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A SPECIAL REDUCEDNESS IN NEAR-RINGS

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ABSTRACT. A near-ring N is reduced if, for $a \in N$, $a^2 = 0$ implies $a = 0$, and N is left strongly regular if for all $a \in N$ there exists $x \in N$ such that $a = xa^2$. Mason introduced this notion and characterized left strongly regular zero-symmetric unital near-rings. Several authors ([2], [5], [7]) studied these properties in near-rings. Reddy and Murty extended some results in Mason to the non-zero symmetric case. In this paper, we will define a concept of strong reducedness and investigate a relation between strongly reduced near-rings and left strongly regular near-rings.

1. Introduction

Throughout this paper we work with right near-rings. So all our near-rings are right near-rings.

A near-ring N is reduced if, for $a \in N$, $a^2 = 0$ implies $a = 0$, and N is left strongly regular if for all $a \in N$ there exists $x \in N$ such that $a = xa^2$. Right strong regularity is defined in a symmetric way. Mason [4] introduced this notion and characterized left strongly regular zero-symmetric unital near-rings. Several authors ([2], [5], [7]) studied these properties in near-rings.

Reddy and Murty [7] extended some results in Mason [4] to the non-zero symmetric case. They observed that every left strongly

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regular near-ring has the following interesting property (*). In this paper we will consider this property. Let N be a near-ring and let N_c denote the constant part of N. We will define a special reduced near-ring N, what is called strongly reduced.

We will show that every strongly reduced near-ring N is reduced, and that strong reducedness is a general concept of the following property (*):

(i) for any $a, b \in N$, $ab = 0$ implies $ba = b0$.

(ii) for $a \in N$, $a^3 = a^2$ implies $a^2 = a$ in Reddy and Murty [7].

Left or right strongly regular near-rings form one of the important class of strongly reduced near-rings. We investigate some properties of strongly reduced near-rings. Using strong reducedness, we characterize left strongly regular near-rings.

For notations and basic concepts, we shall refer to Pilz's book [6].

2. Results

For any near-ring N, N_c denotes the constant part of N, that is, $N_c = \{a \in N \mid a = a0\}$. A near-ring N is said to be *strongly reduced* if, for $a \in N$, $a^2 \in N_c$ implies $a \in N_c$. Obviously N is strongly reduced if and only if, for $a \in N$ and any positive integer n, $a^n \in N_c$ implies $a \in N_c$. We will show that a strongly reduced near-ring is reduced, that is, for $a \in N$, $a^2 = 0$ implies $a = 0$.

A subnear-ring H of a near-ring N is called *left invariant* if $NH \subseteq H$, *right invariant* if $HN \subseteq H$ and *invariant* if it is both left and right invariant. For a subset S of N, $< S|$, $|S >$ and $< S >$ stand for the left invariant, right invariant and invariant subnear-rings of N generated by S respectively. For any element $a \in N$, $< a|$, $|a >$ and $< a >$ are called *the principal left invariant, principal right invariant* and *principal invariant* subnear-rings of N generated by a, respectively.

There are slightly generalized new concepts of left strong regularity and right strong regularity. A near-ring N is called *quasi left strongly regular* if $a \in < a^2|$ for each $a \in N$, *quasi right strongly regular* if $a \in |a^2 >$ for each $a \in N$.

There are lots of quasi left (right resp.) strongly regular near-rings which are not left (right resp.) strongly regular.

First we introduce an equationally defined classes of strongly reduced near-rings which are easily proved by the definition of strongly reduced near-rings as the following Lemma.

LEMMA 1. *(1) The direct product of strongly reduced near-rings is strongly reduced, and vice versa.*

(2) Every subring of a strongly reduced near-ring is strongly reduced.

(3) Every homorphic image of a strongly reduced contant near-ring is strongly reduced.

Now we give some sufficient conditions for quasi left strongly regular near-rings or quasi right strongly regular near-rings to be strongly reduced.

PROPOSITION 2. *(1) The direct sum or a subdirect product of strongly reduced near-rings is strongly reduced.*

(2) All quasi left strongly regular near-rings and quasi right strongly regular near-rings are strongly reduced. In particular, all right or left strongly regular near-rings are strongly reduced.

(3) Every integral near-ring N is strongly reduced. Hence a subdirect product of integral near-rings is strongly reduced.

Proof. (1) Obvious from Lemma 1.

(2) Note that the constant part N_c is an invariant subnear-ring of N. Suppose $a \in < a^2 >$ for each $a \in N$. If $a^2 \in N_c$ then $a \in < a^2 > \subseteq N_c$.

(3) Let $a \in N$ with $a^2 \in N_c$. Then $(a - a^2)a = 0$, and hence $a = a^2 \in N_c$. □

We state some basic properties of a special reduced near-ring.

PROPOSITION 3. *Let N be a strongly reduced near-ring and let $a, b \in N$. Then we have the following.*

(1) N is reduced.

(2) If $ab^n \in N_c$ for any positive integer n, then $\{ab, ba\} \cup aNb \cup bNa \subseteq N_c$. In particular, $ab \in N_c$ implies $ba \in N_c$, $aNb \in N_c$ and $bNa \in N_c$.

(3) If $ab^n = 0$ for any positive integer n, then $ab = 0$ and $ba = b0$. In particular, $ab = 0$ implies $ba = b0$ (Reddy and Murty's property (*) (i), [7]).

Proof. (1) Assume that $a^2 = 0$. Then $a^2 \in N_c$, and hence $a \in N_c$. Then we see $a = a0 = a0a = aa = 0$.

(2) First suppose $ab \in N_c$. Then $(ba)^2 = baba = bab0a = bab0 \in N_c$. Since N is strongly reduced, we have $ba \in N_c$. Then we obtain $xba \in N_c$ for each $x \in N$, whence $(axb)^2 \in N_c$. By the strong reducibility of N, we obtain $axb \in N_c$ for each $x \in N$. Since $ba \in N_c$, we also obtain $bNa \subseteq N_c$. Now suppose $ab^n \in N_c$. Then $(ab)^n \in N_c$ by the above argument. Since N is strongly reduced, this implies $ab \in N_c$. Hence by the first paragraph, the claim is proved.

(3) If $ab^n = 0$ for some $n \geq 1$, then $ab \in N_c$ by (2). Hence $ab = abb^{n-1} = ab^n = 0$. Then $(ba)^2 = baba = b0 \in N_c$. Hence $ba \in N_c$. Therefore $(ba)^2 - ba \in N_c$. Then $(ba)^2 - ba = \{(ba)^2 - ba\}b = babab - bab = b0 - b0 = 0$. Hence we obtain $ba = (ba)^2 = b0$.□

In case N is a zero-symmetric near-ring, clearly N is strongly reduced if and only if N is reduced. A near-ring N is called a (P_0)-near-ring if, for each $a \in N$, there exists an integer $n > 1$ such that $a = a^n$ (see [6, 9.4, p.289]).

EXAMPLE 4. (1). Every left strongly regular near-ring or right strongly regular near-ring is strongly reduced.

(2). Every quasi left strongly regular near-rings or quasi right strongly regular near-rings is strongly reduced.

(3). Every integral near-ring or constant near-ring is strongly reduced. In particular, every Boolean near-ring or (P_0)-near-ring is strongly reduced.

The following examples show that a reduced near-ring is not necessarily strongly reduced.

EXAMPLE 5. Let $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ with addition modulo 6 and define multiplication as follows (Pilz [6] near-rings of low order; $\mathbb{Z}_6$ No. 21):

$\cdot$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	3	3	1	3	1	1
2	0	0	2	0	2	2
3	3	3	3	3	3	3
4	0	0	4	0	4	4
5	3	3	5	3	5	5

Obviously this is a reduced near-ring. The constant part of $\mathbb{Z}_6$ is $\{0, 3\}$. Since $1^2 = 3$ is a constant element but 1 is not, this near-ring is not strongly reduced. Also note that $1^n \neq 1$ for any integer $n > 1$.

EXAMPLE 6. Let $V = \{0, a, b, c, \}$ be a Klein's four group under addition.

(1) We define multiplication as follows (Pilz [6] near-rings of low order;V No. 20):

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	a	b	c
c	a	0	c	b

The constant part of this near-ring is $\{0, a\}$. Clearly, this near-ring is reduced and strongly reduced.

(2) We have multiplication table as follows (Pilz [6] near-rings of low order;V No. 19):

$\cdot$	0	a	b	c
0	0	0	0	0
a	a	a	a	a
b	0	0	0	b
c	a	a	a	c

The constant part of this near-ring is $\{0, a\}$. Obviously, this near-ring is not reduced, for $b^2 = 0$ and also not strongly reduced.

Now we consider polynomial near-rings over commutative unital rings and polynomial near-rings on groups (Lausch and Nöbauer [3], 8.11, 9.11, Pilz [6], 7.61). Let R be a commutative ring with unity 1, G an additive group, x an indeterminate, $R[x]$ the set of all polynomials over R and

$$G[x] = \{a_0 + n_1 x + a_1 + n_2 x + a_2 + \cdots + a_{t-1} + n_t x + a_t | t \in \mathbb{N}_0,$$
$$a_i \in G,\ n_i \in \mathbb{Z}^* \ and\ a_1 \neq 0, a_2 \neq 0, \cdots, a_{t-1} \neq 0\}.$$

Then $(R[x],\ +,\ \circ)$ and $(G[x],\ +,\ \circ)$ are near-rings with unity x respectively, where $\circ$ is substitution. In this case, we say that $R[x]$ is a polynomial near-ring over R and $G[x]$ is a polynomial near-ring on G. We see that $(R[x])_c = R$ and $(R[x])_0 = \{\sum_{i=1}^{n} a_i x^i | n \in \mathbb{Z}^+\}$ such that $R[x] = (R[x])_c + (R[x])_0$.

Next, for any $f(x) \in R[x]$, a map $f : R \longrightarrow R$ by $a \rightsquigarrow f(x) \circ a = f(a)$ is called the *polynomial function induced by f(x).* We denote that $P(R) = \{f | f(x) \in R[x]\}$ the set of all polynomial functions on R. Similarly, one can define f for $f(x) \in G[x]$ and $P(G)$ the set of all polynomial functions on G. It is well known that $P(R)$ and $P(G)$ are subnear-rings of $M(R)$ ($M(G)$, respectively), and are called *near-rings of polynomial functions* on R (on G, respectively)([6, 7.65, 7.66]).

EXAMPLE 7. (1). Consider a group $(\mathbb{Z}_2,\ +)$ and a commutative ring $(\mathbb{Z}_2,\ +,\ \cdot)$. All 3-kinds of near-rings on a group $(\mathbb{Z}_2,\ +)$ are strongly reduced. Thus every non-zero symmetric near-field is strongly reduced.

$\mathbb{Z}_2[x]$ and $P(\mathbb{Z}_2) = \{0,\ 1,\ x,\ x+1\}$ are strongly reduced.

(2). All 12-kinds of near-rings on a group $(\mathbb{Z}_4,\ +)$ are strongly reduced. However, $\mathbb{Z}_4[x]$ and $P(\mathbb{Z}_4) = \{0,\ 1,\ x,\ x+1\}$ are not strongly reduced.

We give equivalent conditions for a near-ring N to be strongly reduced.

THEOREM 8. *The following statements are equivalent for a near-ring N:*

(1) N is strongly reduced.

(2) For $a \in N$, $a^3 = a^2$ implies $a^2 = a$.

(Reddy and Murty's property (*) (ii), [7]).

(3) If $a^{n+1} = xa^{n+1}$ for $a, x \in N$ and some nonnegative integer n, then $a = xa = ax$.

Proof. (1) $\Longrightarrow$ (2). Assume that $a^3 = a^2$. Then $(a^2 - a)a = 0$, whence $a(a^2 - a) = a0 \in N_c$ by Proposition 3 (3). Then $(a^2 - a)a^2 = (a^3 - a^2)a = 0a = 0$. Again by Proposition 3 (3) $a^2(a^2 - a) = a^2 0 \in N_c$. Hence $(a^2 - a)^2 = a^2(a^2 - a) - a(a^2 - a) = a^2 0 - a0 = (a^2 - a)0 \in N_c$. This implies $a^2 - a \in N_c$. Hence $a^2 - a = (a^2 - a)0 = (a^2 - a)a = 0$.

(2) $\Longrightarrow$ (1). Assume $a^2 \in N_c$. Then $a^3 = a^2 a = a^2$. By condition (2), this implies $a = a^2 \in N_c$.

(1) $\Longrightarrow$ (3). Suppose $a^{n+1} = xa^{n+1}$ for some $n \geq 0$. Then $(a - xa)a^n = 0$. Hence $(a - xa)a = 0$ by Proposition 3 (3), and so $(a - xa)^2 \in N_c$ by Proposition 3 (2). Since N is strongly reduced, we have $a - xa \in N_c$. Then $a - xa = (a - xa)a = 0$, that is $a = xa$. Now $(a - ax)a = a^2 - axa = a^2 - a^2 = 0 \in N_c$. Hence $(a - ax)^2 = a(a - ax) - ax(a - ax) \in N_c$ by Proposition 3 (2), and so $a - ax \in N_c$. Therefore $a - ax = (a - ax)a = 0$.

(3) $\Longrightarrow$ (2). This is obvious. □

Left strongly regular near-rings are studied by several authors ([3]-[5], [7] etc.) Since all left strongly regular near-rings are strongly reduced, we can use it to study left strongly regular near-rings.

The following is a generalization of [7, Theorem 3].

LEMMA 9. *Let N be a strongly reduced near-ring and let $a, x \in N$. If $a^n = xa^{n+1}$ for some positive integer n, then $a = xa^2 = axa$ and $ax = xa$.*

Proof. Assume that $a^n = xa^{n+1}$ for some $n \geq 1$. By Theorem 8 (3), $a = xa^2 = axa$. Then $(ax - xa)a = 0$. Hence, by Theorem 8 (2),

$(ax - xa)^2 = ax(ax - xa) - xa(ax - xa) \in N_c$. Since N is strongly reduced, $ax - xa \in N_c$. Hence $ax - xa = (ax - xa)a = 0$. □

A near-ring N is said to be *left strongly π-regular* if, for each $a \in N$, there exists a positive integer n and an element $x \in N$ such that $a^n = xa^{n+1}$, this equation is equivalent to $a^n = xa^{2n}$. Here we give some characterizations of left strongly regular near-rings.

THEOREM 10. *Let N be a near-ring. Then the following statements are equivalent:*
(1) N is left strongly regular.
(2) N is strongly reduced and left strongly π-regular.
(3) For each $a \in N$, there exists $x, y \in N$ such that $a = xa^2ya$.
(4) For each $a \in N$, $a \in < a^2 > \cap aNa$.

Proof. (1) $\Longrightarrow$ (2) - (4). By Proposition 2 (1), left or right strongly regular near-ring is strongly reduced. Hence this follows from Lemma 9.

(2) $\Longrightarrow$ (1). This also follows from Lemma 9.

(3) $\Longrightarrow$ (1). By hypothesis, N is strongly reduced. If $a = xa^2ya$, then $ya = yxa^2(ya)$. By Theorem 8, $ya = yayxa^2$. Thus $a = xa^2yayxa^2$. This implies that N is left strongly regular.

(4) $\Longrightarrow$ (1). Since $a \in < a^2 >$ for each $a \in N$, N is strongly reduced by Proposition 2 (1). Hence N satisfies (4) in Theorem 8. Since $a \in aNa$, there exists $x \in N$ such that $a = axa$. Hence $a = (ax)a = a(ax) = a^2x$. Then we have $a = axa = (a^2x)xa = a^2x^2a$. Then, by the same way as in □

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