal{F}, \text{ for each } \mathcal{F} \in F(X),$$

where

$$A \in \mathcal{V}_q^\alpha(x) \iff x \in I_q^\alpha(A), \text{ and}$$

$$I_q^\alpha(A) = \begin{cases} I_q(I_q^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists,} \\ \cap_{\beta<\alpha} I_\nu^\beta(A), & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

In 1996, Park [4] studied the n-th pretopological modification $\pi^n q$ and quotient map for a convergence space q.

In 1999, for a convergence space (X, q) with a second convergence structure p, Wilde [5] introduced that (X, q) is "p-topological" iff $\mathcal{F} \xrightarrow{q} x$ implies $\mathcal{V}_p(\mathcal{F}) \xrightarrow{q} x$. Also they showed that there is a finest p-topological convergence structure $\tau_p q$ on X coarser than q and $\mathcal{F} \xrightarrow{\tau_p q} x$ iff there exist $\mathcal{G} \xrightarrow{q} x$ such that $\mathcal{F} \ge \mathcal{V}_p^n(\mathcal{G})$, for some $n \in N$. Furthermore, they induced the topological series for q, the descending ordinal sequence $\{\tau_\alpha q : \alpha \text{ is an ordinal }\}$ defined recursively on X as follows:

$\tau_0 q = q$

$\tau_1 q \;:\; \mathcal{F} \xrightarrow{\tau_1 q} x \iff \exists \mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}_q^n(\mathcal{G})$

$\tau_2 q \;:\; \mathcal{F} \xrightarrow{\tau_2 q} x \iff \exists \mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}_{\tau_1 q}^n(\mathcal{G})$

$\tau_3 q \;:\; \mathcal{F} \xrightarrow{\tau_3 q} x \iff \exists \mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}_{\tau_2 q}^n(\mathcal{G})$

$\vdots$

$\vdots$

$\tau_\alpha q \;:\; \mathcal{F} \xrightarrow{\tau_\alpha q} x \iff \exists \mathcal{G} \xrightarrow{q} x$, $n \in N$ and $\beta < \alpha$ such that $\mathcal{F} \geq \mathcal{V}_{\tau_\beta q}^n(\mathcal{G})$.

In this paper, we will show some relations between decomposition series $\{\pi^\alpha q : \alpha$ is an ordinal $\}$ and topological series $\{\tau_\alpha q : \alpha$ is an ordinal $\}$ for a convergence structure q and the formular $\pi^\beta(\tau_\alpha q) = \pi^{\omega^\alpha \beta} q$, where ω is the first limit ordinal and α and $\beta(\geq 1)$ are ordinals.

2. Decomposition Series, the Neighborhood and Interior Filter of a Filter

We shall summarize some results from [3] and other sources using more modern notation and terminology. we are mainly interested in comparing properties of *decomposition series* with those of the *topological series*, which will be introduced in [5].

Let (X, q) be a convergence space. For $A \subseteq X$, we recall that $I_q^0(A) = A$, $I_q^1 = I_q(A) = \{x : A \in \mathcal{V}_q(x)\}$

Given an ordinal number $\alpha \geq 1$, let I_q^α and cl_q^α denote the αth iterations of interior operator and closure operator for q, respectively. For $A \subseteq X$, we inductively define:

$$I_q^\alpha(A) = \begin{cases} I_q(I_q^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists,} \\ \cap_{\beta<\alpha} I_q^\beta(A), & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

$$cl_q^\alpha(A) = \begin{cases} cl_q(cl_q^{\alpha-1}(A)), & \text{if } \alpha - 1 \text{ exists,} \\ \cup_{\beta<\alpha}(cl_q^\beta(A)), & \text{if } \alpha \text{ is a limit ordinal.} \end{cases}$$

PROPOSITION 2.1. ([5]). *For every ordinal* α *and* $A \subseteq X$, $X \setminus cl_q^\alpha(A) = I_q^\alpha(X \setminus A)$.

If (X, q) *is a convergence space and* $\alpha \geq 1$, *let* $\pi^\alpha q$ *be the pretopology on* X *whose neighborhood filter is* $\mathcal{V}_q^\alpha(x)$, *that is,* $\mathcal{V}_{\pi^\alpha q}(x) = \mathcal{V}_q^\alpha(x)$, *where* $A \in \mathcal{V}_q^\alpha(x)$ *iff* $x \in I_q^\alpha(A)$. *Since* $\beta < \alpha$ *implies* $I_q^\alpha(A) \subseteq I_q^\beta(A)$, *it follows that* $\mathcal{V}_q^\alpha(x) \leq \mathcal{V}_q^\beta(x)$, *and consequently* $\pi^\alpha q \leq \pi^\beta q$.

Definition 2.2. ([3], [5]). *The descending chain* $\{\pi^\alpha q : \alpha \geq 1\}$ of pretopologies on X is called *the decomposition series* of (X, q).

Clearly $\pi^1 q = \pi q$ is the pretopological modification of q, which is the finest pretopological convergence structure on X coarser than q.

Definition 2.3. ([5]). For any ordinal α, $p \in C(X)$ and $\mathcal{G} \in F(X)$, we define the *neighborhood filter* $\mathcal{V}_p(\mathcal{G})$ and the *interior filter* $I_p(\mathcal{G})$ of $\mathcal{G}$, respectively, as follows:

$$\mathcal{V}_p^1(\mathcal{G}) = \mathcal{V}_p(\mathcal{G}), \quad \mathcal{V}_p^\alpha(\mathcal{G}) = \{A \subseteq X : I_p^\alpha(A) \in \mathcal{G}\}.$$

$$I_p^1(\mathcal{G}) = I_p(\mathcal{G}), \quad I_p^\alpha(\mathcal{G}) = [\{I_p^\alpha(G) : G \in \mathcal{G}\}] \text{ if } I_p(G) \neq \emptyset,\ \forall G \in \mathcal{G},$$

where $[\mathcal{B}]$ means the filter generated by $\mathcal{B}$ if $\mathcal{B}$ is a a filter base.

Then we know that if $\alpha < \beta$, then $\mathcal{V}_p^\beta(\mathcal{G}) \leq \mathcal{V}_p^\alpha(\mathcal{G}) \leq \mathcal{G} \leq I_p^\alpha(\mathcal{G}) \leq I_p^\beta(\mathcal{G})$.

PROPOSITION 2.4. *For any ordinals* α, β, $x \in X$ *and* $A \subseteq X$,
(1) $I_q^{\alpha+\beta}(A) = I_q^\beta(I_q^\alpha(A))$.
(2) $\mathcal{V}_q^{\alpha+\beta}(x) = \mathcal{V}_q^\alpha(\mathcal{V}_q^\beta(x))$.

Proof. (1) Let α be a fixed ordinal. We use transfinite induction on β. If $\beta = 1$, $I_q^{\alpha+1} = I_q(I_q^\alpha(A))$ follows by definition. Next, let β be any arbitrary ordinal.

Case 1. Assume that there exists $\bar{\beta}$ such that $\bar{\beta} + 1 = \beta$. By the induction hypothesis, $I_q^{\alpha+\bar{\beta}}(A) = I_q^{\bar{\beta}}(I_q^\alpha(A))$, and so $I_q^{\alpha+\beta}(A) = I_q^{\alpha+\bar{\beta}+1}(A) = I_q(I_q^{\alpha+\bar{\beta}}(A)) = I_q(I_q^{\bar{\beta}}(I_q^\alpha(A))) = I_q^\beta(I_q^\alpha(A))$.

Case 2. Assume that β is a limit ordinal. $I_q^{\alpha+\beta}(A) = \cap_{\gamma<\beta} I_q^{\alpha+\gamma}(A) = \cap_{\gamma<\beta} I_q^\gamma(I_q^\alpha(A)) = I_q^\beta(I_q^\alpha(A))$.

(2) $A \in \mathcal{V}_q^{\alpha+\beta}(x) \iff x \in I_q^{\alpha+\beta}(A) \iff x \in I_q^{\beta}(I_q^{\alpha}(A)) \iff I_q^{\alpha}(A) \in \mathcal{V}_q^{\beta}(x) \iff A \in \mathcal{V}_q^{\alpha}(\mathcal{V}_q^{\beta}(x))$. □

COROLLARY 2.5. *For any ordinals α, β, and $\mathcal{F} \in F(X)$,*
(1) $I_q^{\alpha+\beta}(\mathcal{F}) = I_q^{\beta}(I_q^{\alpha}(\mathcal{F}))$ if these are filters.
(2) $\mathcal{V}_q^{\alpha+\beta}(\mathcal{F}) = \mathcal{V}_q^{\alpha}(\mathcal{V}_q^{\beta}(\mathcal{F}))$.

3. p-Topological Convergence Spaces

In this section, we will summary some propositions about p-topological convergence space of [5] and [6], and change two propositions, which are the following Theorem 3.4 and 3.7.

Henceforth (X, q) means a convergence space equipped with a second convergence structure p.

Definition 3.1. ([5]). A convergence space (X, q) is *p-topological* iff $\mathcal{F} \xrightarrow{q} x$ implies that there exists a $\mathcal{G} \xrightarrow{q} x$ such that $\mathcal{F} \geq I_p(\mathcal{G})$.

PROPOSITION 3.2. ([5]). *(X, q) is p-topological, iff $\mathcal{F} \xrightarrow{q} x \implies \mathcal{V}_p(\mathcal{F}) \xrightarrow{q} x$.*

PROPOSITION 3.3. ([5]). *Let (X, q) be a pretoplogical convergence. Then (X, q) is p-topological iff $\mathcal{V}_q(x) = I_p(\mathcal{V}_q(x))$.*

Proof. ($\implies$) Since $\mathcal{V}_q(x) \xrightarrow{q} x$ and (X, q) is p-topological, there exists $\mathcal{G} \xrightarrow{q} x$ such that $\mathcal{V}_q(x) \geq I_p(\mathcal{G})$. Then $\mathcal{G} \geq \mathcal{V}_q(x)$, so $\mathcal{G} \geq I_p(\mathcal{G})$. This implies $\mathcal{G} = \mathcal{V}_q(x) = I_p(\mathcal{G}) = I_p(\mathcal{V}_q(x))$

($\impliedby$) Let $\mathcal{F} \xrightarrow{q} x$. Then $\mathcal{F} \geq \mathcal{V}_q(x) = I_p(\mathcal{V}_q(x))$. Thus, (X, q) is p-topological, since $\mathcal{V}_q(x) \xrightarrow{q} x$. □

THEOREM 3.4. *If (X, q) is a pretopological and p-topological, then $q \leq \pi^{\omega} p$.*

Proof. Since (X, q) is a pretopological and p-topological, $\mathcal{V}_q(x) = I_p(\mathcal{V}_q(x))$.

Claim: $\mathcal{V}_q(x) \leq \mathcal{V}_p^\omega(x)$. Let $V \in \mathcal{V}_q(x)$. Then $I_p(V) \in I_p(\mathcal{V}_q(x)) = \mathcal{V}_q(x)$. By Induction, $I_p^n(V) \in \mathcal{V}_q(x)$ for all $n \in N$, so $x \in I_p^n(V)$ for all $n \in N$. Thus $x \in \cap_{n<\omega} I_p^n(V) = I_p^\omega(V)$, and hence $V \in \mathcal{V}_p^\omega(x)$. Thus the Claim is proved.

From $\mathcal{V}_p^\omega(x) = \mathcal{V}_{\pi^\omega p}(x)$, we obtain $q \leq \pi^\omega p$. □

PROPOSITION 3.5. ([5]). *Let p and q be topological. Then (X, q) is p-topological iff $q \leq p$.*

Proof. Since q is topological, $\mathcal{V}_q(x)$ has a filter base of q-open sets.

($\Longrightarrow$) Since (X, q) is p-topological and topological, by Theorem 3.4, $q \leq \pi^\omega p = p$.

($\Longleftarrow$) Let $q \leq p$. Then $I_q(A) \subseteq I_p(A) \subseteq A$. This implies that each q-open set is p-open, so $I_p(\mathcal{V}_q(x)) = \mathcal{V}_q(x)$, by Proposition 3.3. (X, q) is p-topological. □

PROPOSITION 3.6. ([5]). *If (X, q) is p-topological and $p < p'$, then (X, q) is p'-topological.*

Proof. It follows from $p < p'$ implies $I_p(\mathcal{G}) \geq I_{p'}(\mathcal{G})$. □

Note that for $q \in C(X)$, $\tau_q = \{A \subseteq X : I_q(A) = A\}$ is a topology on X and τq is the convergence structure defined by

$$\tau q(\mathcal{F}) \xrightarrow{q} x \iff \mathcal{V}_{\tau_q}(x) \leq \mathcal{F}, \text{ for each } \mathcal{F} \in F(X),$$

where $\mathcal{V}_{\tau_q}(x)$ is the τ_q-neighborhood filter at $x \in X$. Then τq is the finest topological convergence structure on X coarser than q.([5]). Now, we obtain the following theorem, which is different from Corollary 2.4 of [6].

THEOREM 3.7. *If (X, q) is p-topological, then:*
(1) $(X, \pi q)$ is p-topological and $\tau q \leq \pi q \leq \pi^\omega p$.
(2) $(X, \tau q)$ is p-topological.

Proof. (1) Let $\mathcal{F} \xrightarrow{q} x$; then there exists a $\mathcal{G} \xrightarrow{q} x$ such that $\mathcal{F} \geq I_p(\mathcal{G}) \geq I_p(\mathcal{V}_q(x))$. This holds for every $\mathcal{F} \xrightarrow{q} x$, so

$$\mathcal{V}_{\pi q}(x) = \mathcal{V}_q(x) = \cap\{\mathcal{F} \in F(X) : \mathcal{F} \xrightarrow{q} x\} \geq I_p(\mathcal{V}_q(x)) = I_p(\mathcal{V}_{\pi q}(x)).$$

Thus $(X, \pi q)$ is p-topological, so the first part is proved.

It is clear that $\tau q \leq \pi q$. Since $(X, \pi q)$ is p-topological and pre-topological, by Theorem 3.4, $\pi q \leq \pi^{\omega} p$.

(2) Since $(X, \tau q)$ is τq-topological and $\tau q \leq \pi q \leq \pi^{\omega} p \leq p$, by Proposition 3.6, $(X, \tau q)$ is p-topological. □

Definition 3.8 For $q,\ p \in C(X)$, $\tau_p q$ is defined by:

$$\mathcal{F} \xrightarrow{\tau_p q} x \iff \exists \mathcal{G} \xrightarrow{q} x \text{ and } n \in N \text{ such that } \mathcal{F} \geq \mathcal{V}_p^n(\mathcal{G}).$$

PROPOSITION 3.9. *For $q,\ p \in C(X)$, $(X, \tau_p q)$ is p-topological.*

Proof. Let $\mathcal{F} \xrightarrow{\tau_p q} x$. Then there exists $\mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}_p^n(\mathcal{G})$, so $\mathcal{V}_p(\mathcal{F}) \geq \mathcal{V}_p(\mathcal{V}_p^n(\mathcal{G})) = \mathcal{V}_p^{n+1}(\mathcal{G})$, [5]. Thus $\mathcal{V}_p(\mathcal{F}) \xrightarrow{\tau_p q} x$. This means $(X, \tau_p q)$ is p-topological. □

4. Relations between Decomposition Series and Topological Series of Convergence Spaces

In this section, we will remind "topological series" defined by [5] and show relations between decomposition series and supratopological series, the formular $\pi^{\beta}(\tau_{\alpha} q) = \pi^{\omega^{\alpha}\beta} q$, where ω is the first limit ordinal and α and $\beta(\geq 1)$ are ordinals.

Let $q \in C(X)$ and $\alpha \geq 0$ ordinal number. The *topological series* for q is the descending ordinal sequence $\{\tau_{\alpha} q\}$ defined recursively on X as follows:

$\tau_0 q = q$

$\tau_1 q\ :\ \mathcal{F} \xrightarrow{\tau_1 q} x \iff \exists \mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}_q^n(\mathcal{G})$

$\tau_2 q\ :\ \mathcal{F} \xrightarrow{\tau_2 q} x \iff \exists \mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}^n_{\tau_1 q}(\mathcal{G})$

$\tau_3 q\ :\ \mathcal{F} \xrightarrow{\tau_3 q} x \iff \exists \mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}^n_{\tau_2 q}(\mathcal{G})$

$\vdots$

$\vdots$

$\tau_\alpha q\ :\ \mathcal{F} \xrightarrow{\tau_\alpha q} x \iff \exists \mathcal{G} \xrightarrow{q} x$, $n \in N$ and $\beta < \alpha$ such that $\mathcal{F} \geq \mathcal{V}^n_{\tau_\beta q}(\mathcal{G})$,

where we know that $\tau_1 q = \tau_q q$, $\tau_2 q = \tau_{\tau_1 q} q = \tau_{\tau_q q} q$, $\cdots$, etc. Also, we know that if there exists α' such that $\alpha = \alpha' + 1$, then

$\mathcal{F} \xrightarrow{\tau_\alpha q} x \iff \exists \mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}^n_{\tau_{\alpha'} q}(\mathcal{G})$,

PROPOSITION 4.1. ([5]). *For $q \in C(X)$, there exists $\widetilde{q}$ which is the finest q-topological convergence structure on X, and $\mathcal{F} \xrightarrow{\widetilde{q}} x$ iff $\mathcal{F} \geq \mathcal{V}^n_q(x)$ for some $n \in N$.*

LEMMA 4.2. *If $\mathcal{G} \xrightarrow{q} x$, then $\mathcal{V}^{n+1}_q(x) \leq \mathcal{V}^n_q(\mathcal{G})$.*

Proof. $A \in \mathcal{V}^{n+1}_q(x) \implies x \in I^{n+1}_q(A) \implies x \in I_q(I^n_q(A)) \implies I^n_q(A) \in \mathcal{V}_q(x) \implies I^n_q(A) \in \mathcal{G}$, since $\mathcal{G} \xrightarrow{q} x \implies \mathcal{G} \geq \mathcal{V}_q(x)$. Thus $A \in \mathcal{V}^n_q(\mathcal{G})$. □

PROPOSITION 4.3. $\widetilde{q} = \tau_1 q$.

Proof. We have already known $\widetilde{q} \geq \tau_1 q$, so it remain to show $\tau_1 q \geq \widetilde{q}$.

Let $\mathcal{F} \xrightarrow{\tau_1 q} x$. Then there exists $\mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}^n_q(\mathcal{G})$.

By the above Lemma, $\mathcal{F} \geq \mathcal{V}^n_q(\mathcal{G}) \geq \mathcal{V}^{n+1}_q(x)$, so $\mathcal{F} \xrightarrow{\widetilde{q}} x$. □

PROPOSITION 4.4. *(1) $q \geq \pi^n q \geq \widetilde{q} \geq \pi^\omega q$. (2) $\pi(\tau_1 q) = \pi^\omega q$.*

Proof. (1) It is clear that $q \geq \pi^n q$. Let $n \in N$ and $\mathcal{F} \in F(X)$. Then $\mathcal{F} \xrightarrow{\pi^n q} x \iff \mathcal{F} \geq \mathcal{V}^n_q(x) \implies \mathcal{F} \xrightarrow{\widetilde{q}} x$, since $\dot{x} \xrightarrow{q} x$. Thus, $\pi^n q \geq \widetilde{q}$ for each $n \in N$.

Also, $\mathcal{F} \xrightarrow{\widetilde{q}} x \implies \exists n \in N$ such that $\mathcal{F} \geq \mathcal{V}_q^n(x) \geq \cap_{m<\omega} \mathcal{V}_q^m(x) = \mathcal{V}_q^\omega(x) = \mathcal{V}_{\pi^\omega q}(x) \implies \mathcal{F} \xrightarrow{\pi^\omega q} x$.

(2) Since $\widetilde{q} = \tau_1 q$, by (1), $\pi(\tau_1 q) \geq \pi(\pi^\omega q) = \pi^\omega q$. While, by Theorem 3.7, $\pi(\tau_1 q) \leq \pi^\omega q$, since $\tau_1 q$ is a q-topological. Thus, $\pi(\tau_1 q) = \pi^\omega q$. □

We know that for $q \in C(X)$, the first term in the toplogical series for q is $\tau_1 q = \widetilde{q}$. $\tau_1 q$ is the finest topological convergence structure on X and also the lower q-topological modification of q, since $\tau_1 q = \widetilde{q} \leq \pi^n q \leq q$. Note that q has no upper q-topological modification unless q is a topology. We next show that that $\tau_2 q$ is related to $\tau_1 q$ exactly as $\tau_1 q$ is related to q. Note that the lower $\tau_1 q$-topological modification of $\tau_1 q$ is $\widetilde{\tau_1 q}$ defined by:

$$\mathcal{F} \xrightarrow{\widetilde{\tau_1 q}} x \iff \exists \mathcal{G} \xrightarrow{\tau_1 q} x \text{ and } n \in N \text{ such that} \mathcal{F} \geq \mathcal{V}_{\tau_1 q}^n(\mathcal{G}).$$

Proposition 4.5. *For any $q \in C(X)$, $\tau_2 q = \widetilde{\tau_1 q}$.*

Proof. $\mathcal{F} \xrightarrow{\tau_2 q} x \implies \exists \mathcal{G} \xrightarrow{q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}_{\tau_1 q}^n(\mathcal{G})$. But $\mathcal{G} \xrightarrow{\tau_1 q} x$ since $\tau_1 q \leq q$. Thus $\mathcal{F} \xrightarrow{\widetilde{\tau_1 q}} x$.

Conversely, $\mathcal{F} \xrightarrow{\widetilde{\tau_1 q}} x \implies \exists \mathcal{G} \xrightarrow{\tau_1 q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}_{\tau_1 q}^n(\mathcal{G})$. Also, $\mathcal{G} \xrightarrow{\tau_1 q} x \implies \exists \mathcal{H} \xrightarrow{q} x$ and $m \in N$ such that $\mathcal{G} \geq \mathcal{V}_q^m(\mathcal{H})$. Thus $\mathcal{F} \geq \mathcal{V}_{\tau_1 q}^n(\mathcal{V}_q^m(\mathcal{H})) \geq \mathcal{V}_{\tau_1 q}^n(\mathcal{V}_{\tau_1 q}^m(\mathcal{H})) = \mathcal{V}_{\tau_1 q}^{n+m}(\mathcal{H})$. Thus $\mathcal{F} \xrightarrow{\tau_2 q} x$. □

Proposition 4.6. *$\pi(\tau_1 q) = \pi^\omega q$ and $\pi(\tau_2 q) = \pi^\omega(\tau_1 q)$.*

Proof. The first equality follows from the Proposition 4.4. The second equality follows from $\pi(\tau_2 q) = \pi(\widetilde{\tau_1 q}) = \pi^\omega(\tau_1 q)$. □

Proposition 4.7. *If α is a limit ordinal, $\mathcal{V}_q^\alpha(x) = \cap_{\beta<\alpha} \mathcal{V}_q^\beta(x)$.*

Proof. $A \in \mathcal{V}_q^{\alpha}(x) \iff x \in I_q^{\alpha}(A) = \cap_{\beta<\alpha} I_q^{\beta}(A) \iff x \in I_q^{\beta}(A),\ \forall \beta < \alpha \iff A \in \mathcal{V}_q^{\beta}(x),\ \forall \beta < \alpha \iff A \in \cap_{\beta<\alpha} \mathcal{V}_q^{\beta}(x)$. □

PROPOSITION 4.8. *$\mathcal{V}_{\tau_n q}(x) = \mathcal{V}_q^{\omega^n}(x)$ and $\mathcal{V}_{\tau_\omega q}(x) = \mathcal{V}_q^{\omega^\omega}(x)$ for all $x \in X$.*

Proof. As we showed in Proposition 4.6, $\pi(\tau_2 q) = \pi^{\omega}(\tau_1 q)$. Thus for any $x \in X$, $\mathcal{V}_{\tau_2 q}(x) = \mathcal{V}_{\tau_1 q}^{\omega}(x)$. Also, by Proposition 4.4, $\mathcal{V}_{\tau_1 q}(x) = \mathcal{V}_q^{\omega}(x)$. By Corollary 2.5, $\mathcal{V}_{\tau_1 q}^2 q(x) = \mathcal{V}_{\tau_1 q}(\mathcal{V}_{\tau_1 q}(x)) = \mathcal{V}_q^{\omega}(\mathcal{V}_q^{\omega}(x)) = \mathcal{V}_q^{\omega+\omega}(x) = \mathcal{V}_q^{\omega 2}(x)$. Similarly, $\mathcal{V}_{\tau_1 q}^n(x) = \mathcal{V}_q^{\omega n}(x)$. Thus $\mathcal{V}_{\tau_1 q}^{\omega}(x) = \cap_{n<\omega} \mathcal{V}_q^{\omega n}(x) = \mathcal{V}_q^{\omega^2}(x)$.
Expending the reasoning of Proposition 4.6, we have $\mathcal{V}_{\tau_3 q}(x) = \mathcal{V}_{\tau_2 q}^{\omega}(x)$, for all $x \in X$, since $\pi(\tau_3 q) = \pi^{\omega}(\tau_2 q)$. $\mathcal{V}_{\tau_2 q}^2(x) = \mathcal{V}_{\tau_2 q}(\mathcal{V}_{\tau_2 q}(x)) = \mathcal{V}_q^{\omega^2}(\mathcal{V}_q^{\omega^2}(x)) = \mathcal{V}_q^{\omega^2 2}(x)$. Similarly, $\mathcal{V}_{\tau_2 q}^n(x) = \mathcal{V}_q^{\omega^2 n}(x)$, so $\mathcal{V}_{\tau_3 q}(x) = \mathcal{V}_{\tau_2 q}^{\omega}(x) = \cap_{n<\omega} \mathcal{V}_{\tau_2 q}^n(x) = \cap_{n<\omega} \mathcal{V}_q^{\omega^2 n}(x) = \mathcal{V}_q^{\omega^3}(x)$. Likewise, we obtain $\mathcal{V}_{\tau_n q}^n(x) = \mathcal{V}_q^{\omega^n}(x)$. This implies that $\mathcal{V}_{\tau_\omega q}(x) = \cap_{n<\omega} \mathcal{V}_q^{\omega^n}(x) = \mathcal{V}_q^{\omega^\omega}(x)$. □

For $q \in C(X)$ and any ordinal α, let $\tau_\alpha q$ and $\sigma_\alpha q$ be defined inductively by $\tau_0 q = \sigma_0 q$ and:
$\mathcal{F} \xrightarrow{\tau_\alpha q} x \iff \exists \mathcal{G} \xrightarrow{q} x,\ n \in N$ and $\beta < \alpha$ such that $\mathcal{F} \geq \mathcal{V}_{\tau_\beta q}^n(\mathcal{G})$,
$\mathcal{F} \xrightarrow{\sigma_\alpha q} x \iff \exists \mathcal{G} \xrightarrow{\sigma_\beta q} x,\ n \in N$ and $\beta < \alpha$ such that $\mathcal{F} \geq \mathcal{V}_{\sigma_\beta q}^n(\mathcal{G})$,

Note that $\tau_1 q = \sigma_1 q$ is the lower q-topological modification of q. If $\alpha + 1$ is any non-limit ordinal, $\sigma_{\alpha+1} q = \tau_1(\sigma_\alpha q)$; in other words, $\sigma_{\alpha+1} q$ is the lower $\sigma_\alpha q$-topological modification of $\sigma_\alpha q$. If α is a limit ordinal, $\sigma_\alpha q = \inf\{\sigma_\beta q : \beta < \alpha\}$. Our first goal is to prove $\sigma_\alpha q = \tau_\alpha q$ for every ordinal α.

PROPOSITION 4.9. *For any ordinal α, $\tau_\alpha q \geq \sigma_\alpha q$.*

Proof. Assume that $\tau_\beta q \geq \sigma_\alpha q$ for every ordinal $\beta < \alpha$. Then $\mathcal{F} \xrightarrow{\tau_\alpha q} x \implies \exists \mathcal{G} \xrightarrow{q} x$ and $\beta < \alpha$ such that $\mathcal{F} \geq \mathcal{V}^n_{\tau_\beta q}(\mathcal{G}) \geq \mathcal{V}^n_{\sigma_\beta q}(\mathcal{G})$.

Also, since $\mathcal{G} \xrightarrow{q} x$, $\mathcal{G} \xrightarrow{\sigma_\alpha q} x$. Thus $\mathcal{F} \xrightarrow{\sigma_\alpha q} x$. □

PROPOSITION 4.10. *For any ordinal α, $\tau_\alpha q = \sigma_\alpha q$.*

Proof. The result is known for $\alpha = 1$. Assume the equality holds for $\beta < \alpha$. By Proposition 4.9, it remains to show that $\mathcal{F} \xrightarrow{\sigma_\alpha q} x \implies \mathcal{F} \xrightarrow{\tau_\alpha q} x$.

Case 1. $\exists \alpha'$ such that $\alpha = \alpha' + 1$. Let $\mathcal{F} \xrightarrow{\sigma_\alpha q} x$. Then there exists $\mathcal{F} \xrightarrow{\sigma_{\alpha'} q} x$ and $n \in N$ such that $\mathcal{F} \geq \mathcal{V}^n_{\sigma_{\alpha'} q}(\mathcal{G}) = \mathcal{V}^n_{\tau_{\alpha'} q}(\mathcal{G})$. Also, by induction hypothesis. $\mathcal{G} \xrightarrow{\tau_{\alpha'} q} x$, so there exists $\mathcal{H} \xrightarrow{q} x$, $\beta < \alpha'$ and $m \in N$ such that $\mathcal{G} \geq \mathcal{V}^m_{\tau_\beta q}(\mathcal{H})$. Thus, $\mathcal{F} \geq \mathcal{V}^n_{\tau_{\alpha'} q}(\mathcal{G}) \geq \mathcal{V}^n_{\tau_{\alpha'} q}(\mathcal{V}^m_{\tau_\beta q}(\mathcal{H})) \geq \mathcal{V}^n_{\tau_{\alpha'} q}(\mathcal{V}^m_{\tau_{\alpha'} q}(\mathcal{H})) \geq \mathcal{V}^{n+m}_{\tau_{\alpha'} q}(\mathcal{H})$, and hence $\mathcal{F} \xrightarrow{\tau_\alpha q} x$.

Case 2. α is a limit ordinal. Then by induction hypothesis, $\tau_\beta q = \sigma_\beta q$ for $\beta < \alpha$, so $\sigma_\alpha q = \inf\{\sigma_\beta q : \beta < \alpha\} = \inf\{\tau_\beta q : \beta < \alpha\} = \tau_\alpha q$. □

PROPOSITION 4.11. *For any ordinal α, $\tau_1(\tau_\alpha q) = \tau_{\alpha+1} q$. Thus $\mathcal{V}_{\tau_{\alpha+1} q}(x) = \mathcal{V}^\omega_{\tau_\alpha q}(x)$ for all $x \in X$.*

Proof. The first assertion follows by Proposition 4.10 and the note preceding Proposition 4.9. The second follows Proposition 4.6, since $\pi(\tau_1 p) = \pi^\omega p$ holds for any convergence structure p, letting $p = \tau_\alpha q$. □

PROPOSITION 4.12. *For any ordinal α and $x \in X$, $\mathcal{V}_{\tau_\alpha q}(x) = \mathcal{V}^{\omega^\alpha}_q(x)$.*

Proof. We will use induction on α. For $\alpha = 1$, the result follows by Proposition 4.11. Assume the equality holds for every $\beta < \alpha$.

Case 1. Assume that there exists α' such that $\alpha = \alpha' + 1$. then by Proposition 4.11, $\mathcal{V}_{\tau_\alpha q}(x) = \mathcal{V}^\omega_{\tau_{\alpha'} q}(x)$, where by induction hypothesis,

$\mathcal{V}_{\tau_{\alpha'}q}(x) = \mathcal{V}_q^{\omega^{\alpha'}}(x)$. Thus $\mathcal{V}^2_{\tau_{\alpha'}q}(x) = \mathcal{V}^1_{\tau_{\alpha'}q}(\mathcal{V}^1_{\tau_{\alpha'}q}(x)) = \mathcal{V}_q^{\omega^{\alpha'}2}(x)$, and similarly $\mathcal{V}^n_{\tau_{\alpha'}q}(x) = \mathcal{V}_q^{\omega^{\alpha'}n}(x)$. Thus $\mathcal{V}_{\tau_\alpha q}(x) = \mathcal{V}^\omega_{\tau_{\alpha'}q}(x) = \cap_{n<\omega}\mathcal{V}^n_{\tau_{\alpha'}q}(x) = \cap_{n<\omega}\mathcal{V}_q^{\omega^{\alpha'}n}(x) = \mathcal{V}_q^{\omega^{\alpha'}\omega}(x) = \mathcal{V}_q^{\omega^{\alpha'+1}}(x) = \mathcal{V}_q^{\omega^\alpha}(x)$.

Case 2. Assume that α is a limit ordinal. By induction hypothesis, $\mathcal{V}_{\tau_\beta q}(x) = \mathcal{V}_q^{\omega^\beta}(x)$ for $\beta < \alpha$. Thus $\mathcal{V}_{\tau_\alpha q}(x) = \cap_{\beta<\alpha}\mathcal{V}_q^{\omega^\beta}(x) = \mathcal{V}_q^{\omega^\alpha}(x)$. □

Consequently, our last result is the following Theorems.

THOEREM 4.13. *For every ordinal α and $\beta \geq 1$ and every $x \in X$,*
(1) $\mathcal{V}^\beta_{\tau_\alpha q}(x) = \mathcal{V}_q^{\omega^\alpha\beta}(x)$.
(2) $\pi^\beta(\tau_\alpha q) = \pi^{\omega^\alpha\beta}q$.

Proof. (1) We will use induction on β. For $\beta = 1$, the result follows by Proposition 4.12. Assume the equality holds for every $\gamma < \beta$.

Case 1. $\exists\beta'$ such that $\beta = \beta'+1$. then by Corollary 2.5, $\mathcal{V}^\beta_{\tau_\alpha q}(x) = \mathcal{V}^{\beta'+1}_{\tau_\alpha q}(x) = \mathcal{V}^{\beta'}_{\tau_\alpha q}(\mathcal{V}_{\tau_\alpha q}(x)) = \mathcal{V}_q^{\omega^\alpha\beta'}(\mathcal{V}_q^{\omega^\alpha}(x)) = \mathcal{V}_q^{\omega^\alpha\beta'+1}(x) = \mathcal{V}_q^{\omega^\alpha\beta}(x)$.

Case 2. β is a limit ordinal. By induction hypothesis, $\mathcal{V}^\gamma_{\tau_\alpha q}(x) = \mathcal{V}_q^{\omega^\alpha\gamma}(x)$ for $\gamma < \beta$. Thus $\mathcal{V}^\beta_{\tau_\alpha q}(x) = \cap_{\gamma<\beta}\mathcal{V}^\gamma_{\tau_\alpha q}(x) = \cap_{\gamma<\beta}\mathcal{V}_q^{\omega^\alpha\gamma}(x) = \mathcal{V}_q^{\omega^\alpha\beta}(x)$.

(2) By (1), it is clear. □

Finally, we define the lengths of decomposition series and topological series of $q \in C(X)$, l_Dq, and l_Tq, respectively by:
l_Dq=inf$\{\lambda : \lambda$ is an ordinal such that $\pi^\lambda q = \pi^{\lambda+1}q\}$,
l_Tq =inf$\{\lambda : \lambda$ is an ordinal such that $\tau_\lambda q = \tau_{\lambda+1}q\}$,

We know that l_Dq=inf$\{\lambda : \lambda$ is an ordinal s.t. $I_q^\lambda(A) = I_q^{\lambda+1}(A)$, $\forall A \subseteq X\}$ =inf$\{\lambda : \lambda$ is an ordinal such that $\pi^\lambda q = \tau q\}$,

PROPOSITION 4.14. *For $q \in C(X)$ and an ordinal α,*
(1) if $l_Tq \leq \alpha$, then $\tau_\alpha q = \tau q$;
(2) if $l_Tq \leq \alpha$, then $l_Dq \leq \omega^\alpha$.

Proof. (1) Let $\lambda = l_T q$. Then $\tau_\lambda q = \tau_{\lambda+1} q = \tau q$. Since $\lambda \leq \alpha$, $\tau_\lambda q \geq \tau_\alpha q \geq \tau q$. Thus $\tau_\alpha q = \tau q$.

(2) Since $l_T q \leq \alpha$, $\tau_\alpha q = \tau q$. Thus $\pi(\tau_\alpha q) = \pi(\tau q)$, so $\pi^{\omega^\alpha} q = \tau q$. Finally, $l_D q \leq \omega^\alpha$. □

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Department of Mathematics,
Colleage of Natural Science,
Geongsang National University
Jinju 660-701 Korea
E-mail: sanghop@nongae.gsnu.ac.kr