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QUADRATIC TRANSFORMATIONS INVOLVING HYPERGEOMETRIC FUNCTIONS OF TWO AND HIGHER ORDER

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Abstract. By applying various known summation theorems to a general transformation formula based upon Bailey's transformation theorem due to Slater, Exton has obtained numerous and new quadratic transformations involving hypergeometric functions of order greater than two (some of which have typographical errors). We aim at *first* deriving a general quadratic transformation formula due to Exton and *next* providing a list of quadratic formulas (including the corrected forms of Exton's results) and some more results.

1. Introduction and Preliminaries

As a consequence of Bailey's transform method, Slater [4, p. 60] gave the following general hypergeometric transform

$$
\begin{aligned}
(1.1)\quad & \sum_{n=0}^{\infty} \frac{((a))_n\,((d))_n\,((v))_{2n}\,x^n\,y^n}{((h))_n\,((g))_n\,((f))_{2n}\,n!} \\
& \qquad \times {}_{U+D+V}F_{E+F+G}\left[\begin{matrix}(u),\ (d)+n,\ (v)+2n \\ (e),\ (g)+n,\ (f)+2n\end{matrix}\ ;\ x\right]
\end{aligned}
$$

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$$= \sum_{n=0}^{\infty} \frac{((d))_n\,((u))_n\,((v))_n\,x^n}{((g))_n\,((e))_n\,((f))_n\,n!}$$

$$\times_{A+E+V+1} F_{U+H+F} \left[\begin{matrix}(a),\ 1-(e)-n,\ (v)+n,\ -n\\ (h),\ \ 1-(u)-n,\ \ (f)+n\end{matrix}\,;\,(-1)^{1+E-U}\,y\right],$$

where the notation is that used as standard in the theory of hypergeometric functions (see, *e.g.*, Slater [4]).

In (1.1), if we take $D = 2$, $F = 1$, $U = V = E = G = 0$ with $d_1 = d - 1/2$, $d_2 = d$ and $f = 2d$, we obtain the following *corrected* version of the general quadratic transformation given by Exton [2, Eq. (2.1)]:

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{((a))_n\,(d-\frac{1}{2})_n\,x^n\,y^n}{((h))_n\,(d+\frac{1}{2})_n\,2^{2n}\,n!}\,{}_2F_1\left[\begin{matrix}d+n-\frac{1}{2},\ d+n\\ 2d+2n\end{matrix}\,;\,x\right]\\ &\quad= \sum_{n=0}^{\infty} \frac{(d-\frac{1}{2})_n\,(d)_n\,x^n}{(2d)_n\,n!}\,{}_{A+1}F_{H+1}\left[\begin{matrix}(a),\ \ -n\\ (h),\ 2d+n\end{matrix}\,;\,-y\right].\end{aligned} \tag{1.2}$$

By using the well-known result (see [3, p. 70])

$${}_2F_1\left[\begin{matrix}a-\frac{1}{2},\ a\\ 2a\end{matrix}\,;\,z\right] = \left(\frac{1}{2}+\frac{1}{2}\sqrt{1-z}\right)^{1-2a}, \tag{1.3}$$

we get another result due to Exton [2]:

$$\begin{aligned}&\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d}\\ &\quad\times \sum_{n=0}^{\infty} \frac{((a))_n\,(d-\frac{1}{2})_n}{((h))_n\,(d+\frac{1}{2})_n\,n!}\left\{\frac{xy}{(1+\sqrt{1-x})^2}\right\}^n\\ &\quad= \sum_{n=0}^{\infty} \frac{(d-\frac{1}{2})_n\,(d)_n\,x^n}{(2d)_n\,n!}\,{}_{A+1}F_{H+1}\left[\begin{matrix}(a),\ \ -n\\ (h),\ 2d+n\end{matrix}\,;\,-y\right],\end{aligned} \tag{1.4}$$

which can also be rewritten in the following form:

$$\begin{aligned}(1.5)\qquad &\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_{A+1}F_{H+1}\left[\begin{matrix}(a),\ d-\frac{1}{2}\\ (h),\ d+\frac{1}{2}\end{matrix};\ \frac{xy}{(1+\sqrt{1-x})^2}\right]\\ &=\sum_{n=0}^{\infty}\frac{(d-\frac{1}{2})_n\,(d)_n\,x^n}{(2d)_n\,n!}\ {}_{A+1}F_{H+1}\left[\begin{matrix}(a),\quad -n\\ (h),\ 2d+n\end{matrix};\ -y\right].\end{aligned}$$

Note that if the inner hypergeometric function on the right-hand side of (1.5) can be summed by means of certain known summation theorems, then their resulting quadratic transformations are obtained.

For this, the following known summation theorems (see Appendix III in [4]) will be required in our present investigation:

$$(1.6)\qquad {}_2F_1\left[\begin{matrix}a,\ -n\\ c\end{matrix};\ 1\right]=\frac{(c-a)_n}{(c)_n}.$$

$$(1.7)\qquad {}_2F_1\left[\begin{matrix}a,\quad -n\\ 1+a+n\end{matrix};\ -1\right]=\frac{(1+a)_n}{(1+\frac{1}{2}a)_n}.$$

$$(1.8)\qquad {}_3F_2\left[\begin{matrix}a,\quad b,\quad -n\\ 1+a-b,\ 1+a+n\end{matrix};\ 1\right]=\frac{(1+a)_n\,(1+\frac{1}{2}a-b)_n}{(1+\frac{1}{2}a)_n\,(1+a-b)_n}.$$

$$(1.9)\qquad {}_3F_2\left[\begin{matrix}a,\ 1+\frac{1}{2}a,\ -n\\ \frac{1}{2}a,\quad b\end{matrix};\ 1\right]=\frac{(b-a-1)_n\,(2+a-b)_n}{(b)_n\,(1+a-b)_n}.$$

$$(1.10)\qquad {}_4F_3\left[\begin{matrix}a,\quad 1+\frac{1}{2}a,\quad b,\quad -n\\ \frac{1}{2}a,\ 1+a-b,\ 1+a+n\end{matrix};\ 1\right]=\frac{(1+a)_n\,(\frac{1}{2}+\frac{1}{2}a-b)_n}{(\frac{1}{2}+\frac{1}{2}a)_n\,(1+a-b)_n}.$$

$$(1.11)\qquad {}_4F_3\left[\begin{matrix}a,\quad 1+\frac{1}{2}a,\quad b,\quad -n\\ \frac{1}{2}a,\ 1+a-b,\ 1+a+n\end{matrix};\ -1\right]=\frac{(1+a)_n}{(1+a-b)_n}.$$

$$\begin{aligned}(1.12)\qquad &{}_5F_4\left[\begin{matrix}a,\quad 1+\frac{1}{2}a,\quad b,\quad c,\quad -n\\ \frac{1}{2}a,\ 1+a-b,\ 1+a-c,\ 1+a+n\end{matrix};\ 1\right]\\ &=\frac{(1+a)_n\,(1+a-b-c)_n}{(1+a-b)_n\,(1+a-c)_n}.\end{aligned}$$

We aim at *first* deriving a general quadratic transformation formula (1.2) by following a different method from that of Exton and *next* providing a list of quadratic formulas (including the corrected forms of Exton's results) and some more results.

2. Main quadratic transformations

The following quadratic transformations involving hypergeometric functions of order two or greater than two will be established.

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_2F_1\left[\begin{matrix} a,\ d-\frac{1}{2} \\ d+\frac{1}{2}\end{matrix};-\frac{x}{(1+\sqrt{1-x})^2}\right] = {}_3F_2\left[\begin{matrix} d-\frac{1}{2},\ d-\frac{1}{2}a,\ d-\frac{1}{2}a+\frac{1}{2} \\ 2d-a,\quad d+\frac{1}{2}\end{matrix};x\right]. \tag{2.1}$$

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_2F_1\left[\begin{matrix} 2d-1,\ d-\frac{1}{2} \\ d+\frac{1}{2}\end{matrix};\frac{x}{(1+\sqrt{1-x})^2}\right] = {}_2F_1\left[\begin{matrix} d-\frac{1}{2},\ d \\ d+\frac{1}{2}\end{matrix};x\right]. \tag{2.2}$$

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} 2d-1,\ b,\ d-\frac{1}{2} \\ 2d-b,\ d+\frac{1}{2}\end{matrix};-\frac{x}{(1+\sqrt{1-x})^2}\right] = {}_3F_2\left[\begin{matrix} d-\frac{1}{2},\ d,\ d-b+\frac{1}{2} \\ 2d-b,\quad d+\frac{1}{2}\end{matrix};x\right]. \tag{2.3}$$

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix} a,\ 1+\frac{1}{2}a,\ d-\frac{1}{2} \\ \frac{1}{2}a,\ d+\frac{1}{2}\end{matrix};-\frac{x}{(1+\sqrt{1-x})^2}\right] = {}_3F_2\left[\begin{matrix} d-\frac{1}{2},\ d-\frac{1}{2}a-\frac{1}{2},\ d-\frac{1}{2}a \\ d+\frac{1}{2},\quad 2d-a\end{matrix};x\right]. \tag{2.4}$$

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_2F_1\left[\begin{matrix} 2d-1,\ b \\ 2d-b\end{matrix};-\frac{x}{(1+\sqrt{1-x})^2}\right] = {}_2F_1\left[\begin{matrix} d-\frac{1}{2},\ d-b \\ 2d-b\end{matrix};x\right]. \tag{2.5}$$

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_2F_1\left[\begin{matrix}2d-1,\ b\\ 2d-b\end{matrix};\frac{x}{(1+\sqrt{1-x})^2}\right] = {}_2F_1\left[\begin{matrix}d-\frac{1}{2},\ d\\ 2d-b\end{matrix};x\right]. \tag{2.6}$$

$$\left(\frac{1}{2}+\frac{1}{2}\sqrt{1-x}\right)^{1-2d} {}_3F_2\left[\begin{matrix}2d-1, & b, & c\\ 2d-b, & 2d-c\end{matrix};-\frac{x}{(1+\sqrt{1-x})^2}\right] = {}_3F_2\left[\begin{matrix}d-\frac{1}{2}, & d, & 2d-b-c\\ 2d-b, & 2d-c\end{matrix};x\right]. \tag{2.7}$$

3. Derivations

We begin by deriving the result (1.2). For this, let us denote the left-hand side of (1.2) by L, express the ${}_2F_1$ as a series, by using

$$(\alpha)_m\,(\alpha+m)_n=(\alpha)_{m+n} \quad \text{and} \quad (\alpha)_{2m}=2^{2m}\left(\frac{\alpha}{2}\right)_m\left(\frac{\alpha}{2}+\frac{1}{2}\right)_m,$$

we have

$$\begin{aligned}
L &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a))_m\,(d-\frac{1}{2})_m\,(d-\frac{1}{2}+m)_n\,(d+m)_n\,x^{m+n}\,y^m}{((h))_m\,(d+\frac{1}{2})_m\,(2d+2m)_n\,2^{2m}\,m!\,n!}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{((a))_m\,(d-\frac{1}{2})_{m+n}\,(d)_{m+n}\,x^{m+n}\,y^m}{((h))_m\,(2d)_{2m+n}\,m!\,n!}\\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{((a))_m\,(d-\frac{1}{2})_n\,(d)_n\,x^n\,y^m}{((h))_m\,(2d)_{m+n}\,m!\,(n-m)!}.
\end{aligned}$$

By applying $(n-m)!=(-1)^m\,n!/(-n)_m$ to the last equality, we get

$$\begin{aligned}
L &= \sum_{n=0}^{\infty}\frac{(d-\frac{1}{2})_n\,(d)_n\,x^n}{(2d)_n\,n!}\sum_{m=0}^{n}\frac{((a))_m\,(-n)_m\,(-y)^m}{((h))_m\,(2d+n)_m\,m!}\\
&= \sum_{n=0}^{\infty}\frac{(d-\frac{1}{2})_n\,(d)_n\,x^n}{(2d)_n\,n!}\,{}_{A+1}F_{H+1}\left[\begin{matrix}(a), & -n\\ (h), & 2d+n\end{matrix};-y\right],
\end{aligned}$$

which is the right-hand side of (1.2).

Next we come back to derive the quadratic transformations (2.1) to (2.7). It is observed that the inner hypergeometric function on the right-hand side of (1.5) can be summed by means of the formulas (1.6) to (1.12) and the quadratic transformations (2.1) to (2.7) can be obtained. Here, we shall mention only the proof of the result (2.3) in some detail and other things are presented without proof. For this, if we take $A = 2$, $H = 1$, $y = -1$, $a_1 = 1$, $a_2 = b$ and $h = 1 + a - b$ in (1.5), then the inner hypergeometric function on the right-hand side of (1.5) takes the form

$$ {}_3F_2 \left[\begin{matrix} 2d-1, & b, & -n \\ 2d-b, & 2d+n \end{matrix} ; 1 \right] $$

which, upon using (1.8), reduces to

$$ \frac{(2d)_n \, (\frac{1}{2} + d - b)_n}{(d + \frac{1}{2})_n \, (2d - b)_n}. $$

On inserting this result into the resulting equation obtained from (1.5) by the above-mentioned substitutions, we get (2.3). In a similar manner, the results (2.1), (2.2), (2.4) to (2.7) can be established from (1.6), (1.7), (1.9) to (1.12), respectively.

We conclude this paper by remarking that results (2.1) and (2.7) are given by Exton [2], and (2.3) and (2.4) are also corrected versions of Exton's results [2].

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